American Journal of Theoretical and Applied Statistics
2025; 14(6): 267-276
http://www.sciencepublishinggroup.com/j/ajtas
doi: 10.11648/j.ajtas.20251406.12

Recursive Estimation of the Intensity Function of the Non-homogeneous Poisson Process

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To cite this article:
Marcel Sihintoe Badiane, Siba Kalivogui, Bakary D Coulibaly, Aguemon Wiwegnon Uriel-Longin. (2025). Recursive Estimation of the Intensity Function of the Non-homogeneous Poisson Process. *American Journal of Theoretical and Applied Statistics, 14*(6), 267-276. https://doi.org/10.11648/j.ajtas.20251406.12

Received: 07 October 2025 ; **Accepted:** 25 October 2025 ; **Published:** 19 December 2025

Abstract: This study focuses on the recursive nonparametric estimation of the intensity function associated with a non-homogeneous Poisson process. Accurately estimating the intensity function is crucial for understanding the dynamics of events in fields such as finance, neuroscience, and environmental monitoring. While traditional nonparametric estimators are theoretically robust, their reliance on the entire dataset for every update makes them impractical for real-time applications. To overcome this limitation, we introduce a recursive estimator that supports efficient, online updates as new data becomes available. This approach significantly lowers computational overhead while maintaining strong statistical reliability. We thoroughly analyze the asymptotic behavior of the proposed estimator, paying particular attention to the Asymptotic Mean Integrated Squared Error (AMISE), a key measure of estimation accuracy. Additionally, we compare the performance of our recursive estimator with Cucala's non-recursive method. The results reveal that our approach achieves equivalent or superior accuracy in terms of AMISE, particularly in large-sample scenarios. A computational performance comparison further underscores the advantages of the proposed method, demonstrating its substantial reduction in execution time and its suitability for applications requiring rapid processing.

Keywords: Nonparametric Estimation, Recursive Kernel Estimators, Intensity Function

1. Introduction

Point processes form a key area within spatial statistics, primarily concerned with analyzing the geometrical structure of patterns formed by objects referred to as events that are randomly distributed in both number and space. Such data arise across a wide range of disciplines, including ecology [9, 25], epidemiology [10, 11], astronomy [5, 12], forestry [13], and seismology [14–17]. Comprehensive theoretical foundations and classical applications of point processes can be found in works such as [8, 18–21].

A central objective in point process theory is to model the first-order intensity function. One common approach involves assuming a parametric form for the intensity and estimating it using model selection tools like the Akaike Information Criterion (AIC) [19, 20, 22], or through pseudolikelihood methods [23]. In the Bayesian framework, [24] introduced models based on log-Gaussian Cox processes. However, these parametric techniques may yield unreliable estimates when the assumed model poorly reflects the true underlying intensity function.

To address this, nonparametric alternatives have been proposed, such as quadrat count methods and kernel-based estimators. In particular, Diggle, P. J introduced the first kernel estimator for intensity, adapting the structure of the classical kernel density estimators developed by [26–28], with an added edge correction. While Diggle's estimator is valuable for exploratory analysis, its lack of consistency has limited its broader applicability [44]. To overcome this issue, Cucala developed an asymptotic framework for Diggle's estimator by introducing the concept of the idensity of event locations, which relies on the idea that intensity and density functions differ only by a multiplicative constant [29].

Recursive kernel approaches were introduced and abundantly studied in the non-spatial case. We refer, for instance, to the pioneer works by Wolverton, Deheuvels and Wegman, who have paid attention to nonparametric recursive density estimation for time processes ([30–32]).

