

Research Article

Impact of Varying Thermal Conductivity and Viscosity on Natural Convection Flow Along a Vertical Flat Plate with Heat Conduction and Viscous Dissipation

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Abstract

This study examines how natural convection flow down a vertical flat plate is affected by varying viscosity and thermal conductivity, taking into account the effects of heat conduction and viscous dissipation. The fluid flow and heat transfer are described mathematically, accounting for temperature-dependent changes in thermal conductivity and viscosity. Using suitable boundary conditions, the governing equations—such as the momentum, energy, and continuity equations—are numerically solved. The impact of these changes on temperature distributions, velocity profiles, Nusselt number, and total heat transfer efficiency is the main focus of the analysis. The findings show that changes in thermal conductivity and viscosity have a major effect on the establishment of thermal boundary layers, heat transfer efficiency, and flow characteristics. The research was conducted to improve the comprehension and prediction of heat transfer processes in a variety of engineering applications by examining the effect of varying thermal conductivity and viscosity on natural convection flow along a vertical flat plate with heat conduction and viscous dissipation. Natural convection is essential in situations where heat transmission occurs without external mechanical aid, including cooling systems, electronic gadgets, and building ventilation. Researchers seek to create more precise models for predicting fluid flow and heat transfer behaviour under varying settings by examining these changes.

Keywords

Steady State, Variation of Viscosity, Variation of Thermal Conductivity, Heat Generation, Viscous Dissipation, Joule Heating

1. Introduction

Investigating intricate relationships between several factors has long been a topic of interest and practical significance in the fields of fluid dynamics and heat transfer phenomena. Among these parameters, viscosity and thermal conductivity play crucial roles in determining the properties of fluid flow and heat transfer processes. The importance of examining how varying viscosity and thermal conductivity affect natural convection flow has been acknowledged by researchers more

and more in recent years, especially in situations involving vertical flat plates heated by Joule heating and heat conduction. The complex interactions between explored in this study in a novel way. This work intends to contribute to the growing body of knowledge in fluid dynamics and heat transfer by elucidating the subtle nuances of changing viscosity and thermal conductivity impacts on natural convection through sophisticated computational tools and creative methodologies.

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In order to fulfil the demands of modern technological and environmental issues, this research aims to offer important insights that can guide the creation of more effective and sustainable engineering solutions by clarifying the underlying mechanisms and behaviors. In order to clarify the complex connections between variable viscosity, thermal conductivity, natural convection flow, heat conduction, and Joule heating in conjunction with vertical flat plates, we will examine the theoretical framework, numerical methodology, and main conclusions of this investigation in the sections that follow. We work to improve our comprehension of these phenomena and open the door for further developments in fluid dynamics and heat transfer research by means of thorough analysis changing viscosity, thermal conductivity, and their effects on natural convection flow along a vertical flat plate are and careful inquiry. From various angles, the effects of a vertical flat plate with heat conduction, variable viscosity, and dependent thermal conductivity on free convection flow are noteworthy. For their objective, researchers are interested in the technology and procedure. Sarker et al. [1] investigated the Variable viscosity and thermal conductivity's effects on magneto hydrodynamic (MHD) natural convection flow in a vertical flat plate. Alam et al. [2] examined the impact of a vertical flat plate with heat conduction, pressure stress work, and viscous dissipation in natural convection flow. Alim et al. [3] examined the effect of Joule heating on the coupling of conduction with magneto hydrodynamic (MHD) free convection flow from a vertical flat plate. Rahman et al. [4] demonstrated the effects of temperature-dependent thermal conductivity on magneto hydrodynamic (MHD) free convection flow and a vertical flat plate with heat conduction.

