International Journal of Discrete Mathematics
2025; 7(1): 14-19
http://www.sciencepublishinggroup.com/j/dmath
doi: 10.11648/j.dmath.20250701.12

On Sequentially Perfect and Sequentially Uniform One Factorizations of Product of Cycles

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To cite this article:

Jamuna Loganathan, Hemalatha Palanisamy. (2025). On Sequentially Perfect and Sequentially Uniform One Factorizations of Product of Cycles. *International Journal of Discrete Mathematics, 7*(1), 14-19. https://doi.org/10.11648/j.dmath.20250701.12

Received: 12 August 2025; **Accepted:** 28 August 2025; **Published:** 25 September 2025

Abstract: Let G be an n-regular graph. A sequentially perfect one-factorization is a decomposition of G into one-factors, such that union of any two consecutive one-factors gives a hamiltonian cycle, and also if the union of two consecutive one-factors forms a two regular graph, then this property is termed as sequentially uniform. For any two graphs G and H their product [Cartesian, Tensor, Wreath] has the vertex set $\{(g, h);\ g$ belongs to $V(G)$ and h belongs to $V(H)\}$. The Cartesian Product G Cartesian Product H has the edge set such that two vertices (g_1, h_1) and (g_2, h_2) are adjacent if they differ in exactly one coordinate, and the corresponding vertices in that graph are adjacent; in the tensor product G Tensor Product H two vertices (g_1, h_1) and (g_2, h_2) are adjacent only if both g_1 is adjacent to g_2 in G and h_1 is adjacent to h_2 in H; in the wreath product G Wreath Product H the edge set is $E(G$ Wreath Product $H) = \{(g_1, h_1)(g_2, h_2) : g_1g_2$ belongs to $E(G)$ or $g_1 = g_2$ and h_1h_2 belongs to $E(H)\}$. In this paper it is proved that for all even $m > 2$ and odd $n \geq 3$, there exists sequentially perfect and sequentially uniform one-factorizations in C_m Cartesian Product C_n, C_m Tensor Product C_n and C_m Wreath Product C_n.

Keywords: Cycles, Cartesian Product of Graphs, Tensor Product of Graphs, Wreath Product of Graphs, One-factorization, Sequentially Perfect One-factorization, Sequentially Uniform One-factorization

1. Introduction

Here we consider finite, undirected graphs only. For two graphs G and H, their *Cartesian Product* $G\square H$, *Tensor Product* $G \times H$ and the *Wreath Product* $G \otimes H$ has the same vertex set $V(G) \times V(H)$ = {(g, h); g $\in$ V(G) and h $\in$ V(H) } and edge set defined by E($G\square H$) = {(g, h) ($g^{'}$, $h^{'}$): g=$g^{'}$ and h$h^{'}$$\in$E(H) or h=$h^{'}$ and g$g^{'}$$\in$E(G)}, E($G \times H$) = {(g, h) ($g^{'}$, $h^{'}$): g$g^{'}$$\in$E(G) and h$h^{'}$$\in$E(H)} and E($G \otimes H$) = {(g, h) ($g^{'}$, $h^{'}$): g$g^{'}$$\in$E(G) or g=$g^{'}$ and h$h^{'}$$\in$E(H)}. An one-factor in a graph G is an one-regular spanning subgraph. If G can be decomposed into set of one-factors say, $\{F_1, F_2, ..., F_n\}$ then it is called an one-factorization of G. An one-factorization of G is said to be perfect if the union of any pair of one-factors is a hamiltonian cycle of G. A one-factorization of a graph is called uniform if the union of each pair of one-factors is isomorphic to a two-regular subgraph. An one-factorization of a graph G is said to be k-semi-perfect, if $F_i \cup F_j$ forms a hamilton cycle for every $1 \leq i \leq k$ and every $k + 1 \leq j \leq n$. An one-factorization is sequentially uniform, if the one-factors can be ordered as $(F_1, F_2, ..., F_k)$ so that the graphs G_i with edge set $F_i \cup F_{i+1}, i = 1, 2, 3, ..., k - 1$ forms a 2-regular graph. Of course G_i is non-hamiltonian, whenever G_i is sequentially uniform. If $F_i \cup F_{i+1}$, for $i = 1, 2, 3, ..., k - 1$ is a Hamiltonian cycle, then the one-factorization is said to be sequentially perfect. If G can be factorized into p cycles of length l, then it is said to be of type (l, l, ...,l). The concept of one-factorization has been introduced in 1891 by Petersen [10]. A detailed study on one-factorizations in complete graphs, regular graphs and product graphs can be seen in [1, 11–14]. The study on perfect one-factorizations in complete graph K_n for specific values of n can be seen in [2–4, 7, 15, 16]. Further, the semi-perfect one-factorization of cartesian product of graphs has been studied in [9]. In [5] Dinitz et al, proved that there exist sequentially perfect and sequentially uniform one-factorizations in complete graphs. In this paper, we use the concept of one-factor of jump i and the graph labeling technique to find sequentially perfect and sequentially uniform one-factorizations in Cartesian product, Tensor product and Wreath product of cycles. Also, we have generated an algorithm (see Appendix) to get the order of one-factors required to construct sequentially perfect and sequentially uniform one-factorizations.

