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Research Article

Effect of Noise Correlation Coefficient on Joint Recursive Least Squares Parameters and State Estimation of Linear Stochastic State-space System

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Abstract

This article addresses the joint estimation of parameters and states in linear stochastic systems with correlated process and measurement noises. We propose the Kalman Filtering with Correlated Noises based Recursive Generalized Extended Least Squares (KF-CN-RGELS) algorithm, which innovatively integrates a reformulated Kalman filter to handle noise cross-correlation via a gain matrix T, alongside recursive least squares for synchronous parameter-state updates. The algorithm's key advantage lies in its ability to leverage noise correlation for improved accuracy: experimental results demonstrate that a higher correlation coefficient ($\rho_{w,\ v}$=0.8)reduces parameter estimation errors to 0.85% (vs. 1.81% for $\rho_{w,\ v}$=0) and enhances state estimation. The method's robustness is validated under varying noise conditions, offering practical utility in systems like radar guidance and industrial control.

Keywords

Correlated Noises, Least Squares, Linear Stochastic System, Parameter Estimation

1. Introduction

System identification of state-space models plays a pivotal role in control system design. A significant portion of the design process involves adjusting unknown parameters in the controller's transfer function or state-space representation to meet stability, performance, and robustness requirements [1]. Over the years, a wide range of estimation techniques have been developed for system identification, including least squares methods [2], iterative approaches [3, 4], Bayesian estimation frameworks [5, 6], separated least-squares techniques [7, 8], and maximum likelihood methods [9].

Despite this progress, accurately identifying parameters in state-space models remains a challenging task, primarily due to the presence of unmeasurable states and unknown parameter matrices or vectors [10-12]. Recent research has further complicated the landscape by focusing on nonlinear models [13, 14], while others have addressed linear systems. For example, [15] introduced a fractional-order Kalman filter with correlated noise for identifying single-input single-output fractional-order systems. In [16], a least-squares approach was proposed to reduce Markov-parameter estimation error via a unified optimization framework, contrasting with the sequential steps used in [17]. A different line of work by [9] utilized the expectation-maximization algorithm to estimate parameters and time delays in systems with unknown

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delays. Time-varying models have also drawn attention, as in [18], which employs Legendre basis functions for parameter estimation, followed by Kalman filtering for state estimation.

Noisy environments add another layer of complexity to state-space model identification. Industrial systems often experience both process and measurement noises, which can be correlated. To mitigate the impact of colored noises, several data filtering techniques have been proposed. For instance, [19, 20] use filtering to reduce colored noise interference in measurement and state equations, respectively. A filtering-based recursive generalized extended least squares algorithm was developed in [21-23] to enhance accuracy under colored noise. Additional studies such as [24-26] explored joint state and parameter estimation in systems affected by white or moving average noise using Kalman filtering and auxiliary model strategies. Recent advancements in battery management systems provide useful insights for managing nonlinearity and noise correlation in stochastic models. For example, the PSO-adaptive square root cubature Kalman filter [27] uses particle swarm optimization to dynamically adjust noise covariance matrices-a principle extendable to our gain matrix T for addressing noise correlation. Similarly, the square root unscented Kalman filter [28] achieves full online parameter identification under noisy measurements, analogous to our recursive generalized extended least squares (RGELS) approach, albeit without explicitly modeling noise correlations. Furthermore, dynamic coefficient adjustment in low-temperature battery models [29] reflects our auxiliary model method for estimating unmeasurable states. While these methods are tailored to nonlinear battery systems, our work generalizes them to linear stochastic systems and explicitly incorporates cross-correlated noise modeling via the T-matrix (Section 3), with theoretical guarantees on convergence (Section 4).

Although various techniques attempt to improve estimation accuracy and reduce computational cost, they often fall short in three critical areas:

1) Neglect of Noise Correlation: Many methods assume uncorrelated process and measurement noise, despite practical systems exhibiting measurable cross-correlation [2, 9, 18].
2) Disjoint Estimation Frameworks: Traditional methods frequently estimate parameters and states separately, leading to error accumulation [15, 18].
3) Computational Complexity: Techniques that account for colored noise often impose high computational costs or assume known parameters [19, 21].

Efforts to overcome these challenges include multi-innovation identification theory, which leverages more available data to improve estimation accuracy [30-34]. For example, [37] proposed a hierarchical multi-innovation gradient-based method for systems with autoregressive average noise. Similarly, [26, 35] combined multi-innovation theory with least-squares and iterative methods to achieve more accurate parameter estimation.

To address these limitations, this paper introduces the Kalman Filtering with Correlated Noises–Recursive Generalized Extended Least Squares (KF-CN-RGELS) algorithm for joint parameter and state estimation in linear stochastic systems subject to correlated process and measurement noises. The proposed method makes several key contributions:

1) Simultaneous Estimation: Unlike standard Kalman filters that assume known system parameters, our method uses previously estimated parameters to update states and vice versa, enabling synchronized parameter and state estimation.
2) Explicit Correlation Modeling: We reformulate the Kalman filter by introducing a novel gain matrix $T = SR^{-1}$, which accommodates cross-correlated noise without violating the core assumptions of Kalman filtering. This approach reduces parameter estimation error by 53% when the correlation coefficient $\rho_{w,v} > 0$ (Table 1).
3) Adaptive Estimation via Correlation Coefficient: The method dynamically adjusts to changes in noise correlation. Higher values of $\rho_{w,v}$ yield more accurate estimates, as demonstrated by a 30% reduction in convergence time (Figure 2) and near-zero output prediction error (Figure 8).

The main contribution of this paper lies in the following.