Alim et al. [5] investigated the combined effects of viscous dissipation and Joule heating on the coupling of conduction and free convection. Molla et al. [6] examined the natural convection laminar flow with temperature-dependent viscosity and thermal conductivity and a vertically wavy surface. Safiqul Islam et al. [7] demonstrated the effects of temperature-dependent thermal conductivity on natural convection flow along a vertical flat plate with heat generation. Kabir et al. [8] examined the effects of viscous dissipation on magneto hydrodynamic (MHD) spontaneous convection flow along a vertical wavy surface. Hossain [9] examines the effects of viscous and Joule heating on magneto hydrodynamic (MHD) free convection flow with varying plate temperature. Soundalgekar et al. [10] investigate the transient free convection on an isothermal flat plate using finite difference analysis. Elbashbeshy et al. [11] analyse a steady free convection flow down a vertical plate with changing viscosity and thermal diffusivity. Kafoussius et al. [12] explore the numerical investigation of the mixed free and forced convective laminar boundary layer flow past a vertical isothermal flat plate with temperature dependent viscosity.

Anwar Hossain et al. [13] present the influence of radiation on the free convection flow of fluid with changing viscosity from a porous vertical plate. In the situation of unsteady flow,

Seddeek [14] investigates the impact of varying viscosity on a magneto hydrodynamic (MHD) free convection flow past a semi-infinite flat plate with an aligned magnetic field. G. Palani Kwan et al. [15] calculated A numerical investigation on a vertical plate with changing viscosity and heat conductivity. Siattery JC [16] investigated Momentum, energy and mass transfer in continua. Ockendon H, Ockendon JR. [17] investigated Variable viscosity flows in heated and cooled channels. Elbashbeshy EMA, Dimian MF [18] computed the effects of radiation on the flow and heat transfer over a wedge with variable viscosity. Seddeek MA, Abdelmeguid MS. [19] investigated the effects of radiation and thermal diffusivity on heat transfer over a stretching surface with variable heat flux Balamurugan, K. and Karthikeyan, R [20] studied viscous dissipation effect on steady free convection flow past a semi-infinite flat plate in the presence of magnetic field. Borah, G. and Hazarika, G. C. [21] calculated the effects of variable viscosity & thermal conductivity on steady free convection flow along a semi-infinite vertical plate (in presence of uniform transverse magnetic field). Abdel-Rahman, G. M. [22] investigated the effects of variable viscosity and thermal conductivity on unsteady MHD flow of non-Newtonian fluid over a stretching porous sheet.

Alam, M. M., Alim, M. A. and Chowdhury, M. M. K. [23] investigated free convection from a vertical permeable circular cone with pressure work and non-uniform surface temperature. Aktar, S., Mahmuda Binte Mostafa Ruma and Alim, M. A. [24] calculated the conjugate effects of heat and mass transfer on natural convection flow along an isothermal sphere with radiation heat loss was. Nasrin, R. and Alim, M. A. [25] studied the MHD free convection flow along a vertical flat plate with thermal conductivity and viscosity depending on temperature. Miraj et al. [26] calculated the effects of viscous dissipation and radiation on natural convection flow on a sphere in presence of heat generation. Haque et al. [27] investigated the effects of viscous dissipation on Natural convection flow over a sphere with temperature dependent thermal conductivity. Gebhart, B. [28] studied the effects of viscous dissipation in natural convection. Based on experimental analysis, an analytical solution for the dependent thermal conductivity and variable viscosity in natural convection flow over a vertical flat plate in the presence of heat conduction will be created in this work. A. Pozzi, and M. Lupo, [29] investigated the coupling of conduction with laminar natural convection along a flat plate. Al-Mahasne Mayas Mohammad et al. [30] calculated the variable Temperature Plate Heat Transfer: MHD Fluid Natural Convection Flow in Porous Medium. Journal of Advanced Research in Numerical Heat Transfer. Iqbal Athal et al. [31] studied the Viscosity dissipation and mixed convection flow in a vertical double-passage channel with permeable fluid. Md. Farhad Hasan et al. [32] calculated the Natural Convection Flow over a Vertical Permeable Circular Cone with Uniform Surface Heat Flux in Temperature-Dependent Viscosity with Three-Fold Solutions within the Boundary Layer. Sujit

Mishra et al. [33] studied Thermal performance of nanofluid flow along an isothermal vertical plate with velocity, thermal, and concentration slip boundary conditions employing buongiorno's revised non-homogeneous model. East European Journal of Physics. Uzma Ahmad et al. [34] investigated the Effects of temperature dependent viscosity and thermal conductivity on natural convection flow along a curved surface in the presence of exothermic catalytic chemical reaction.