Let X be an inhomogeneous point process defined on $\mathbb{R}^2$, and consider a realization of X, denoted as $X_1, \dots, X_N$, within a bounded region $W \subset \mathbb{R}^2$. For simplicity, events corresponding to point patterns are written in uppercase letters, while lowercase letters are used to represent any point in $\mathbb{R}^2$. If dx refers to an infinitesimal region around the point $x \in \mathbb{R}^2$, the first-order intensity function of X (see [44]) is defined as:

$$\lambda(x) = \lim_{|dx| \to 0} \left(\frac{\mathbf{E}[N(dx)]}{|dx|} \right)$$

where dx is a small region around x, $|dx|$ represents its area, and $N(dx)$ denotes the number of points within dx. Effectively, the product $\lambda(x)|dx|$ can be interpreted as the probability of observing a data point in the infinitesimal region dx. In stationary cases, where $\lambda(x) = \lambda$ is a constant, the intensity can be estimated as $\lambda = \frac{n}{|D|}$, where n counts the number of points in D, the observed bounded region of interest in $\mathbb{R}^2$. However, for non-stationary or inhomogeneous cases, $\lambda(x)$ varies as a function of x.

A fundamental problem when dealing with inhomogeneous point processes is estimating their first-order intensity. Parametric methods can address this by assuming a functional form for the intensity function and estimating its unknown parameters through techniques such as maximum pseudolikelihood [44] or maximum likelihood ([23]). However, if the assumed parametric form deviates significantly from reality, the resulting estimates may lack reliability. As an alternative, non-parametric estimation of the first-order intensity is possible. For example, Diggle [26] proposed the following kernel-based intensity estimator:

$$\hat{\lambda}_h(x) = \frac{1}{p_h(x)} \sum_{i=1}^{N} K_h(x - X_i) = \frac{1}{p_h(x) h^2} \sum_{i=1}^{N} K\left(\frac{x - X_i}{h} \right),$$

where $K(.)$ is a radially symmetric bivariate probability density function (the kernel function), $h > 0$ is the smoothing parameter or bandwidth, and $K_h(.)$ is the scaled or smoothed version of the kernel. The term $p_h(x) = \int_W h^{-2} K\left(\frac{x-y}{h}\right) dy$ represents the edge-correction factor to account for boundary effects.

Bandwidth selection is crucial in kernel estimation, but it has received relatively little attention in the context of point processes. For stationary Cox processes in $\mathbb{R}^2$, Diggle [33] showed that minimizing the mean squared error (MSE) for kernel intensity estimation yields the same bandwidth as least-squares cross-validation (LSCV) in standard density estimation (see [34, 35]). Although this equivalence has not been proven for inhomogeneous processes, Brooks et al. [36] demonstrated that LSCV provides asymptotically optimal bandwidth selection for such cases in $\mathbb{R}^2$. Additionally, smooth bootstrap procedures have been employed to optimize bandwidth in kernel density estimation [37, 38] and kernel hazard rate estimation [39], with some studies reporting superior performance compared to cross-validation. In point processes, Loh [40] applied non-parametric bootstrap methods for bandwidth selection in estimating the two-point correlation function, while Cowling et al. [41] developed smooth bootstrap approaches to construct confidence bands for kernel intensity estimators. Nevertheless, to date, no specific bootstrap procedures for bandwidth selection in kernel intensity estimation have been proposed.

A notable drawback of non-parametric intensity estimators lies in their lack of consistency. As noted by Guan [43], when using a kernel $K(.)$ with finite support centered at the origin, only local information around each point contributes to estimating the intensity. For truly continuous intensity functions, local smoothing can yield an

$$\hat{\lambda}_h(x) = \frac{1}{p_h(x)} \sum_{i=1}^{N} K_{x,h}(X_i). \tag{1}$$

where $x = (x_1, x_2)^T$, $X_i = (X_{i1}, X_{i2})^T$, $i = 1, ..., N$ and $h = (h_1, h_2)^T$ is the vector of bandwidths. Note that $K_{x,h}(.)$ is the bivariate continuous kernel (see [3, 29]).

The number of events in an inhomogeneous Poisson point process, denoted as N, follows a distribution of $Poisson\left(\int_W \lambda(x)\, dx\right) = Poisson(m)$. This allows us to derive a relationship connecting the bivariate density and the first-order intensity of a spatial point process [3, 29].