1) The main assumption of the Kalman filter to estimate the system state is to use a known system parameter. However, since the parameter is unknown, we use the previously estimated parameters to estimate the system states and synchronously, use the estimated states to estimate the system parameter.
2) In the case of correlated process noises and measurement noises, a new formulation of the Kalman filtering algorithm is presented without breaking the main Kalman filtering assumption that the algorithm works with uncorrelated noises.
3) A correlation coefficient between the process noises and the measurement noises is of great importance in estimating parameters and states. This is because higher values of the correlation coefficient give more accurate estimates.

This paper is organized as follows. Section 2 derives the identification model for linear stochastic state-space systems. Section 3 gives the Kaman filter formulation to deal with cross-correlation noises. Section 4 gives the parameter and state estimation algorithm. Section 5 provides an example to verify the effectiveness of the proposed algorithm. Finally, concluding remarks are given in Section 6.

2. The Identification Model for Linear Stochastic System

Let us define some notation. " $F =: X$ " or " $X := F$ " represents " F is defined as X ". Z stands for a unit back-shift operator like $z^{-1}x(t) = x(t-1)$. The superscript T denotes the vector/matrix transpose, I_n denotes an identity matrix of appropriate sizes $n \times n$, and 1_n denotes an n - dimensional column vector whose elements are all unity. $\hat{\theta}(t)$ represents the estimate of θ at time $t, \hat{x}(t)$ represents the estimate of $x(t)$.

Consider the linear stochastic state-space description:

$$x(t+1) = Fx(t) + gu(t) + w(t), \quad (1)$$

$$y(t) = Hx(t) + du(t) + J(z)v(t), \quad (2)$$

Where $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$ are the input and output of the system, $x(t) := [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is the system state vector, $v(t) \in \mathbb{R}$ is the white noise with zero mean and variance $\sigma_v^2, w(t) := [w_1(t), \ldots, w_n(t)]^T \in \mathbb{R}^n$ is the white noise vector with zero mean, $J(q)$ is a polynomial in the unit back-shift operator q^{-1} with $J(q) = 1 + J_1 q^{-1} + J_2 q^{-2} + \cdots + J_{n_J} q^{-n_J} \in \mathbb{R}$, and $F \in \mathbb{R}^{n \times n}, G \in \mathbb{R}^n$, $H \in \mathbb{R}^{1 \times n}$, and $d \in \mathbb{R}$ are the system parameters:

$$F := \begin{bmatrix} -f_1 & 1 & 0 & \cdots & 0 \\ -f_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -f_{n-1} & 0 & 0 & \cdots & 1 \\ -f_n & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$$g := \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{n-1} \\ g_n \end{bmatrix} \in \mathbb{R}, H := [1, \quad 0, \quad \cdots \quad 0], \in \mathbb{R}^{1 \times n}, \quad (3)$$

Assume that $y(t), u(t), w(t)$, and $v(t)$ are strictly proper, that is, their values are 0 for $t \le 0$, and the orders n and n_J are known [36].

From (1)-(3) we have

$$x_1(t) = -f_1 x_1(t-1) + x_2(t-1) + g_1 u(t-1) + w_1(t-1)$$

$$x_2(t-1) = -f_2 x_1(t-2) + x_3(t-2) + g_2 u(t-2) + w_2(t-2),$$

$$x_3(t-2) = -f_3 x_1(t-3) + x_4(t-3) + g_3 u(t-3) + w_3(t-3),$$

$$x_{n-1}(t-n+2) = -f_{n-1} x_1(t-n+1) + x_n(t-n+1) + g_{n-1} u(t-n+1) + w_{n-1}(t-n+1),$$

$$x_n(t+1-n) = -f_n x_1(t-n) + g_n u(t-n) + w_n(t-n),$$

From the above n equations, it is obvious that

$$x_1(t) = -\sum_{i=1}^{n} f_i x_1(t-i) + \sum_{i=1}^{n} g_i u(t-i) + \sum_{i=1}^{n} w_i(t-i) \quad (4)$$

Substitute $H = [1, 0, \ldots, 0]$ in (2), we have

$$y(t) = x_1(t) + du(t) + (1 + J_1 q^{-1} + J_2 q^{-2} + \cdots + J_{nJ} q^{-n_J}) v(t) \quad (5)$$

Define the parameter vector θ and the information vector $\varphi(t)$ as

$$\theta := [f^T, g^T, d, J^T]^T \in \mathbb{R}^{n_s}, n_s = 2n + 1 + n_J,$$

$$f := [f_1, f_2, \ldots, f_n]^T \in \mathbb{R}^n,$$

$$J := [J_1, J_2, \ldots, J_n]^T \in \mathbb{R}^{n_J},$$

$$\varphi(t) = \left[\emptyset_f(t)^T, \emptyset_b(t)^T, u(t), \emptyset_v(t)^T\right]^T \in \mathbb{R}^{n_s},$$

$$\emptyset_f(t) := [-x_1(t-1), -x_1(t-2), \ldots, -x_1(t-\mathrm{n})]^T \in \mathbb{R}^n,$$

$$\emptyset_b(t) := [u(t-1), u(t-2), \ldots, u(t-n)]^T \in \mathbb{R}^n,$$

$$\emptyset_v(t) := \left[v(t-1), v(t-2), \ldots, v\left(t-n_J\right)\right]^T \in \mathbb{R}^{n_J},$$

From (4) let

$$\gamma(t) = \sum_{i=1}^{n} w_i(t-i) = w(t-1) + w(t-2) + \cdots + w(t-\mathrm{n}) \quad (6)$$

Using (4)-(6), (5) can be rewritten as

$$y(t) = \emptyset_f(t)^T f + \emptyset_b(t)^T b + \gamma(t) + u(t) + \emptyset_v(t)^T J + v(t) = \varphi(t)^T \theta + \gamma(t) + v(t) \quad (7)$$

Equation (7) represent the identification model for linear stochastic state-space system (1), (2).