By approximating functions and derivatives in terms of the central differences in both coordinate directions, the momentum and energy equations are discretized in terms of the non-dimensional coordinates x and y in order to state the equations in finite difference form. The computer code for the current problem, which employs an effective implicit finite-difference method, was developed as a result of numerical simulations of these equations. It is known as the Crank-Nicolson scheme. The results of data analysis have been developed for velocity profile, temperature profile, local skin friction, local Nusselt number, average skin friction, and average Nusselt number for a variety of parameters, including variable viscosity, dependent thermal conductivity, heat formation, viscous dissipation, joule heating, and Prandtl's number.

This study aims to give a thorough grasp of the fundamental principles regulating such intricate fluid dynamics and heat transfer phenomena by incorporating heat conduction and Joule heating effects into the analysis. This study is important because it has the ability to clarify basic concepts of fluid flow and heat transfer while also providing useful information with applications in a variety of domains, including environmental science, thermal engineering, and renewable energy systems. In order to improve system efficiency, mitigate thermal problems in a variety of engineering applications, and optimize heat transfer processes, it is essential to comprehend how changing viscosity and thermal conductivity affect natural convection flow over vertical flat plates.

2. Mathematical Analysis

This paper considers the unsteady flow of a viscous incompressible fluid across a semi-infinite vertical plate. The x -axis is taken vertically upward with the plate and the y -axis

is picked at the leading-edge perpendicular to the plate (Shown in Figure 1). It is assumed that the leading edge of the plate is where the x -axis originates. All fluid physical properties are assumed to be constant, with the exception of the density variation in the body force term in the momentum equation, where the Boussinesq approximation is applied, the thermal conductivity, which varies linearly with fluid temperature, and the fluid viscosity, which varies exponentially with fluid temperature.

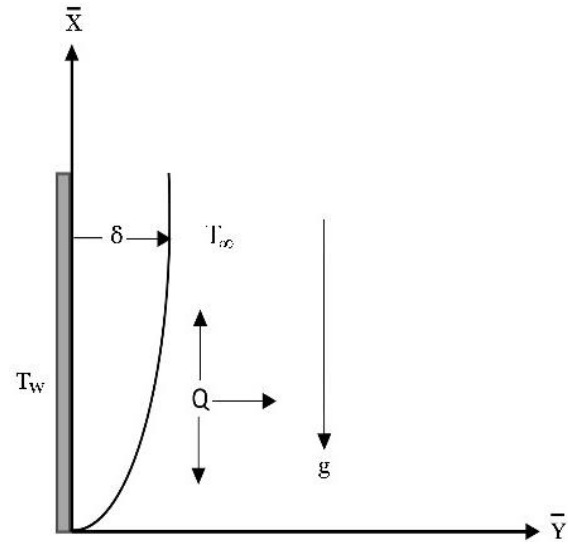


Figure 1. Geometry.

The mathematical statement of the basic conservation laws of mass, momentum and energy for the steady viscous incompressible and electrically conducting flow, after simplifying we have

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \quad (1)$$

$$\frac{\partial \bar{u}}{\partial t'} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = \frac{1}{\rho} \frac{\partial}{\partial \bar{y}} \left(\mu \frac{\partial \bar{u}}{\partial \bar{y}} \right) + g\beta(T' - T_\infty) \quad (2)$$

$$\frac{\partial T'}{\partial t'} + \bar{u} \frac{\partial T'}{\partial \bar{x}} + \bar{v} \frac{\partial T'}{\partial \bar{y}} = \frac{1}{\rho C_p} \frac{\partial}{\partial \bar{y}} \left(\kappa \frac{\partial T'}{\partial \bar{y}} \right) + \frac{Q_0}{\rho C_p} (T' - T_\infty) + \frac{\mu}{\rho C_p} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 + \frac{\sigma \beta_0^2}{\rho C_p} (\bar{u})^2 \quad (3)$$