The relationship is expressed as:

$$f(x) = \frac{\lambda(x)}{\int_W \lambda(x)\, dx} = \frac{\lambda(x)}{m}.$$

Based on this formulation, Cucala introduced the density of event locations as:

$$\lambda_0(x) = \frac{\lambda(x)}{m}.$$

When considering a vector of bandwidths h, the kernel estimator of $\lambda_0(.)$ is represented as:

$$\hat{\lambda}_{0,h}(x) = \frac{\hat{\lambda}_h(x)}{N} \mathbf{1}_{[N \neq 0]}.$$

Recent studies have explored the asymptotic properties of recursive kernel density estimators for dependent data. Amiri, A. introduced a generalized framework encompassing several variations of recursive estimators, characterized by a parameter $\ell \in [0, 1]$ specific to each estimator [2]. For fixed values such as $\ell = 1/2$ and $\ell = 1$, prior analyses investigated asymptotic normality under negative association by Liang [47], while Masry [48] studied mean-square convergence and asymptotic normality under α-mixing conditions. Building on these findings, Mezhoud, H., et al. extended Amiri's results to η-dependent time processes [2].

This article focuses on the estimation of the intensity function for non-homogeneous Poisson processes through a recursive kernel smoothing approach, which generalizes prior methodologies like those proposed by [29]. The problem of intensity function estimation has been extensively examined across diverse contexts in existing literature. While nonparametric estimators, including Cucala's intensity function estimator, provide effective solutions for intensity estimation, they face challenges related to computational complexity, particularly in high-dimensional scenarios [29].

In this work, we aim to extend Amiri's framework to address non-homogeneous Poisson processes. Specifically, we introduce a recursive kernel estimator for the intensity function, broadening the scope of previous studies in the domain of non-recursive estimators.

The remainder of the paper is organized as follows. In Section 2, we delve into the recursive kernel intensity for the inhomogeneous Poisson process. Section 3 outlines our key assumptions and presents the results of the intensity function analysis. Finally, Section 4 provides a comprehensive simulation and conclusion.

2. The Recursive Kernel Intensity Function

Let $\{X_1,\ldots,X_N\}$ denote a realization of the inhomogeneous spatial point process X, observed over the region $W \subset \mathbb{R}^2$. For this study, we assume the point process follows a Poisson distribution (as discussed in [44]). The assumption of a kernel is essential for developing an asymptotic theory and ensuring the consistency of the kernel estimator for the density of event locations. Nevertheless, both the kernel density estimator and the bandwidth selection method introduced in this work can also be employed for non-Poisson point processes.

Considering the relationship between density and intensity function described by Cucala [29], along with the density estimator proposed by [1], the recursive estimator for the intensity function can be expressed as:

$$\hat{\lambda}_0^{\ell}(x) := \frac{1}{\sum_{i=1}^{N} h_i^{2(1-l)}} \sum_{i=1}^{N} \frac{1}{h_i^{2\ell}} K\left(\frac{x-X_i}{h_i}\right) \mathbb{1}_{N\neq 0}(x),\ x\in\mathbb{R}^2, \ell\in[0,1]. \tag{2}$$

Our family of estimators can be computed recursively by:

$$\hat{\lambda}_{0,+1}^{\ell}(x) = \frac{\sum_{i=1}^{N} h_i^{2(1-\ell)}}{\sum_{i=1}^{N+1} h_i^{2(1-\ell)}} \hat{\lambda}_0^{\ell}(x) + K_{N+1}^{\ell}(x - X_{N+1}), \tag{3}$$

$$K_i^{\ell}(\cdot) := \frac{1}{h_i^{2\ell}\sum_{j=1}^{i} h_j^{2(1-\ell)}} K\left(\frac{\cdot}{h_i}\right) \tag{4}$$

$$\hat{\lambda}_0^{\ell}(x) := \frac{B_{N,2(1-l)}^{-1}}{N h_N^{2(1-l)}} \sum_{i=1}^{N} \frac{1}{h_i^{2l}} K\left(\frac{x-X_i}{h_i}\right) \mathbb{1}_{N\neq 0}, \quad x\in\mathbb{R}^2, \quad \ell\in[0,1]. \tag{5}$$

$$B_{N,2(1-\ell)} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{h_i}{h_n}\right)^{2(1-\ell)}$$