2. Preliminaries

2.1. One-factor of Jump i [6]

Let G be a k-regular bipartite graph with bipartition (X, Y) where $X = \{x_0, x_1, \ldots, x_{r-1}\}$, $Y = \{y_0, y_1, \ldots, y_{r-1}\}$. Let the edge set of G be $E(G) = \{F_i[X, Y] = x_j y_{i+j},\ 0 \leq j \leq r-1\}$ for $i \in \{1, 2, \ldots, k\}$. Then for a particular value of $i \in \{1, 2, \ldots, k\}$, the edges of $F_i[X, Y]$ are called the edges with jump i and the one-factor thus formed is known as an one-factor of jump i. Since G is k-regular we have k such one-factors $F_i[X, Y]$ in G for $i \in \{1, 2, \ldots, k\}$. If $G = K_{r,r}$ then $E(G) = \bigcup_{i=0}^{r-1} F_i[X, Y]$.

2.2. Edge Labeling [8]

Edge labeling of G is nothing but assigning labels to the edges of G. A proper edge labeling of a graph G is an edge labeling of G in which no two adjacent edges receive the same label. Clearly, if G is an n-regular and has proper edge labeling with n different labels, then every edge label gives a 1-factor of G. Thus a proper edge labeling of an n-regular graph G with n labels produces a 1-factorization of G.

3. Sequentially Perfect and Sequentially Uniform One-Factorizations in Cartesian Product of Cycles

3.1. Theorem

For given even positive integers $m, n > 2$, $C_m \Box C_n$ admits a Sequentially-Uniform one-factorization.

Proof. Let $G = C_m \Box C_n$, *i.e.*, G is an m-partite graph with m layers of n vertices each. Let $V(G) = (u_{ij})_{m \times n}$ and $E(G) = \{\bigoplus_{j=1}^{n} u_{ij} u_{i(j+1)},\ \ i \in \{1, 2, ..., m\}\} \cup \{\bigoplus_{i=1}^{m} u_{ij} u_{(i+1)j},\ \ j \in \{1, 2, ..., n\}\}$, where the subscripts $j+1$ is taken modulo n and the subscripts $i+1$ is taken modulo m. Since m and n are even we have 4 one-factors F_k of G for $k = 1, 2, 3, 4$. To get a Sequentially Uniform one-factorization in G, we proceed with the labeling of vertices and edges of G to obtain the order of one-factors as follows: Let $f : V(G) \to \{1, 2, 3, \ldots, mn\}$ be a labeling of G defined by,

$$f[V(G) = \left[\begin{array}{c|c|c|c} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} & \begin{bmatrix} 2m+1 & 2m+3 \\ 2m+2 & 2m+4 \end{bmatrix} & \cdots & \begin{bmatrix} (n-2)m+1 & (n-2)m+3 \\ (n-2)m+2 & (n-2)m+4 \end{bmatrix} \\ \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix} & \begin{bmatrix} 2m+5 & 2m+7 \\ 2m+6 & 2m+8 \end{bmatrix} & \cdots & \begin{bmatrix} (n-2)m+5 & (n-2)m+7 \\ (n-2)m+6 & (n-2)m+8 \end{bmatrix} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \begin{bmatrix} 2m-3 & 2m-1 \\ 2m-2 & 2m \end{bmatrix} & \begin{bmatrix} 4m-3 & 4m-1 \\ 4m-2 & 4m \end{bmatrix} & \cdots & \begin{bmatrix} nm-3 & nm-1 \\ nm-2 & nm \end{bmatrix} \end{array}\right]$$