Where, $\bar{u}$ and $\bar{v}$ are the velocity components along with the $\bar{x}$ and $\bar{y}$ axis respectively, t' is the time, T' is the temperature of the fluid in the boundary layer and T_∞ is the fluid temperature far away from the plate, g is the acceleration due to gravity, κ is the thermal conductivity of the fluid, ρ is the density, C_p is the specific heat at con-

stant pressure and μ is the variable dynamic co-efficient of viscosity of the fluid. The amount of heat generated or absorbed per unit volume is $Q_0(T - T_\infty)$, Q_0 being a constant, which may take either positive or negative and the hydrostatic pressure $\frac{\partial P}{\partial \bar{x}} = -\rho_e g$ where, $\rho_e = \rho$. The source term represents the heat formation when $Q_0 > 0$ and the heat ab-

sorption when $Q_0 < 0$. $k(T)$ is the thermal conductivity of the fluid depending on the fluid temperature T' , σ_0 is the electric conduction and β_0 is the magnetic field strength.

The initial and boundary conditions are

$$\begin{aligned} t' \leq 0: \bar{u} = 0, \bar{v} = 0, T' = T'_\infty \text{ for all } y \\ t' > 0: \bar{u} = 0, \bar{v} = 0, T' = T'_w \text{ at } y = 0 \\ t' > 0: \bar{u} = 0, T' = T'_\infty \text{ at } x = 0 \\ t' > 0: \bar{u} = 0, T' \rightarrow T'_\infty \text{ as } y \rightarrow \infty \end{aligned} \quad (4)$$

On introducing the following non-dimensional quantities in equations (1) to (4)

$$\begin{aligned} x = \frac{\bar{x}}{l}, y = \frac{\bar{y}}{l} Gr^{\frac{1}{4}}, u = \frac{l\bar{u}}{\nu} Gr^{\frac{-1}{2}}, v = \frac{\bar{v}l}{\nu} Gr^{\frac{-1}{4}}, \\ t = \frac{\nu t'}{l^2} Gr^{\frac{1}{2}}, T = \frac{T' - T'_\infty}{T'_w - T'_\infty}, Gr = \frac{g\beta l^3 (T'_w - T'_\infty)}{\nu^2}, \\ p_r = \frac{\mu_0 C_p}{\kappa_0}, \nu = \frac{\mu_0}{\rho} \end{aligned} \quad (5)$$

In this case, l stands for plate length, kinematic viscosity, Gr for Grashof number, and Pr for Prandtl number. The literature has a variety of viscosity and thermal conductivity variations with dimensionless temperature. Statterry [16], Ockendon and Ockendon [17], El-bashbeshy and Ibrahi [18], and Seddeek and Abdelmeguid [19] have all proposed the following types.

$$\frac{\mu}{\mu_0} = e^{-\lambda T} \quad (6)$$

$$\frac{\kappa}{\kappa_0} = 1 + \gamma T \quad (7)$$

Where λ and γ denote the viscosity and thermal conductivity variation parameters respectively, depended on the nature of the fluid. Here μ_0 and k_0 are the viscosity and the thermal conductivity at temperature T'_w .

The equation of continuity is

$$\begin{aligned} \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} &= 0 \\ \Rightarrow \frac{\partial \left(\frac{\nu}{l} Gr^{\frac{1}{2}} u \right)}{\partial (xl)} + \frac{\partial \left(\frac{\nu}{l} Gr^{\frac{1}{4}} v \right)}{\partial \left(\frac{yl}{Gr^{\frac{1}{4}}} \right)} &= 0 \\ \Rightarrow \frac{\nu}{l} Gr^{\frac{1}{2}} \frac{\partial u}{\partial x} + \frac{\nu}{l} Gr^{\frac{1}{4}} \frac{\partial v}{\partial y} &= 0 \\ \therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned} \quad (8)$$

Now momentum equation (2) can be reduced by applying the non-dimensional transformation (5) and (6), we have