The condition imposed on the intensity function λ_0 is classical in this field, and has been used, for example, by [7] for density estimation with the Parzen-Rosenblatt estimator. We also consider kernels K satisfying the following assumptions:

Assumptions H

(H.1) $K:\mathbb{R}^2\to\mathbb{R}$ is a probability density function, strictly positive, symmetric, and bounded;

(H.2)
$$\lim_{\|x\|\to+\infty} \|x\|^2 K(x) = 0, \quad \forall x\in\mathbb{R}^2;$$

(H.3)
$$\int_{\mathbb{R}^2} |v_i v_j| K(v)\,dv < \infty, \quad i,j = 1,\ldots,2.$$

Assumptions *H.1-H.3* are classical in nonparametric estimation, and are satisfied in particular by the Epanechnikov and Gaussian kernels.

The bandwidth h_m satisfies the following conditions:

(*H.4*) $h_m \downarrow 0$ and $mh_m^4 \to \infty$ as $m \to \infty$;

(*H.5*) For all $r \in (-\infty, 4]$,

$$B_{m,r} := \frac{1}{m}\sum_{i=1}^{m}\left(\frac{h_i}{h_m}\right)^r \to \alpha_r < \infty \quad \text{as } m \to \infty.$$

3. Main Results

We can now establish the exact asymptotic expressions for the bias, variance, and mean squared error (MSE) of $\hat{\lambda}_0^\ell(x)$, as a function of the parameter ℓ

Theorem 3.1. Under assumptions *H.1-H.5*, we have:

(a) For all $\ell \in [0,1]$,

$$h_m^{-2}\left[\mathbb{E}\hat{\lambda}_0^\ell(x) - \lambda_0(x)\right] \longrightarrow \frac{-e^{-m}\lambda_0(x)}{h_m^2} + (1-e^{-m})Q_{\lambda_0}^2(x) \quad \text{when} \quad m \to \infty,$$

with :

$$Q_{\lambda_0}(x) := \frac{1}{2}\frac{B_{m,2(1-\ell)}}{m}\sum_{i=1}^{m}\left(\frac{h_i}{h_m}\right)^{2(1-\ell)+2}\sum_{i,j=1}^{2}\frac{\partial^2\lambda_0(x)}{\partial x_i \partial x_j} \times \int_{\mathbb{R}^2} u_i u_j K(u)\, du. \tag{6}$$

(b) For all $\ell \in [0,1]$,

$$mh_m^2 \operatorname{Var}\left(\hat{\lambda}_0^\ell(x)\right) \longrightarrow A(m)R_\ell(x)^2 \quad \text{when} \quad m \to \infty,$$

with:

$$R_\ell(x)^2 := \frac{\beta_{2(1-2\ell)}}{\beta_{2(1-\ell)}^2}\lambda_0(x)\int_{\mathbb{R}^2} K^2(u)\, du. \tag{7}$$

$$A(m) = \mathbb{E}\left[\frac{1}{N}\mathbb{1}_{N\neq 0}\right] = e^{-m}\sum_{k=1}^{\infty}\frac{m^k}{kk!} < e^{-m}\sum_{k=0}^{\infty}\frac{2m^k}{(k+1)!} = \frac{2}{m} \longrightarrow 0$$

when $m \longrightarrow \infty$

(c) If

$$h_m = C_m m^{-\frac{1}{6}}, \quad \text{with } C_m \downarrow c > 0,$$

then:

$$m^{\frac{4}{6}}\,\mathbb{E}\left[\hat{\lambda}_0^\ell(x) - \lambda_0(x)\right]^2 \longrightarrow \frac{c^4(4+2\ell)^2}{4}Q_{\lambda_0}^2(x) + \frac{(4+2\ell)^2\lambda_0(x)\|K\|_2^2}{c^2(4+2)(2+2\ell)}, \quad \text{as } m \to \infty, \tag{8}$$

at every point x where $\lambda_0(x) > 0$.