i.e., $V(G)$ can be taken as $\frac{mn}{4}$ blocks in which each block has 4 vertices and the vertices of each block is assigned with labels of consecutive integers. Then assign an edge label by the surjective function $f^{'} : E(G) \to \{1, 2, 3, 2m-3, 2m-2, ((n-2)m+2)\}$, such that $f^{'}(uv) = |f(u) - f(v)|$, for $u, v \in V(G)$. Then we get 4 one-factors $F_k, k = 1, 2, 3, 4$ with edges having labels as given below:

$$F_1 = \text{Edges with labels } ((n-2)m+2) \text{ and } 2m-2 \quad (1)$$
$$F_2 = \text{Edges with labels } 1 \quad (2)$$
$$F_3 = \text{Edges with labels } 2 \quad (3)$$
$$F_4 = \text{Edges with labels } 3 \text{ and } 2m-3 \quad (4)$$

Then the union of consecutive one-factors $F_k \cup F_{k+1}$, where $k = 1, 2, 3$ forms a $2-$regular graph. Hence the proof.

3.2. Theorem

For even n > 2, $C_4 \Box C_n$ admits a Sequentially-Perfect one-factorization.

Proof. Let $G = C_4 \Box C_n$, with vertex set $V(G)$ and edge set $E(G)$ as defined in Theorem 3.1. Then the one-factors F_k, $k = 1, 2, 3, 4$ of G are given below:

$$F_1 = \left\{ \bigoplus_{i=1}^{2} u_{i1} u_{in} \bigoplus_{j=2}^{n-1} u_{1j} u_{2j} \bigoplus_{i=3}^{m} u_{ij} u_{i(j+1)},\ j \in \{1, 3, 5, \ldots, (n-1)\} \right\}$$

$$F_2 = \left\{ u_{1n} u_{4n} \bigoplus_{i=1,3} u_{i1} u_{(i+1)1} \bigoplus_{j=2}^{n-2} u_{1j} u_{1(j+1)} \bigoplus_{j=2,4,6,\ldots}^{n-2} u_{4j} u_{4(j+1)} \bigoplus_{j=2}^{n} u_{2j} u_{3j} \right\}$$

$$F_3 = \left\{ \bigoplus_{i=1}^{2} u_{ij}u_{i(j+1)},\ j \in \{1,3,5,\dots,n-1\} \bigoplus_{i=3}^{4} u_{i1}u_{in} \bigoplus_{j=2}^{n-1} u_{3j}u_{4j} \right\}$$

$$F_4 = \left\{ \bigoplus_{j=1}^{n-1} u_{ij}u_{4j} \bigoplus u_{21}u_{31} \bigoplus u_{in}u_{(i+1)n},\ i = 1,3 \bigoplus_{i=2}^{3} u_{ij}u_{i(j+1)},\ j \in \{2,4,6,\dots,n-2\} \right\}$$

Then $F_k \cup F_{k+1}$ for $k = 1,3$ form two Hamiltonian cycles. Hence G has a Sequentially Perfect one-factorization.

3.3. Theorem

For given even positive integers $m, n > 2$ and $m \neq 4$, $C_m \square C_n$ admits a Sequentially-Perfect one-factorization.

Proof. Let $G = C_m \square C_n$. Then the 4 one-factors which form a Sequentially Perfect one-factorization in G are given below:

$$F_1 = \left\{ \bigoplus_{i=1}^{2} u_{i1}u_{in} \bigoplus_{j=2}^{n-1} u_{1j}u_{2j} \bigoplus_{i=3}^{m} u_{i1}u_{i2} \bigoplus_{\substack{i=m-1 \\ j\in\{3,5,\dots,n-1\}}}^{m} u_{ij}u_{i(j+1)} \bigoplus_{\substack{j=3 \\ i\in\{3,5,\dots,m-3\}}}^{n} u_{ij}u_{(i+1)j} \right\}$$

$$F_2 = \left\{ u_{1n}u_{mn} \bigoplus_{\substack{i=1 \\ i \text{ odd}}}^{m-1} u_{i1}u_{(i+1)1} \bigoplus_{j=2}^{n-2} u_{1j}u_{1(j+1)} \bigoplus_{\substack{j=2 \\ j \text{ even}}}^{n-2} u_{mj}u_{m(j+1)} \bigoplus_{j=2}^{n} u_{ij}u_{(i+1)j} \right\}$$