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t'} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} &= \frac{1}{\rho} \frac{\partial}{\partial \bar{y}} \left(\mu \frac{\partial \bar{u}}{\partial \bar{y}} \right) + g\beta (T' - T'_\infty) \\ \Rightarrow \frac{Gr\nu^2}{l^3} \frac{\partial u}{\partial t} + \frac{Gr\nu^2}{l^3} u \frac{\partial u}{\partial x} + \frac{Gr\nu^2}{l^3} v \frac{\partial u}{\partial y} &= \frac{1}{\rho} \frac{\nu Gr \mu_0}{l^3} \frac{\partial}{\partial y} \left(e^{-\lambda T} \frac{\partial u}{\partial y} \right) + g\beta (T'_w - T'_\infty) \\ \Rightarrow \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \left[e^{-\lambda T} \frac{\partial^2 u}{\partial y^2} - \lambda e^{-\lambda T} \frac{\partial T}{\partial y} \frac{\partial u}{\partial y} \right] + T \end{aligned} \quad (9)$$

Again, the energy equation (3) can be reduced by the above similarity transformation (5) and (7), we have

$$\begin{aligned} \Rightarrow \frac{\partial T'}{\partial t'} + \bar{u} \frac{\partial T'}{\partial \bar{x}} + \bar{v} \frac{\partial T'}{\partial \bar{y}} &= \frac{1}{\rho C_p} \frac{\partial}{\partial \bar{y}} \left(\kappa \frac{\partial T'}{\partial \bar{y}} \right) + \frac{Q_0}{\rho C_p} (T' - T'_\infty) + \frac{\mu}{\rho C_p} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 + \frac{\sigma \beta_0^2}{\rho C_p} (\bar{u})^2 \\ \frac{\partial \{T'_\infty + T(T'_w - T'_\infty)\}}{\partial \left(\frac{tl^2}{\nu Gr^{\frac{1}{2}}} \right)} + \frac{\nu u}{l} Gr^{\frac{1}{2}} \frac{\partial \{T'_\infty + T(T'_w - T'_\infty)\}}{\partial (xl)} &+ \frac{\nu v}{l} Gr^{\frac{1}{4}} \frac{\partial \{T'_\infty + T(T'_w - T'_\infty)\}}{\partial \left(\frac{yl}{Gr^{\frac{1}{4}}} \right)} \end{aligned}$$