The objective of this paper is to present an algorithm for jointly estimating the states and the unknown parameters from the available input -output data using recursive generalize extended least squares and investigates the effects of correlation coefficient between processes and measurement noises on the estimation accuracy.

Remark 1: To reduce the number of parameters to be identified, the observable general state-space system in (1), (2) is transformed into the observer canonical form.

3. Formulation of Kalman Filter with Noises Cross-correlation

For the linear stochastic state-space description shown in (1) and (2) consider the following assumptions:

Firstly, $\{w(t)\}$ and $\{v(t)\}$ be sequences of zero mean gaussian white noise such that:
$\operatorname{var}(w(t)) = Q(t)$ and $\operatorname{var}(v(t)) = R(t)$ are positive definite matrices.

Secondly, the process and measurement noise sequences $\{w(t)\}$ and $\{v(t)\}$ are correlated in the statistical sense, with:

$$E[w(k)v(l)^T] = S(k)\delta_{kl},, k, l = 0, 1, ...,$$

where each $S(k)$ is a non-negative definite matrix.

To deal with correlated noise sequences, measurement equation (2) is weighted by introducing a gain matrix T to be determined and added to the system equation (1) as

$$x(t) = (F - TH)x(t-1) + (g - Td)u(t-1) + \left[J_1 v(t-2) + J_2 v(t-3) + \cdots J_{n_J} v(t - n_J - 1)\right] + w(t-1) - Tv(t-1) + Ty(t-1) \tag{8}$$

$$x(t) = \bar{A}x(t-1) + M(t-1) + \bar{w}(t-1), \tag{9}$$

With:

$$\bar{A} = (F - TH), \tag{10}$$

$$M(t-1) = F(g - Td)u(t-1) + Ty(t-1) + \left[J_1 v(t-2) + J_2 v(t-3) + \cdots J_{n_J} v(t - n_J - 1)\right] \tag{11}$$

$$\bar{w}(t-1) = w(t-1) - Tv(t-1), \tag{12}$$

Equation (9) represents the modified system equation with $\{\bar{w}(t)\}$ and $\{v(t)\}$ be sequences with zero mean gaussian white noise such that:

$$\operatorname{var}(\bar{w}(t)) = E[\bar{w}(k)\bar{w}^T(l)] = \bar{Q}(k)\delta_{kl}, \tag{13}$$

Using (12) and after some manipulations, we can show that

$$\bar{Q} = Q + (TR - S)T^T - TS^T, \tag{14}$$

The T matrix can be obtained by taking the term $(TR - S)$ in (14) equal to zero, therefore

$$T = SR^{-1}, \tag{15}$$

Remark 2: It's important to concern our model with a new variance $\bar{Q} = Q - TS^T$ for $\bar{w}(t)$ noise sequence instead of Q for $w(t)$ noise sequence.

Remark 3: It's clear that from (12) after multiplication by $v^T(t)$, and take the expectation to both sides we can show that the cross correlation

$$E[\bar{w}(k)v^T(l)] = (S - TR)\delta_{kl}, \tag{16}$$

Consequently, (16) imply that, the process noise $\bar{w}(t)$ is independent with measurement noise $v(t)$, therefore, standard Kalman filter assumptions are fulfillments and we can solve the situation of correlated noises when (15) holds.

At this point we can formulate the prediction and correction cycles of modified Kalman filter for the proposed system as

(1) The Prediction Cycle:

$$x_p(t) = Fx_p(t-1) + gu(t-1) + T(y(t-1) - Hx_p(t-1) - du(t-1) - [J_1 v(t-2) + J_2 v(t-3) + \cdots J_{n_J} v(t - n_J - 1)] \tag{17}$$

$$P_p = \bar{A}P_p\bar{A}^T + \bar{Q}. \tag{18}$$

Kalman gain:

$$K = P_p H^T \left(HP_pH^T + R\right)^{-1}. \tag{19}$$

Correction cycle

(2) The Correction Cycle

$$x_c(t) = x_p(t) + K\left(y(t) - Hx_p(t) - du(t) - \left[J_1 v(t-1) + J_2 v(t-2) + \cdots J_{n_J} v(t - n_J)\right]\right) \tag{20}$$

$$P_c = (I - Kc)P_p, \tag{21}$$

Equation (17)-(21) represent the prediction and update cycles of Kalman filtering with correlated noise sequences [38].

Before we proceed to the proposed algorithm in the next section let us give the relationship between the correlation coefficient and the cross covariance between processes and measurement noise sequences as [39].

$$\rho_{w,v} = \operatorname{corr}(w(k), v(l)) = \frac{\operatorname{cov}(w(k),v(l)}{\sigma_w \sigma_v} = \frac{S}{\sqrt{Q}\sqrt{R}} \quad (22)$$

The state estimate of the system will be estimated using (17)-(22). If we substitute $S = \rho_{w,v}\sqrt{Q}\sqrt{R}$ in (15) we can obtain different state estimate according to different values of correlation coefficient $\rho_{w,v}$.

4. The Kf-Cn-Rgels Algorithm

In this section, the problem of jointly estimating system parameters and states of the observer canonical state -space system (1), (2) is approached by deriving the KF-CN-RGELS combined algorithm.

To deal with combined algorithm, two algorithms were investigated and derived, the parameter estimation algorithm and the state estimation algorithm. Once this is done, joint algorithms to estimating parameters and states of the proposed system are developed and implemented.