$$F_3 = \left\{ \bigoplus_{\substack{i=1 \\ j\in\{1,3,\dots,n-1\}}}^{2} u_{ij}u_{i(j+1)} \bigoplus_{i=3}^{m} u_{i1}u_{in} \bigoplus_{j=2}^{n-1} u_{(m-1)j}u_{mj} \bigoplus_{\substack{i=3 \\ j\in\{2,4,\dots,n-2\}}}^{m-2} u_{ij}u_{i(j+1)} \right\}$$

$$F_4 = \left\{ \bigoplus_{j=1}^{n-1} u_{ij}u_{mj} \bigoplus_{\substack{i=2 \\ i \text{ even}}}^{m-2} u_{i1}u_{(i+1)1} \bigoplus_{\substack{i=2 \\ j\in\{2,4,\dots,n-2\}}}^{m-1} u_{ij}u_{i(j+1)} \bigoplus_{\substack{i=3 \\ i \text{ odd}}}^{m-3} u_{i2}u_{(i+1)2} \bigoplus_{i=1}^{m-1} u_{in}u_{(i+1)n} \bigoplus_{\substack{i=3 \\ j\in\{3,5,\dots,n-1\}}}^{m-2} u_{ij}u_{i(j+1)} \right\}$$

Then $F_k \cup F_{k+1}$ for $k = 1,3$ form two Hamiltonian cycles. Hence G has a Sequentially Perfect one-factorization.

4. Sequentially Perfect and Sequentially Uniform one-Factorizations in Tensor Product of Cycles

4.1. Theorem

For given even positive integers m > 2 and odd $n \geq 3$, $C_m \times C_n$ admits a Sequentially-Perfect one-factorization.

Proof. Let $G = C_m \times C_n$. Since G is a 4-regular graph, it has 4 one-factors. First we find a one-factorization of $C_m \times C_n = G$ using jump one-factors [6]. Now it is clear that G is a multipartite graph having m rows and n columns, i.e., $V[G] = [b_i^j]_{m\times n}$. Name each row of G by B_i, where $i \in \{1,2,3,\dots,m\}$. And let $E[G] = E[G_1] \cup E[G_2]$, where G_1 and G_2 are defined below:

By the definition of jump one-factor [9], let G_1 be a 2-regular graph with $m/2$ bipartitions (B_l, B_{l+1}), $l \in \{1,3,5,\dots,m-1\}$ where $B_l = \{b_0^l, b_1^l, b_2^l, \dots, b_{n-1}^l\}$ and $B_{l+1} = \{b_0^{l+1}, b_1^{l+1}, b_2^{l+1}, \dots, b_{n-1}^{l+1}\}$. Let the edge set of G_1 be $E(G_1) = \{F_i[(B_l, B_{l+1})] = b_j^l b_{i+j}^{l+1},\ 0 \leq j \leq n-1\}$ for $i \in \{1, n-1\}$ and the subscript $i+j$ is taken modulo n with residues $\{1,2,\dots,n\}$. Then for a particular value of i, the edges of $F_i[(B_l, B_{l+1})]$ are the edges with jump i and the one-factor thus formed is termed as a one-factor of jump i.

Let G_2 be a 2-regular graph with $m/2$ bipartitions $(B_1, B_m) \bigcup\limits_{k=2,4,\dots}^{m-2} (B_k, B_{k+1})$. Then the edge set of G_2 is $E(G_2) = \{F_i'[(B_k, B_{k+1}) \cup (B_1, B_m)] = b_j^k b_{i+j}^{k+1},\ 0 \leq j \leq n-1\}$ for $i \in \{1, n-1\}$ and the subscript $i+j$ is taken modulo n with residues $\{1,2,\dots,n\}$. Then for a particular value of i, the edges of $F_i'[(B_k, B_{k+1})]$ are the edges with jump i and the one-factors then formed are termed as one-factors of jump i.

Now $\{F_1, F_1', F_{n-1}, F_{n-1}'\}$ are the ordered one-factors of $G(= G_1 \cup G_2)$ to be sequentially perfect, since the consecutive one-factors with jumps 1 and $n-1$ namely $F_1 \cup F_1'$ and $F_{n-1} \cup F_{n-1}'$ form two Hamiltonian cycles. Hence the proof.