$$\begin{aligned}
& \Rightarrow \frac{1}{\rho C_p} \frac{\partial}{\partial \left(\frac{yl}{Gr^{\frac{1}{2}}} \right)} \left(\kappa \frac{\partial \{T'_w + T(T'_w - T'_\infty)\}}{\partial \left(\frac{yl}{Gr^{\frac{1}{4}}} \right)} \right) + \frac{Q_0}{\rho C_p} (T' - T'_\infty) + \frac{\mu}{\rho C_p} \left(\frac{\partial \left(\frac{\nu u Gr^{\frac{1}{2}}}{l} \right)}{\partial \left(\frac{yl}{Gr^{\frac{1}{2}}} \right)} \right)^2 \\
& + \frac{\sigma \beta_0^2}{\rho C_p} \left(\frac{u^2 v^2}{l^2} Gr \right) \frac{\nu Gr^{\frac{1}{2}} (T'_w - T'_\infty)}{l^2} \frac{\partial T}{\partial t} + \frac{\nu Gr^{\frac{1}{2}} (T'_w - T'_\infty)}{l^2} u \frac{\partial T}{\partial x} + \frac{\nu Gr^{\frac{1}{2}} (T'_w - T'_\infty)}{l^2} v \frac{\partial T}{\partial y} \\
& = \frac{Gr^{\frac{1}{2}} (T'_w - T'_\infty)}{l^2 \rho C_p} \frac{\partial}{\partial y} \left\{ \kappa_0 (1 + \gamma T) \frac{\partial T}{\partial y} \right\} + \frac{Q_0}{\rho C_p} (T' - T'_\infty) + \frac{\mu}{\rho C_p} \frac{\nu^2 Gr Gr^{\frac{1}{2}} (T'_w - T'_\infty)}{l^2 l^2} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\sigma \beta_0^2 \nu^2 Gr}{\rho C_p l^2} u^2 \\
& \Rightarrow \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\kappa_0}{\mu_0 C_p} \left\{ (1 + \gamma T) \frac{\partial^2 T}{\partial y^2} + \gamma \left(\frac{\partial T}{\partial y} \right)^2 \right\} + \frac{Q_0 l^2 (T' - T'_\infty)}{\rho C_p \nu Gr^{\frac{1}{2}} (T'_w - T')} + \\
& \frac{\mu \nu^2 Gr^{\frac{3}{2}}}{\rho C_p l^4} \left(\frac{\partial u}{\partial y} \right)^2 \frac{l^2}{\nu Gr^{\frac{1}{2}} (T'_w - T'_\infty)} + \frac{\sigma \beta_0^2 \nu^2 Gr}{\rho C_p l^2} u^2 \frac{l^2}{\nu Gr^{\frac{1}{2}} (T'_w - T'_\infty)} \\
& \Rightarrow \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{P_r} \left\{ (1 + \gamma T) \frac{\partial^2 T}{\partial y^2} + \gamma \left(\frac{\partial T}{\partial y} \right)^2 \right\} + \frac{Q_0 l^2 (T' - T'_\infty)}{\rho C_p \nu Gr^{\frac{1}{2}} (T'_w - T')} + \frac{\mu \nu Gr}{\rho C_p l^2 (T'_w - T'_\infty)} \left(\frac{\partial u}{\partial y} \right)^2 \\
& + \frac{\sigma \beta_0^2 \nu^2 Gr}{\rho C_p \nu Gr^{\frac{1}{2}} (T'_w - T'_\infty)} u^2 \\
& \Rightarrow \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{P_r} \left[\gamma \left(\frac{\partial T}{\partial u} \right)^2 \right] + \frac{1}{P_r} \left[(1 + \gamma T) \frac{\partial^2 T}{\partial y^2} \right] + QT + \frac{\mu \nu g \beta l^3}{\rho C_p l^2 (T'_w - T'_\infty) \nu^2} (T'_w - T'_\infty) \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\sigma \beta_0^2 \nu Gr^{\frac{1}{2}}}{\rho C_p (T'_w - T'_\infty)} u^2 \\
& \Rightarrow \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{P_r} \left[\gamma \left(\frac{\partial T}{\partial y} \right)^2 \right] + \frac{1}{P_r} \left[(1 + \gamma T) \frac{\partial^2 T}{\partial y^2} \right] + QT + \frac{\mu g \beta l}{\rho C_p \nu} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\sigma \beta_0^2 \nu Gr^{\frac{1}{2}}}{\rho C_p (T'_w - T'_\infty)} u^2 \\
& \Rightarrow \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{P_r} \left[\gamma \left(\frac{\partial T}{\partial u} \right)^2 \right] + \frac{1}{P_r} \left[(1 + \gamma T) \frac{\partial^2 T}{\partial y^2} \right] + QT + \frac{\mu g \beta l}{\rho C_p \frac{\mu}{\rho}} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\sigma \beta_0^2 \nu Gr^{\frac{1}{2}}}{\rho C_p (T'_w - T'_\infty)} u^2 \\
& \Rightarrow \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{P_r} \left[\gamma \left(\frac{\partial T}{\partial u} \right)^2 \right] + \frac{1}{P_r} \left[(1 + \gamma T) \frac{\partial^2 T}{\partial y^2} \right] + QT + \frac{g \beta l}{C_p} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\sigma \beta_0^2 \nu Gr^{\frac{1}{2}}}{\rho C_p (T'_w - T'_\infty)} u^2 \\
& \Rightarrow \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{P_r} \left[\gamma \left(\frac{\partial T}{\partial u} \right)^2 \right] + \frac{1}{P_r} \left[(1 + \gamma T) \frac{\partial^2 T}{\partial y^2} \right] + QT + N \left(\frac{\partial u}{\partial y} \right)^2 + Jul(u^2) \quad (10)
\end{aligned}$$

The corresponding initial conditions and boundary conditions in a dimensionless form are as follows

$$t \leq 0 : u = 0, v = 0, T = 0, \text{ for all } y$$

$$t > 0 : u = 0, v = 0, T = 1, \text{ at } y = 0$$

$$u = 0, T = 0 \text{ at } x = 0$$

$$u \rightarrow 0, T \rightarrow 0 \text{ at } y \rightarrow \infty \quad (11)$$

The free convective unstable laminar boundary layer flow with variable viscosity and thermal conductivity and an isothermal semi-infinite vertical plate is described by equations (8) to (10) with the boundary condition (11).