Theorem 3.2. Under assumptions *H.1-H.5*, if for all $\alpha > 0$:

$$\lim_{m\to\infty}\frac{mh_m^2}{(\ln\ln m)^{2(\alpha+1)}\ln m} = \infty \quad \text{and} \quad \lim_{m\to\infty}\frac{\ln h_m}{\ln m} < \infty \tag{9}$$

then for all x such that $\lambda_0(x) > 0$:

(a) *Almost sure convergence:*

$$\lim_{m\to\infty}\sqrt{\frac{mh_m^2}{\ln\ln m}}\left(\hat{\lambda}_0^\ell(x)-\mathbb{E}[\hat{\lambda}_0^\ell(x)]\right)=R_\ell(x)\sqrt{2}\quad \text{a.s.}$$

where R_ℓ^2 is defined in equation (7).

(b) *Furthermore, the choice:*

$$h_m=C_m\left(\frac{\ln\ln m}{m}\right)^{\frac{1}{6}},\quad \text{with } C_m\downarrow c>0,$$

implies:

$$\overline{\lim_{m\to\infty}}\left(\frac{m}{\ln\ln m}\right)^{\frac{1}{3}}\left(\hat{\lambda}_0^\ell(x)-\lambda_0(x)\right)$$
$$=R_\ell(x)\sqrt{2}c^2+\frac{1}{c^{2\beta+2(1-\ell)+2}}Q_{\lambda_0}(x)\quad \text{a.s.}$$

where $\hat{\lambda}_0^\ell(x)$ is defined in equation (2).

Lemma 3.1. Under the assumptions of Theorem 3.1, the choice

$$h_m=m^{-\nu},\quad 0<\nu<\frac{1}{4}$$

implies that, for all x such that $\lambda_0(x)>0$,

$$\lim_{m\to\infty}\frac{m^{1-2\nu}}{\ln\ln m}\left(\hat{\lambda}_0^\ell(x)-\lambda_0(x)\right)\longrightarrow\frac{1}{2}\left[1-2\nu(1-\ell)\right]^2$$
$$c^2\left[1-2\nu(1-2\ell)\right]\lambda_0(x)\|K\|_2^2\quad \text{a.s.}$$

Theorem 3.3. Under the assumptions *H.1-H.5*, the variance of the estimate is given by

$$Var(\hat{\lambda}_0^\ell(x))=\frac{A(m)}{mh_m^2}\frac{\beta_{2(1-2\ell)}}{\beta_{2(1-\ell)}^2}\lambda_0(x)\int_{\mathbb{R}^2}K^2(u)du. \tag{10}$$

and the mean integrate squared error is given by

$$\begin{aligned}
MSE(\hat{\lambda}_0^\ell(x))&=Bias^2(\hat{\lambda}_0^\ell(x))+Var(\hat{\lambda}_0^\ell(x))\\
&=e^{-2m}\lambda_0^2(x)-(e^{-m}-e^{-2m})\frac{\sum_{i=1}^m h_i^{2(1-\ell)+2}}{\sum_{i=1}^m h_i^{2(1-\ell)}}\lambda_0(x)\sum_{i,j=1}^2\frac{\partial^2\lambda_0(x)}{\partial x_i\partial x_j}\\
&\times\int_{\mathbb{R}^2}u_iu_jK(u)\,du+\frac{(1-e^{-m})^2}{4}\left(\frac{\sum_{i=1}^m h_i^{2(1-\ell)+2}}{\sum_{i=1}^m h_i^{2(1-\ell)}}\right)^2\times\left(\sum_{i,j=1}^2\frac{\partial^2\lambda_0(x)}{\partial x_i\partial x_j}\int_{\mathbb{R}^2}u_iu_jK(u)\,du\right)^2\\
&+\frac{A(m)}{mh_m^2}\frac{\beta_{2(1-2\ell)}}{\beta_{2(1-\ell)}^2}\lambda_0(x)\int_{\mathbb{R}^2}K^2(u)du
\end{aligned}\tag{11}$$