4.2. Theorem

For given even positive integers $m > 2$ and odd $n \geq 3$, $C_m \times C_n$ admits a Sequentially-Uniform one-factorization.

Proof. The one-factors of $C_m \times C_n$ can be obtained as in Theorem 4.1. Let $F'' = \{F_1, F_{n-1}, F_1', F_{n-1}'\}$ be the order of one-factors of $G(= G_1 \cup G_2)$ to be sequentially uniform. i.e., the consecutive one-factors in F'' namely $F_1 \cup F_{n-1}$, $F_{n-1} \cup F_1'$, $F_1' \cup F_{n-1}'$, $F_{n-1}' \cup F_1$ form 2-regular graphs. Hence the proof.

5. Sequentially Perfect and Sequentially Uniform One-Factorizations in Wreath Product of Cycles

5.1. Theorem

For given even positive integers $m > 2$ and odd $n \geq 3$, $C_m \otimes C_n$ admits a Sequentially-Perfect 1-factorization.

Proof. Let $G = C_m \otimes C_n$. i.e., G is an m-partite graph with m layers of n vertices each. Let $V(G) = (b_j^i)_{m\times n}$. Since G is a $2n+2$-regular graph, it has $2n+2$ one-factors. Now $G = G_1 \bigoplus G_2 \bigoplus G_3$, where
$V(G_i) = V(G)$, for $i = 1,2,3$;
$E(G_1) = E(C_m \Box C_n)$;
$E(G_2) = \bigcup_{i=1,3,5,...}^{m-1} E(B_l, B_{l+1})$;
$E(G_3) = \bigcup_{k=2,4,...}^{m-2} E(B_k, B_{k+1}) \bigcup E(B_1, B_m)$;

Then G_1 be 4-regular subgraph of G and each G_2 and G_3 are $n-1$-regular subgraphs of G i.e., G_1 has 4 one-factors and $G_2 \cup G_3$ has $2(n-1)$ one-factors. Now the one-factors $F_k,\ \ k = 1,2,3,4$ of G_1 are as follows:

$$F_1 = \left\{ \bigoplus_{i=1,3,.}^{m-1} b_n^i b_n^{i+1} \bigoplus_{j=1,3,.}^{n-2} b_j^i b_{j+1}^i,\ \ i \in \{1,2,3,...,m\} \right\}$$

$$F_2 = \left\{ \bigoplus_{i=1,2,.}^{m} b_1^i b_n^i \bigoplus_{j=2,3,.}^{n-1} b_j^i b_j^{i+1},\ \ i \in \{1,3,5,...,m-1\} \right\}$$

$$F_3 = \left\{ b_1^1 b_1^m \bigoplus_{i=2,4,...}^{m-2} b_1^i b_1^{i+1}, b_{n-1}^2 b_{n-1}^3, b_n^2 b_n^3 \bigoplus_{j=2,4,...}^{n-1} b_j^1 b_{j+1}^1 \right.$$
$$\left. \bigoplus_{j=2,4,...}^{n-1} b_j^i b_{j+1}^i, i \in \{4,5,\ldots,m\} \bigoplus_{j=2,4,...}^{n-3} b_j^i b_{j+1}^i, i \in \{2,3\} \right\}$$

$$F_4 = \left\{ \bigoplus_{j=2,3,...}^{n} b_j^1 b_j^m \bigoplus_{i=1,3,...}^{m-1} b_1^i b_1^{i+1} \bigoplus_{i\in\{2,3\}} b_{n-1}^i b_n^i \bigoplus_{j=2,3,...}^{n-2} b_j^2 b_j^3 \bigoplus_{i=4,6,8,...}^{m-2} b_j^i b_j^{i+1},\ \ j \in \{2,3,4,...,n\} \right\}$$