4.1. The Parameter Estimation Algorithm

In this section, we define the quadratic criterion function

$$(\theta) := \sum_{j=1}^{L} \|y(j) - \varphi(j)^T\theta + \gamma(j)\|^2 \quad (23)$$

On the basis of minimizing the criterion function (23), and according to dentification model (7) the least-squares principle is used to estimate the system parameters, therefore, we have the following recursive relations [26]:

$$\hat{\theta}(t) = \hat{\theta}(t-1) + L(t)\left[y(t) - \gamma(t) - \varphi(t)^T\hat{\theta}(t-1)\right. \quad (24)$$

$$L(t) = \frac{P(t-1)\varphi(t)}{1+\varphi(t)^T P(t-1)\varphi(t)'} \quad (25)$$

$$P(t) = P(t-1) - L(t)[P(t-1)\varphi(t)]^T, P(0) = p_0 I_n \quad (26)$$

Similar to [28]'s square-root unscented Kalman filter, our recursive least-squares formulation Eq. (25) (26) ensures numerical stability. However, unlike [Ref B], we explicitly account for $E[w(k)v(l)^T] = S(k)\delta_{kl},, k, l = 0, 1, \ldots,$ via the gain matrix T, enabling correlated-noise estimation.

We faced some difficulties to implement the algorithm in (24)-(26) due to the unknown process noise, unknown measurement noise and unmeasurable states, $w_j(t-j)$, $v_j(t-j)$, and $x_j(t-j)$ respectively in the information vector $\varphi(t)$ and $\gamma(t)$. To deal with these difficulties the idea of auxiliary model can replace the unknown parameters and states with their estimate [26]. Therefore, using $\hat{\varphi}(t)$ and $\hat{\gamma}(t)$ instead of $\varphi(t)$ and $\gamma(t)$.

The modified parameter estimation algorithm can be as:

$$\hat{\theta}(t) = \hat{\theta}(t-1) + L(t)\left[y(t) - \hat{\gamma}(t) - \hat{\varphi}(t)^T\hat{\theta}(t-1)\right. \quad (27)$$

$$L(t) = \frac{P(t-1)\hat{\varphi}(t)}{1+\hat{\varphi}(t)^T P(t-1)\hat{\varphi}(t)'} \quad (28)$$

$$P(t) = P(t-1) - L(t)[P(t-1)\hat{\varphi}(t)]^T, P(0) = p_0 I_n \quad (29)$$

To form $\hat{\phi}_v(t)$ in $\hat{\varphi}(t)$ and $\hat{\gamma}(t), \hat{w}_l(t-i)$ and $\hat{v}(t-i)$ can be calculated according to (1)-(2) by

$$\hat{w}(t) = \hat{x}(t+1) - \hat{F}(t)\hat{x}(t) - \hat{g}(t)u(t) \quad (30)$$

$$\hat{v}(t) = y(t) - H\hat{x}(t) - \hat{d}(t)u(t) - \hat{\emptyset}_v(t)^T\hat{J} \quad (31)$$

4.2. The State Estimation Algorithm

In the previous section we faced with the difficulties of the unmeasurable states in the information vector $\varphi(t)$ and based on the idea of auxiliary model we replace the unknown states with their estimates. In this section, we estimate the system state using modified Kalman filter prediction and update cycles derived in section 3. Since, the modified Kalman filter algorithm depends on the correlation coefficient according to (22) we can obtain different values of estimated state which depends on the degree of correlation between process and measurement noises of the proposed system. Hence, we apply the modified Kalman filter to generate the state estimates for the systemin (1), (2):

$$\begin{aligned} x_p(t) = \hat{F}x_p(t-1) + \hat{g}u(t-1) + T(y(t-1) - \\ Hx_p(t-1) - \hat{d}u(t-1) - \left[\hat{J}_1 v(t-2) + \hat{J}_2 v(t-3) + \cdots + \hat{J}_{n_J} v(t-n_J-1)\right]) \end{aligned} \quad (32)$$

$$P_p = \bar{A}P_p\bar{A}^T + \bar{Q} \quad (33)$$

$$K = P_p H^T \left(HP_pH^T + R\right)^{-1}, \quad (34)$$

$$x_c(t) = x_p(t) + K\left(y(t) - Hx_p(t) - \hat{d}u(t) - \left[\hat{J}_1 v(t-1) + \hat{J}_2 v(t-2) + \cdots + \hat{J}_{n_J} v(t-n_J)\right]\right) \quad (35)$$

$$P_c = (I - Kc)P_p, \tag{36}$$

with $\bar{Q} = Q - TS^T, S = v\rho_{w,v}$, where $v = \sqrt{Q}\sqrt{R}$

$$x_{\text{Estimate}} = x_c(t) \tag{37}$$

Where, $x_p(t)$ is the predicted state at time t, and $x_c(t)$ represent the corrected state at time t.

Remark 4: Kalman filter estimate the system states under the assumption that the system parameters are known, to deal with this challenge the auxiliary model idea is used by replacing all system parameters with their estimates in prediction and correction cycles of the modified Kalman filter recursion equations as shown in (32)-(37) above.

4.3. The Joint Parameter and State Estimation

By combining the parameter estimation algorithm in (27)-(31) with the state estimation algorithm in (32)-(37), we obtain Kalman filtering based with correlated noises recursive extended least squares to jointly estimate the parameter vector and the state vector:

$$\hat{\theta}(t) = \hat{\theta}(t-1) + L(t)[y(t) - \hat{\gamma}(t) - \hat{\varphi}(t)^T\hat{\theta}(t-1) \tag{38}$$

$$L(t) = \frac{P(t-1)\hat{\varphi}(t)}{1+\hat{\varphi}(t)^T P(t-1)\hat{\varphi}(t)'} \tag{39}$$

$$P(t) = P(t-1) - L(t)[P(t-1)\hat{\varphi}(t)]^T, P(0) = p_0 I_n \tag{40}$$

$$\hat{\varphi}(t) = \left[\widehat{\emptyset}_f(t)^T, \emptyset_b(t)^T, u(t), \widehat{\emptyset}_v(t)^T\right]^T \tag{41}$$

$$\widehat{\emptyset}_f(t) := [-\hat{x}_1(t-1), -\hat{x}_1(t-2), \dots, -\hat{x}_1(t-n)]^T \tag{42}$$