$$\begin{aligned}
MISE(\hat{\lambda}_0^\ell(x))&=e^{-2m}\int_{\mathbb{R}^2}\lambda_0^2(x)dx-(e^{-m}-e^{-2m})\frac{\sum_{i=1}^m h_i^{2(1-\ell)+2}}{\sum_{i=1}^m h_i^{2(1-\ell)}}\int_{\mathbb{R}^2}\lambda_0(x)\sum_{i,j=1}^2\frac{\partial^2\lambda_0(x)}{\partial x_i\partial x_j}dx\int_{\mathbb{R}^2}u_iu_jK(u)\,du\\
&+\frac{(1-e^{-m})^2}{4}\left(\frac{\sum_{i=1}^m h_i^{2(1-\ell)+2}}{\sum_{i=1}^m h_i^{2(1-\ell)}}\right)^2\times\int_{\mathbb{R}^2}\left(\sum_{i,j=1}^2\frac{\partial^2\lambda_0(x)}{\partial x_i\partial x_j}\int_{\mathbb{R}^2}u_iu_jK(u)\,du\right)^2dx+\frac{A(m)}{mh_m^2}\frac{\beta_{2(1-2\ell)}}{\beta_{2(1-\ell)}^2}R(K)\int_{\mathbb{R}^2}\lambda_0(x)dx
\end{aligned}\tag{12}$$

with $R(K)=\int_{\mathbb{R}^2}K^2(u)du$

Theorem 3.4. Under the assumptions *H.1-H.5*, the AMISE is given by

$$AMISE(\hat{\lambda}_0(x)) = \frac{1}{4}\left(\frac{\sum_{i=1}^{m} h_i^{2(1-\ell)+2}}{\sum_{i=1}^{m} h_i^{2(1-\ell)}}\right)^2 \times \int_{\mathbb{R}^2}\left(\sum_{i,j=1}^{2}\frac{\partial^2\lambda_0(x)}{\partial x_i\partial x_j}\int_{\mathbb{R}^2} u_i u_j K(u)\,du\right)^2 dx$$

$$+ \frac{A(m)}{mh_m^2}\frac{\beta_{2(1-2\ell)}}{\beta_{2(1-\ell)}^2}R(K)\int_{\mathbb{R}^2}\lambda_0(x)dx \tag{13}$$

4. Simulation

In this section we study the finite sample properties for the non-parametric intensity estimator. For recursive estimators, we use the recursive formula (2). The efficiency of one estimator is evaluated by its time of computation, the mean integrate square error, defined (for some estimator $\hat{\lambda}_0^\ell$ of λ_0) given by (13). We consider the following tree data generating processes:

This section explores the finite sample properties of the non-parametric intensity estimator. For recursive estimators, the recursive formula (2) is employed. The efficiency of an estimator is assessed based on its computation time and the mean integrated square error, which is defined for a given estimator $\hat{\lambda}_0^\ell$ of λ_0 using (13). The study examines 3 specific data-generating processes:

1. $\lambda_k(x) = b\Phi_{(0.3-0.2x_2,0.02)}(x_1) + 25, \quad x = (x_1, x_2) \in [0,1]^2$,
   where Φ is the univariate normal density with mean $\mu = 0.3 - 0.2x_2$ and standard deviation $\sigma = 0.1$ for $k = 1$ and $\sigma = 0.02$ for $k = 2$, using several values of b (b= 50 or 100 or 200 or 500 or 1000) to obtain $m = \int_W \lambda(x)dx \approx$ 50 or 100 or 200 or 500 or 1000 respectively [4]. This model was used by Barr and al and Badiane and al to analyze the performance of the Voronoi estimator of the first order intensity [3, 6].
2. $\lambda_3(x) = b(\cos(\sqrt{(x_1-3)^2 + (x_2-3)^2}))^2, \quad x = (x_1, x_2) \in [0,1]^2$, using several values of b (b=127 or 330 or 254 or 634 or 1267) to obtain $m = \int_W \lambda(x)dx \approx$ 50 or 100 or 200 or 500 or 1000 respectively [3, 6].