Where, $p_r = \mu_0 C_p / \kappa_0$, the Prandtl's number, $Q_0 l^2 / \mu C_p Gr^{1/2} = Q$ is the heat formation parameter $Ju_l = \frac{\sigma \beta_0^2 \nu Gr^{1/2}}{\rho C_p (T'_w - T'_\infty)}$, is the joule heating parameter and $N = g \beta l / C_p$ is viscous dissipation parameter.

The local shear stress in the plate is defined by

$$\tau_{\bar{x}} = \left(\mu \frac{\partial \bar{u}}{\partial \bar{y}} \right)_{\bar{y}=0} \quad (12)$$

The non-dimensional form of local skin friction, which is obtained by introducing the non-dimensional quantities found in equations (5)–(6) in (12), is provided by

$$\begin{aligned} \tau_{\bar{x}} &= \left(\mu \frac{\partial \bar{u}}{\partial \bar{y}} \right)_{\bar{y}=0} \Rightarrow \frac{l^2}{\nu \mu_0} \tau_{(xl)} = Gr^{3/4} e^{-\lambda} \left[\frac{\partial u}{\partial y} \right]_{y=0} \\ \therefore \bar{\tau}_x &= Gr^{3/4} e^{-\lambda} \left[\frac{\partial u}{\partial y} \right]_{y=0} \end{aligned} \quad (13)$$

The integration of equation (13) from $x = 0$ to gives $x = 1$ the average skin friction and it is given by

$$\therefore \bar{\tau} = e^{-\lambda} Gr^{3/4} \int_0^1 \left(\frac{\partial u}{\partial y} \right)_{y=0} dx \quad (14)$$

The local Nusselt number is defined by

$$\begin{aligned} N_{u_{\bar{x}}} &= \frac{-l \left(\kappa \frac{\partial T'}{\partial \bar{y}} \right)_{\bar{y}=0}}{\kappa_0 (T'_w - T'_\infty)} \\ \Rightarrow N_{u_{(xl)}} &= -l \left[\frac{\kappa_0 (1 + \gamma T)}{\kappa_0 (T'_w - T'_\infty) l Gr^{1/4}} \left(\frac{\partial T}{\partial y} \right)_{y=0} \right] \\ \Rightarrow Gr^{-1/4} (T'_w - T'_\infty) N_{u_{(xl)}} &= -(1 + \gamma) \left(\frac{\partial T}{\partial y} \right)_{y=0} \\ \therefore y = 0, T = 1 \\ \Rightarrow \bar{N}_{u_x} &= -(1 + \gamma) \left(\frac{\partial T}{\partial y} \right)_{x=0} \\ \therefore \bar{N}_{u_x} &= -(1 + \gamma) \left(\frac{\partial T}{\partial y} \right)_{y=0} \end{aligned}$$

$$\therefore \bar{N}_{u_x} = -(1+\gamma) \left(\frac{\partial T}{\partial y} \right)_{y=0} \quad (15)$$

The integration of equation (15) from $x=0$ to $x=1$ gives the average skin friction and it is given by

$$\therefore \bar{N}_{u_x} = -(1+\gamma) \int_0^1 \left(\frac{\partial T}{\partial y} \right)_{y=0} dx \quad (16)$$