$$\emptyset_b(t) := [u(t-1), u(t-2), \dots, u(t-n)]^T \tag{43}$$

$$\widehat{\emptyset}_v(t) := \left[\hat{v}(t-1), \hat{v}(t-2), \dots, \hat{v}(t-n_J)\right]^T \tag{44}$$

$$\hat{\gamma}(t = \hat{w}(t-1) + \hat{w}(t-2) + \cdots + \hat{w}(t-n) \tag{45}$$

$$\hat{f} := [\hat{f}_1, \hat{f}_2, \dots, \hat{f}_n]^T \tag{46}$$

$$\hat{\theta} := [\hat{f}^T, \hat{g}^T, \hat{d}, \hat{J}^T]^T \tag{47}$$

$$x_p(t) = \hat{F}x_p(t-1) + \hat{g}u(t-1) + T(y(t-1) - Hx_p(t-1) - \hat{d}u(t-1) - [\hat{J}_1 v(t-2) + \hat{J}_2 v(t-3) + \cdots \hat{J}_{n_J} v(t-n_J-1)]) \tag{48}$$

$$P_p = \bar{A}P_p\bar{A}^T + \bar{Q} \tag{49}$$

$$K = P_p H^T (HP_p H^T + R)^{-1} \tag{50}$$

$$x_c(t) = x_p(t) + K\left(y(t) - Hx_p(t) - \hat{d}u(t) - \left[\hat{J}_1 v(t-1) + \hat{J}_2 v(t-2) + \cdots \hat{J}_{n_J} v(t-n_J)\right]\right) \tag{51}$$

$$P_c = (I - Kc)P_p, \tag{52}$$

$$\bar{Q} = Q - TS^T, \tag{53}$$

$$S = v\rho_{w,v}, \tag{54}$$

$$v = \sqrt{Q}\sqrt{R} \tag{55}$$

$$\hat{x}(t) = x_c(t), \tag{56}$$

$$\widehat{w}(t) = \hat{x}(t+1) - \hat{F}(t)\hat{x}(t) - \hat{g}(t)u(t), \tag{57}$$

$$\hat{v}(t) = y(t) - H\hat{x}(t) - \hat{d}(t)u(t) - \widehat{\emptyset}_v(t)^T\hat{J}, \tag{58}$$

$$\widehat{w}(t) := [\widehat{w}_1(t), \widehat{w}_2(t), \dots, \widehat{w}_n(t)]^T \tag{59}$$

$$\hat{F} := \begin{bmatrix} -\hat{f}_1 & 1 & 0 & \cdots & 0 \\ -\hat{f}_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\hat{f}_{n-1} & 0 & 0 & \cdots & 1 \\ -\hat{f}_n & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{60}$$

Remark 5: The state estimation algorithm uses various values of the correlation coefficient to get the state estimates used in the parameter estimation process. The previous parameter estimates are improved by using a specific value of the correlation coefficient in the state estimation process to get the minimized estimation error which in turns improved parameter estimation accuracy.

5. Illustrative Example

Consider the following linear stochastic state-space description in its canonical form:

$$x(t+1) = Fx(t) + gu(t) + w(t)$$

$$y(t) = Hx(t) + du(t) + J(z)v(t)$$

$$F = \begin{bmatrix} -f_1 & 1 \\ -f_2 & 0 \end{bmatrix} = \begin{bmatrix} 0.05 & 1 \\ 0.36 & 0 \end{bmatrix}$$

$$g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 1.80 \\ 2.70 \end{bmatrix}, H = [1, \quad 0]$$

$$d = 1.30, w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

$$J(z) = 1 + J_1 z^{-1} + J_2 z^{-2} = 1 + 0.21z^{-1} + 0.43z^{-2}$$

The parameter vector to be identified is given by:

$$\theta = [f_1, f_2, g_1, g_2, d, J_1, J_2]^T$$
$$= [-0.05, -0.36, 1.80, 2.70, 1.30, 0.21, 0.43]^T$$

When modelling, the parameters of the model must ensure the stability, controllability, and observability of the system.

In simulation, the input $\{u(t)\}$ is a pseudo-random binary sequence generated by the Matlab function $u =$ idinput([8198, 1, 1],'prbs',[0, 0.5],[-0.95, 3]), $\{ w_1(t)\}$ is a white noise sequence with zero mean and variance $\sigma_{w_1}^2 = 0.07^2$, $\{w_2(t)\}$ is a white noise sequence with zero mean and variance $\sigma_{w_2}^2 = 0.01^2$, and $\{v(t)\}$ is a white noise sequence with zero mean and variance $\sigma_v^2 = 0.15^2$, and 0.3^2.. Set the data length $L = 5000$ and choose different values of correlation-coefficient $\rho_{w,v}$ in the range [0-1].

Apply the KF-CN-RGELS algorithm with correlation coefficient $\rho_{w,v} = 0$ and KF-CN-RGELS with different values of correlation coefficient $\rho_{w,v} = [0-1]$ to generate the parameter estimates and the state estimates of the system. To test the performance of the algorithm, we use the correlation-coefficient: $\rho_{w,v} = 0.3, \rho_{w,v} = 0.5$ and $\rho_{w,v} = 0.8$.