Table 1. *AMISE for* λ_1.

m	100	200	300	400	500	600	700	800	900	1000
$\ell = 0$	0.09000400	0.05431603	0.04255789	0.03433607	0.03182646	0.02813108	0.02695227	0.02542130	0.02439409	0.02394308
$\ell = 0.25$	0.08702616	0.05255531	0.03996958	0.03122780	0.02914962	0.02620523	0.02448197	0.02326612	0.02245753	0.02128738
$\ell = 0.5$	0.08588044	0.04798087	0.03688177	0.03083313	0.02742870	0.02472486	0.02352746	0.02163888	0.02029571	0.01985511
$\ell = 0.75$	0.09032026	0.04669673	0.03441650	0.02878827	0.02417882	0.02240616	0.02047662	0.01884335	0.01776804	0.01736258
$\ell = 1$	0.08318790	0.04203474	0.03218427	0.02798985	0.02316302	0.02135061	0.01855814	0.01682907	0.01586503	0.01472962
N-Cucala	0.01262058	0.00631250	0.00420980	0.00315846	0.00252765	0.00210711	0.00180672	0.00158144	0.00140621	0.00126603

Table 2. *AMISE for* λ_2.

m	100	200	300	400	500	600	700	800	900	1000
$\ell = 0$	0.07807019	0.03857489	0.02626431	0.01969904	0.01509942	0.01313655	0.01127964	0.09942	0.00977278	0.00854499
$\ell = 0.25$	0.07638976	0.03744009	0.02548391	0.01938836	0.01562648	0.01296909	0.01120782	0.00976252	0.00871051	0.00784083
$\ell = 0.5$	0.07333321	0.03791183	0.02553201	0.01939621	0.01503910	0.01313271	0.01115697	0.00967673	0.00871014	0.00780123
$\ell = 0.75$	0.07813212	0.03867416	0.02625181	0.01922492	0.01559635	0.01312970	0.01123931	0.00977937	0.00838264	0.00786796
$\ell = 1$	0.07332035	0.03523962	0.02215149	0.01857128	0.01571018	0.01284314	0.01121665	0.00970216	0.00828664	0.00783233
N-Cucala	0.01266819	0.00636985	0.00427040	0.00322068	0.00259085	0.00217096	0.00187104	0.00164610	0.00147114	0.00133118

Table 3. *AMISE for* λ_3.

m	100	200	300	400	500	600	700	800	900	1000
$\ell = 0$	0.09572008	0.05627179	0.04499034	0.03977890	0.03521334	0.03287814	0.02958691	0.02925416	0.02782186	0.02725282
$\ell = 0.25$	0.09179052	0.05602660	0.04215537	0.03685550	0.03323482	0.02948652	0.02771347	0.02750318	0.02560075	0.02556891

m	100	200	300	400	500	600	700	800	900	1000
$\ell = 0.5$	0.09458181	0.05199916	0.04069859	0.03387976	0.03021782	0.02802043	0.02626819	0.02494713	0.02360138	0.02331779
$\ell = 0.75$	0.08760614	0.04828332	0.03617066	0.03063066	0.02824850	0.02536253	0.02299810	0.02166223	0.02051792	0.02007490
$\ell = 1$	0.08547880	0.04635842	0.03478291	0.02929131	0.02422125	0.02194968	0.01998724	0.01934042	0.01824520	0.01700774
N-Cucala	0.07001236	0.03511892	0.02348778	0.01767220	0.014182865	0.01185663	0.01019504	0.00894884	0.00797958	0.00720417

Tables 1-3 display the empirical mean integrated squared errors (AMISE) for our recursive estimators across ℓ values $\{0, 0.25, 0.5, 0.75, 1\}$, alongside those of Cucala's estimator. In this particular case, the mean integrated squared errors are observed to be remarkably similar for all ℓ values in the given range. Moreover, the results highlight that the efficiency of our recursive estimators improves as ℓ increases with respect to AMISE. However, Cucala's estimator demonstrates superior performance compared to the recursive estimators.