Next, we find one-factors of G_2 by using jump one-factors. Name each row of G_2 by B_i, where $i \in \{1,2,3,...,m\}$. By definition of jump one-factor [9], let G_2 be a $(n-1)$-regular graph with $m/2$ bipartitions (B_l, B_{l+1}), $l \in \{1,3,5,...m-1\}$ where $B_l = \{b_0^l, b_1^l, b_2^l, ..., b_{n-1}^l\}$ and $B_{l+1} = \{b_0^{l+1}, b_1^{l+1}, b_2^{l+1}, ..., b_{n-1}^{l+1}\}$. Let the edge set of G_2 be $E(G_2) = \{F_i^{'}[B_l, B_{l+1}] = b_j^l b_{i+j}^{l+1},\ \ 0 \le j \le n-1\}$ for some $i \in \{1,2,...,n-1\}$ and the subscript $i+j$ is taken modulo n. Then for a particular value of i, the edges of $F_i^{'}[B_l, B_{l+1}]$ are the edges with jump i and the one-factor thus formed is termed as an one-factor of jump i. Let the edge set of G_3 be $E(G_3) = \{F_i^{''}[(B_k, B_{k+1}) \bigcup (B_1, B_m)] = b_j^k b_{i+j}^{k+1},\ \ 0 \le j \le n-1\}$ for $i \in \{1,2,..n-1\}$ and the subscript $i+j$ is taken modulo n. Then for a particular value of i, the edges of $F_i^{''}[(B_k, B_{k+1}) \bigcup (B_1, B_m)]$ are the edges with jump i and the one-factors thus formed is termed as an one-factor of jump i. To obtain the required one-factors of $F_1^{''}$ in G_3 we take the edges of jump 1 in (B_1, B_m) and the edges of jump $n-1$ in (B_k, B_{k+1}), $k \in \{2,4,...,m-2\}$. Similarly for $F_{n-1}^{''}$ in G_3 we take the edges of jump $n-1$ in (B_1, B_m) and the edges of jump 1 in (B_k, B_{k+1}). Now $F_1, F_2, F_3, F_4, F_i^{'}, F_i^{''}$ be the one-factors of $G(= G_1 \cup G_2 \cup G_3)$. Then $\{F_4, F_1^{'}, F_3, F_{n-1}^{'}, F_1^{''}, F_1, F_{n-1}^{''}, F_2, F_2^{'}, F_2^{''}, F_3^{'}, F_3^{''}, ...,$ $F_{n-2}^{'}, F_{n-2}^{''}\}$ be the ordered one-factors of G to be sequentially perfect as the consecutive one-factors form Hamiltonian cycles.

5.2. Theorem

For given even positive integers $m > 2$ and odd $n \ge 3$, $C_m \otimes C_n$ admits a Sequentially-Uniform one-factorization.

Proof. Now the one-factors of $C_m \otimes C_n$ can be obtained as in Theorem 5.1. Except the one-factors F_3 and F_4 of G_1, the remaining one-factors are defined as in Theorem 3.6 and F_3 and F_4 are defined as follows:

$$F_3 = \left\{ \bigoplus_{i=1,3,.}^{m-1} b_1^i b_1^{i+1} \bigoplus_{j=2,4,.}^{n-1} b_j^i b_{j+1}^i, i \in \{1,2,3,...,m\} \right\}$$

$$F_4 = \left\{ \bigoplus_{j=1,2,.}^{n} b_j^1 b_j^m \bigoplus_{j=1,2,.}^{n} b_j^i b_j^{i+1}, i \in \{2,4,...,m-2\} \right\}$$

Now $F^{'''} = \{F_1, F_2, F_3, F_4, F_i^{'} \cup F_{i+1}^{'}, F_i^{''} \cup F_{i+1}^{''}\}$ be the order of one-factors of $G(= G_1 \cup G_2 \cup G_3)$ to be sequentially uniform as the consecutive one-factors in $F^{'''}$ with jump $i \in \{1,2,...,n-1\}$ form 2-regular graphs.

6. Concluding Remarks

The above results settled the problem of existence of sequentially perfect and sequentially uniform one-factorizations of cartesian product, tensor product and wreath product of cycles. It is proved that for even positive integers $m, n > 2$, $C_m \Box C_n$ has sequentially perfect and sequentially uniform one-factorizations and for even positive integers $m > 2$ and odd $n \ge 3$, $C_m \times C_n$ and $C_m \otimes C_n$ have sequentially perfect and sequentially uniform one-factorizations. The results have been verified using an algorithmic approach given in Appendix.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Appendix

Algorithm for Sequentially Perfect and sequentially Uniform 1-factorization of $C_m \Box C_n$, $C_m \times C_n$ and $C_m \otimes C_n$.