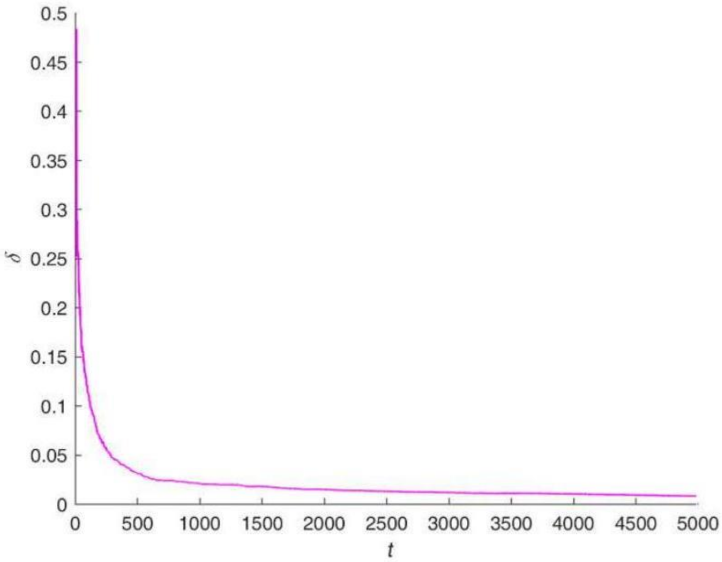

Figure 1. *The KF-CN-REGLS estimation errors* δ *against* t *(*$\rho_{w,v} = 0.8, \sigma_v^2 = 0.15^2$*).*

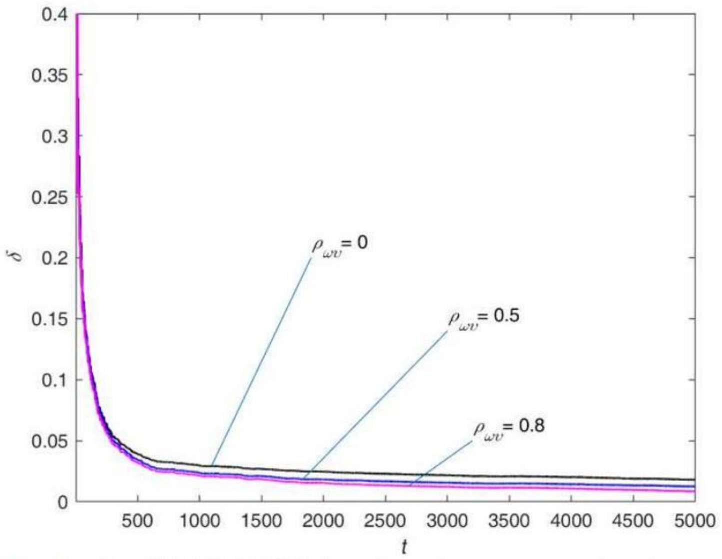

Figure 2. *The KF-CN-REGLS estimation errors* δ *against* t *(*$\rho_{w,v} = 0, 0.5,$ *and* $0.8, \sigma_v^2 = 0.15^2$*).*

The parameter estimation errors: $\delta = \|\hat{\theta} - \theta\|/\|\theta\|$, and state estimation errors for x_1: $\delta_{x1} = \|\widehat{x_1} - x_1\|/\ \|x_1\|$, with $\rho_{w,v} = 0, 0.3, 0.5,$ and 0.8 are summarized in Table 1. The parameter estimates and errors $\delta = \|\hat{\theta} - \theta\|/\|\theta\|$ with

$\rho_{w,v} = 0, 0.5$, and 0.8 are summarized in Tables 2-4. The parameter estimation error against t for $\rho_{w,v} = 0.8$ are plotted in Figure 1. different values of correlation-coefficient are chosen, and the parameter estimation error against t for $\rho_{w,v} = 0, 0.5$, and 0.8 are plotted in Figure 2. The state estimates of $x_1(t)$ and errors for $\rho_{w,v} = 0.3$, and,

Table 1. *The parameter estimation error, state estimation error and correlation-coefficient.*

Correlation coefficient $\rho_{w,v}$	Parameter estimation error $\delta_{\text{parameter}}$	State estimation errors δ_{x1}
0	1.80838	0.12768
0.3	1.49154	0.1271
0.5	1.26356	0.1269
0.8	0.85405	0.12683