Table 4. *CPU-time for intensity estimator for λ_1.*

m	100	200	300	400	500	600	700	800	900	1000
$\ell = 0$	0.442	0.446	0.447	0.462	0.494	0.504	0.525	0.526	0.525	0.532
$\ell = 0.25$	0.450	0.460	0.469	0.526	0.537	0.541	0.542	0.546	0.548	0.573
$\ell = 0.5$	0.443	0.501	0.571	0.626	0.649	0.675	0.731	0.733	0.747	0.783
$\ell = 0.75$	0.444	0.449	0.458	0.497	0.512	0.527	0.549	0.559	0.562	0.582
$\ell = 1$	0.416	0.421	0.425	0.428	0.431	0.433	0.439	0.450	0.453	0.471
N-Cucala	1.388	1.447	1.483	1.492	1.535	1.561	1.613	1.666	1.694	1.795

Table 5. *CPU-time for intensity estimator for λ_2.*

m	100	200	300	400	500	600	700	800	900	1000
$\ell = 0$	0.424	0.427	0.437	0.442	0.446	0.450	0.453	0.496	0.515	0.556
$\ell = 0.25$	0.424	0.429	0.431	0.438	0.441	0.447	0.497	0.524	0.527	0.572
$\ell = 0.5$	0.420	0.427	0.433	0.435	0.437	0.447	0.449	0.519	0.528	0.551
$\ell = 0.75$	0.433	0.434	0.438	0.439	0.444	0.493	0.498	0.511	0.512	0.519
$\ell = 1$	0.432	0.435	0.446	0.448	0.449	0.451	0.453	0.456	0.514	0.535
N-Cucala	1.443	1.480	1.507	1.572	1.582	1.592	1.608	1.627	1.631	1.765

Table 6. *CPU-time for intensity estimator for λ_3.*

m	100	200	300	400	500	600	700	800	900	1000
$\ell = 0$	0.424	0.427	0.437	0.442	0.446	0.450	0.453	0.496	0.515	0.556
$\ell = 0.25$	0.424	0.429	0.431	0.438	0.441	0.447	0.497	0.524	0.527	0.572
$\ell = 0.5$	0.420	0.427	0.433	0.435	0.437	0.447	0.449	0.519	0.528	0.551
$\ell = 0.75$	0.318	0.320	0.321	0.322	0.326	0.331	0.335	0.381	0.391	0.407
$\ell = 1$	0.311	0.315	0.319	0.320	0.325	0.377	0.391	0.399	0.401	0.405
N-Cucala	1.061	1.040	1.114	1.234	1.239	1.287	1.279	1.431	1.556	2.118

Tables 4-6 illustrate the CPU time in seconds required for the recursive estimators for various values of ℓ ($\{0, 0.25, 0.5, 0.75, 1\}$) and for Cucala's estimator. It is noteworthy that computing Cucala's estimator demands more time compared to the $\hat{\lambda}_0^{\ell}(x)$ estimators derived through the recursive method. Additionally, the choice of ℓ does not appear to affect the computation time. The reported CPU times were measured using the following hardware configuration: Processor, MacBook, Hard Drive, 250GB, Memory 8GB..

5. Conclusion

This work focuses on the recursive estimation of the intensity function of a non-homogeneous Poisson process, deriving the asymptotic properties of the proposed estimator, with particular emphasis on the calculation of its Asymptotic Mean Integrated Squared Error (AMISE). A comparative analysis shows that the AMISE of Cucala's estimator is preferable to that of the recursive estimator. However, an evaluation of execution times reveals a significant computational advantage in favor of the recursive approach, highlighting its superior efficiency and establishing it as a method that successfully combines practical performance with reasonable estimation accuracy.

Abbreviations

AMISE	Asymptotic Mean Integrated Squared Error
MSE	Mean Squared Error
LSCV	Least-Squares Cross-Validation
AIC	Akaike Information Criterion
PDF	Probability Density Function

Acknowledgments

The authors express their sincere gratitude to the associate editor and two referees for their thorough review and valuable comments and suggestions, which significantly contributed to enhancing the quality of the paper.

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All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The author declares no conflicts of interest.

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