Appendix I. Pseudo Code for Tensor Product of Cycles

The algorithm with the following pseudo code will generate the order of one-factors using jump one-factor technique and edge labeling technique. Further it will check if there exists hamilton cycles or two-regular graphs of same order for the purpose of proving the graph G to be sequentially perfect and sequentially uniform.

```
Function create_connections(row, column, row_step,
column_step)
connections ← []
no_of_row_operations ← row // 2
for row_operations_index in range (no_of_row_operations):
   row_index ← row_op_index × 2
   pair_row ← (row_index + row + row_step) mod row
   src_index ← min (pair_row, row_index)
   dest_index ← max (pair_row, row_index)
   for col_index in range(column):
      pair_col ← (col_index + column + column_step)
mod column
      node1 ← src_index × column + col_index
      node2 ← dest_index × column + pair_col
      connections.append ((node1, node2))
return connections
```

Appendix II. Example Output

1. Sequentially Perfect 1-factorization (SP1F) of $C_4 \times C_7$

Following Figure 1 is the output obtained using our algorithmic approach. [using python programming]

The graph g_1, g_2, g_3 and g_4 represent one-factors of $C_4 \times C_7$ and h_1, h_2, h_3 and h_4 denotes the hamilton cycle of G formed by the union of two consecutive one-factors.

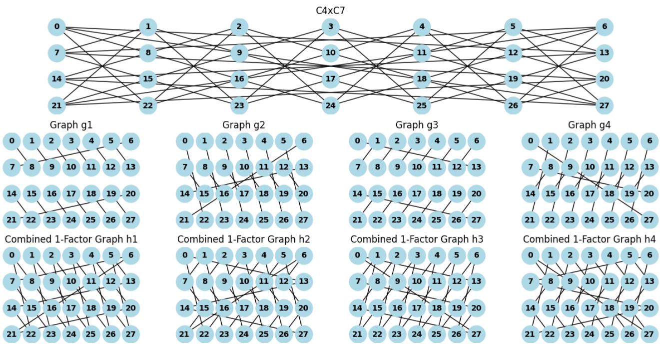

Figure 1. *SP1F of $C_4 \times C_7$.*

2. Sequentially Uniform 1-factorization (SU1F) of $C_4 \times C_7$

Following Figure 2 is the output obtained using our algorithmic approach. [using python programming]

The graphs g_1, g_2, g_3 and g_4 represent one-factors of $C_4 \times C_7$ and h_1 and h_2 denotes the two-regular graph of type $(2m, 2m)$ of G formed by the union of two consecutive one-factors.

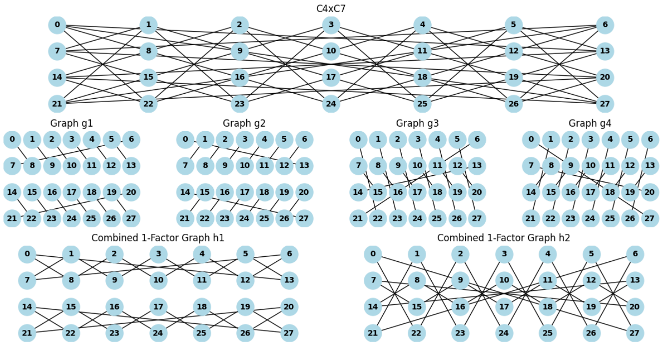

Figure 2. *SU1F of $C_4 \times C_7$.*

Appendix III. Pseudo Code for Cartesian Product of Cycles

```
Function create_connections(row, column, row_step,
column_step)
connections ← [ ]
row_mul_factor ← get_mul_factor(row_step)
column_mul_factor ← get_mul_factor(column_step)
no_of_row_operations ← row ÷ row_mul_factor
for row_operations_index in range(no_of_row_operations):
   row_index ← row_operations_index × row_mul_factor
   pair_row ← (row_index + row + row_step) mod row
   src_index ← min(pair_row, row_index)
   dest_index ← max(pair_row, row_index)
   number_of_column_operations ← column ÷
column_mul_factor
   for col_operations_index in range
(number_of_column_operations):
      col_index ← col_operations_index × column_mul_factor
      pair_col ← (col_index + column + column_step)
mod column
      node1 ← src_index × column + col_index
      node2 ← dest_index × column + pair_col
      connections.append((node1, node2))
Return connections
```

Appendix IV. Pseudo Code for Wreath Product of Cycles

By combining the pseudo codes of tensor product and cartesian product of cycles $C_m \times C_n$ and $C_m \Box C_n$, we can obtain the pseudo code for the wreath product of cycles $C_m \otimes C_n$ to obtain the sequentially perfect and sequentially uniform one-factorizations in $C_m \otimes C_n$.

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