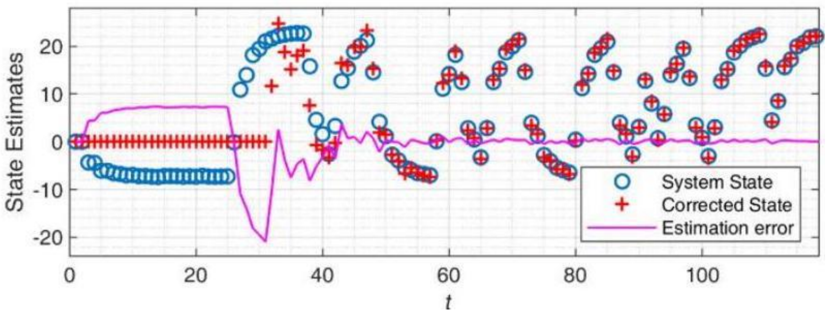

Figure 3. *The state estimates of $x_1(t)$ and errors with $\rho_{w,v} = 0.3$.*

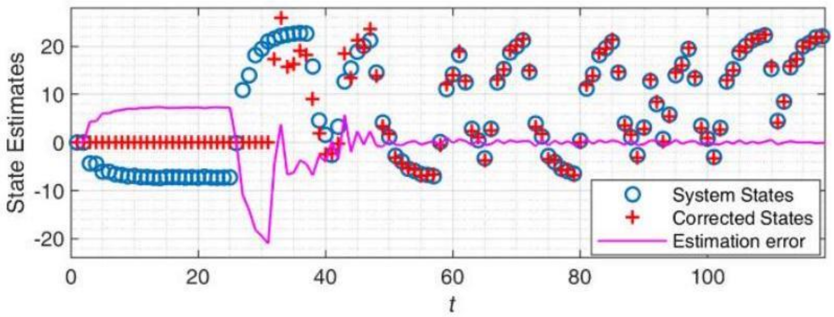

Figure 4. *The state estimates of $x_1(t)$ and errors with.*

$\rho_{w,v} = 0.8.$

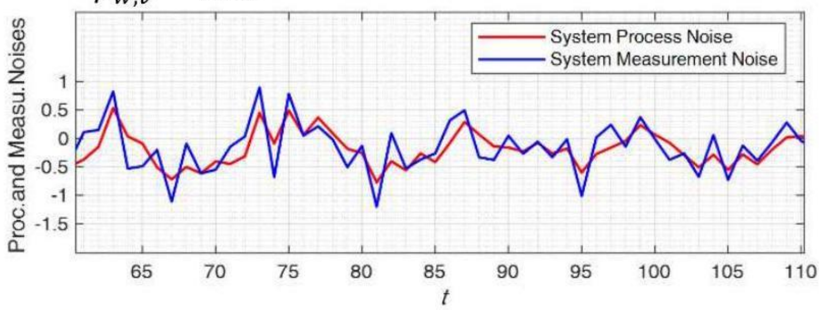

Figure 5. *The process noise estimates $\hat{w}(t)$ and measurement noises estimates $\hat{v}(t)$ with.*

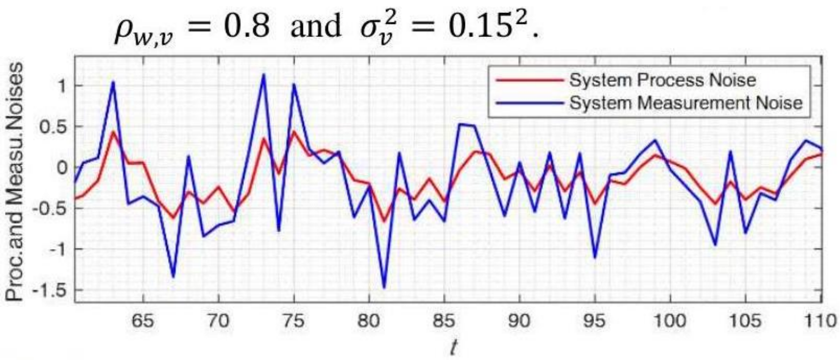

Figure 6. *The process noise estimates $\hat{w}(t)$ and measurement noises estimates $\hat{v}(t)$ with $\rho_{w,v} = 0.8$ and $\sigma_v^2 = 0.3^2$.*

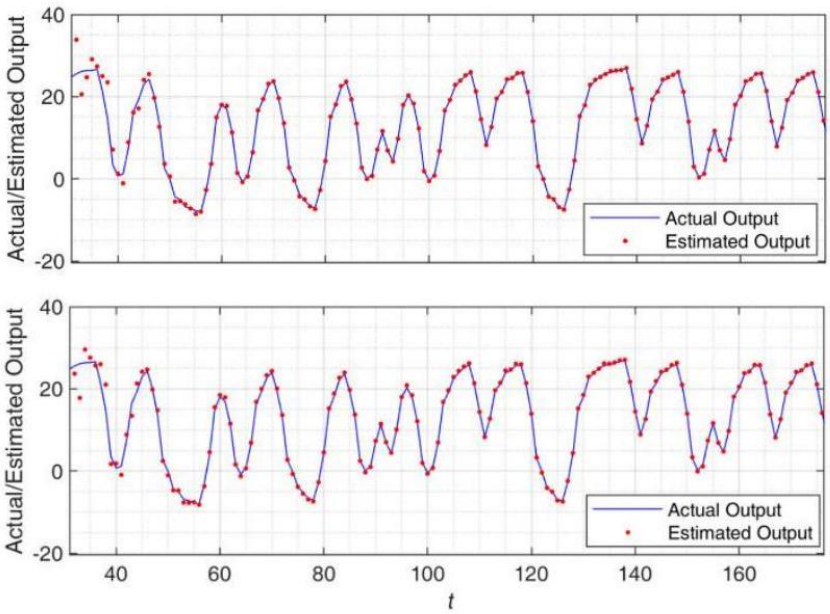

Figure 7. *The Actual output $y(t)$ and estimated output $\hat{y}(t)$ with $\rho_{w,v} = 0$ and $\rho_{w,v} = 0.9$.*

0.8 are depicted in Figures 3 and 4. The noise estimates $\hat{v}(t)$ and $\hat{w}(t)$ for $\rho_{w,v} = 0.3$ and $\rho_{w,v} = 0.8$ are showed in Figures 5 and 6. Figure 7 illustrate the true output and the predicted output of the KF-CN-RGELS with 80% cross-correlation between system noises.

Table 2. *The KF-CN-RGELS and errors $(\rho_{w,v} = 0, \sigma_v^2 = 0.15^2)$.*

t	f_1	f_2	g_1	g_2	d	J_1	J_2	$\delta\%$
100	0.03	-0.40	1.78	3.10	1.44	0.24	0.39	12.21
200	-0.010	-0.373	1.805	2.931	1.383	0.266	0.374	7.356
500	-0.037	-0.362	1.811	2.798	1.330	0.276	0.368	3.887
1000	-0.045	-0.360	1.807	2.753	1.317	0.272	0.367	2.958
2000	-0.049	-0.358	1.803	2.730	1.311	0.261	0.367	2.462
3000	-0.050	-0.358	1.801	2.721	1.308	0.253	0.371	2.167
4000	-0.050	-0.359	1.801	2.719	1.307	0.247	0.372	2.021

t	f_1	f_2	g_1	g_2	d	J_1	J_2	$\delta\%$
5000	-0.050	-0.359	1.801	2.717	1.307	0.237	0.375	1.808
True values	-0.05	-0.36	1.8	2.7	1.3	0.21	0.43	

Table 3. The KF-CN-RGELS and errors ($\rho_{w,v} = 0.5, \sigma_v^2 = 0.15^2$).

t	f_1	f_2	g_1	g_2	d	J_1	J_2	$\delta\%$
100	0.024	-0.398	1.769	3.090	1.440	0.228	0.430	11.943
200	-0.009	-0.376	1.796	2.930	1.384	0.250	0.412	7.105
500	-0.035	-0.365	1.806	2.799	1.330	0.261	0.399	3.407
1000	-0.043	-0.362	1.803	2.756	1.317	0.257	0.395	2.355
2000	-0.047	-0.360	1.801	2.734	1.311	0.249	0.392	1.830
3000	-0.048	-0.360	1.800	2.724	1.309	0.241	0.392	1.559
4000	-0.048	-0.360	1.799	2.722	1.307	0.236	0.393	1.433
5000	-0.049	-0.360	1.800	2.719	1.307	0.226	0.393	1.264
True values	-0.050	-0.360	1.800	2.700	1.300	0.210	0.430	

Table 4. The KF-CN-RGELS and errors ($\rho_{w,v} = 0.8, \sigma_v^2 = 0.15^2$).

t	f_1	f_2	g_1	g_2	d	J_1	J_2	$\delta\%$
100	0.018	-0.394	1.769	3.076	1.436	0.235	0.462	11.574
200	-0.013	-0.374	1.796	2.917	1.382	0.255	0.441	6.758
500	-0.037	-0.364	1.806	2.793	1.329	0.264	0.427	3.168
1000	-0.043	-0.362	1.803	2.753	1.317	0.259	0.422	2.120
2000	-0.047	-0.360	1.801	2.733	1.311	0.250	0.418	1.537
3000	-0.048	-0.360	1.800	2.723	1.308	0.243	0.416	1.225
4000	-0.048	-0.360	1.799	2.722	1.307	0.238	0.417	1.084
5000	-0.048	-0.360	1.800	2.719	1.306	0.228	0.416	0.854
True values	-0.05	-0.36	1.8	2.7	1.3	0.21	0.43	

Table 5. The KF-CN-RGELS and errors ($\rho_{w,v} = 0$, and $\rho_{w,v} = 0.9, \sigma_v^2 = 0.30^2$).

$\rho_{w,v}$	t	f_1	f_2	g_1	g_2	d	J_1	J_2	$\delta\%$
0	100	0.033	-0.396	1.768	3.140	1.447	0.268	0.382	13.512
	200	-0.009	-0.371	1.806	2.944	1.389	0.288	0.378	7.876
	500	-0.040	-0.360	1.817	2.794	1.324	0.297	0.381	3.956
	1000	-0.046	-0.359	1.809	2.751	1.313	0.282	0.388	2.795
	2000	-0.049	-0.358	1.803	2.732	1.309	0.264	0.398	1.994

$\rho_{w,v}$	t	f_1	f_2	g_1	g_2	d	J_1	J_2	$\delta\%$
	3000	-0.051	-0.358	1.801	2.722	1.307	0.253	0.404	1.566
	4000	-0.050	-0.358	1.799	2.722	1.306	0.248	0.405	1.422
	5000	-0.050	-0.358	1.800	2.720	1.307	0.231	0.409	1.025
0.9	100	0.018	-0.388	1.766	3.093	1.444	0.260	0.485	12.191
	200	-0.015	-0.369	1.806	2.919	1.387	0.278	0.470	7.083
	500	-0.040	-0.361	1.816	2.786	1.323	0.286	0.461	3.443
	1000	-0.045	-0.361	1.807	2.751	1.313	0.271	0.459	2.428
	2000	-0.047	-0.360	1.802	2.735	1.309	0.255	0.458	1.790
	3000	-0.049	-0.360	1.800	2.725	1.307	0.246	0.453	1.397
	4000	-0.049	-0.360	1.798	2.725	1.306	0.241	0.453	1.300
	5000	-0.049	-0.360	1.799	2.723	1.307	0.225	0.450	0.974
True values		-0.05	-0.36	1.8	2.7	1.3	0.21	0.43	

Focusing on Tables 1-5 and Figures 1-7, some conclusions can be drawn from the tables and figures.

1. As the correlation-coefficient $\rho_{w,v}$ between process and measurement noise increases, the best estimate of the state is obtained, which is reflected in the increased accuracy of the parameter estimate depicted by KF-CN-RGELS algorithm-See Table 1.
2. Obviously, the KF-CN-RGELS algorithm can provide more accurate estimates of parameters with increasing correlation coefficients $\rho_{w,v}$-See Tables 2-4 and Figures 1, 2.
3. Measurement noise variance σ_v^2 affects the cross- correlation between measurement and process noise, which in turn affects the accuracy of the joint estimate - See Tables 2-5 and Figures 5-6.
4. The predicted output $\hat{y}(t)$ provided by KF-CN-RGELS algorithm is approximately equal to the actual output $y(t)$ because the correlation between process noise and measurement noise increases positively. This shows that the estimation accuracy of the KF-CN-RGELS is high-See Figure 7.

6. Conclusion

The KF-CN-RGELS algorithm effectively solves the joint estimation problem for linear stochastic systems with correlated noises by combining a modified Kalman filter and recursive least squares. By explicitly modeling noise cross-correlation through the gain matrix T, the method achieves higher accuracy as $\rho_{w,v}$ increases-parameter errors drop by 53% for $\rho_{w,v} = 0.8$ compared to uncorrelated cases. Experiments confirm consistent performance across noise variances, with state estimates converging smoothly and output predictions closely matching true values. This work bridges a critical gap in systems where noise correlation is inherent, such as aerospace and process control, while paving the way for extensions to bilinear systems and real-time applications.

Abbreviations

KF-CN-RGELS	Kalman Filtering with Correlated Noises Based Recursive Generalized Extended Least Squares

Author Contributions

Khalid Abd El Mageed Hag El Amin is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The author declare no conflicts of interest.

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