3. Numerical Techniques

The two-dimensional, non-linear, unsteady and coupled partial differential equations (8), (10) under the initial and boundary conditions in equation (11) are solved using an implicit finite difference scheme of Crank-Nicolson type which is the fast convergent and unconditionally stable. The finite difference equation corresponding to the equations (8) to (10) are given by

$$\frac{[u_{i,j}^{k+1} - u_{i-1,j}^{k+1} + u_{i,j}^k - u_{i-1,j}^k + u_{i,j-1}^{k+1} - u_{i-1,j-1}^{k+1} + u_{i,j-1}^k - u_{i-1,j-1}^k]}{4\Delta x} + \frac{[v_{i,j}^{k+1} - v_{i,j-1}^{k+1} + v_{i,j}^k - v_{i,j-1}^k]}{2\Delta y} = 0 \quad (17)$$

$$\begin{aligned} & \frac{[u_{i,j}^{k+1} - u_{i,j}^{k+1}]}{2\Delta t} + u_{i,j}^k \frac{[u_{i,j}^{k+1} - u_{i-1,j}^{k+1} + u_{i,j}^k - u_{i-1,j}^k]}{2\Delta x} v_{i,j}^k \frac{[u_{i,j+1}^{k+1} - u_{i,j-1}^{k+1} + u_{i,j+1}^k - u_{i,j-1}^k]}{4\Delta y} \\ &= \frac{1}{2} [T_{i,j}^{k+1} + T_{i,j}^k] + e^{-\lambda} \left[\frac{T_{i,j}^{k+1} + T_{i,j}^k}{2} \right] \frac{[u_{i,j-1}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j+1}^k + u_{i,j-1}^k - 2u_{i,j}^k + u_{i,j+1}^k]}{2(\Delta y)^2} \\ & - \lambda e^{-\lambda} \left[\frac{T_{i,j}^{k+1} + T_{i,j}^k}{2} \right] \frac{[T_{i,j+1}^{k+1} - T_{i,j-1}^{k+1} + T_{i,j+1}^k - T_{i,j-1}^k]}{4\Delta y} \frac{[u_{i,j+1}^{k+1} - u_{i,j-1}^{k+1} + u_{i,j+1}^k - u_{i,j-1}^k]}{4\Delta y} \left[\frac{T_{i,j}^{k+1} - T_{i,j}^k}{2\Delta t} \right] \end{aligned} \quad (18)$$

$$\begin{aligned} & + u_{i,j}^k \frac{[T_{i,j}^{k+1} - T_{i-1,j}^{k+1} + T_{i,j}^k - T_{i-1,j}^k]}{4\Delta x} + v_{i,j}^k \frac{[T_{i,j-1}^{k+1} - T_{i,j+1}^{k+1} + T_{i,j-1}^k - T_{i,j+1}^k]}{4\Delta y} \\ &= \frac{1+\gamma T_{i,j}^k}{p_r} \frac{[T_{i,j-1}^{k+1} - 2T_{i,j}^{k+1} + T_{i,j+1}^k + T_{i,j-1}^k - 2T_{i,j}^k + T_{i,j+1}^k]}{2(\Delta y)^2} \frac{\gamma}{p_r} \left[\frac{[T_{i,j+1}^{k+1} - T_{i,j-1}^{k+1} + T_{i,j+1}^k - T_{i,j-1}^k]}{4\Delta y} \right] \\ & + \frac{Q}{2} [T_{i,j}^{k+1} + T_{i,j}^k] + Jul \left\{ \frac{1}{2} [u_{i,j}^{k+1} + u_{i,j}^k] \right\}^2 + N \left\{ \frac{[u_{i,j+1}^{k+1} - u_{i,j-1}^{k+1} + u_{i,j+1}^k - u_{i,j-1}^k]}{4\Delta y} \right\}^2 \end{aligned} \quad (19)$$

The region of integration is considerably outside the momentum and energy boundary layers and is represented by a rectangle with sides $x_{\max}$ (=1) and $y_{\max}$ (=10), where $y_{\max}$ corresponds to y approaches to ∞ . After some initial research, the maximum of y was determined to be (6) in order to satisfy the final two boundary conditions (11). In this case, the grid point along the u -direction is indicated by the subscript i , the v -direction by j , and the t -direction by the superscript k . The coefficients $u_{i,j}^k$ and $v_{i,j}^k$ that appear in the difference equations are treated as constants throughout any one-time step.

Because of the beginning conditions, we know the values of u , v , and T at every grid point at $t = 0$. The following is how

the data from the previous time level (fe) are used to calculate u , v , and T at time level $(k + 1)$: The tridiagonal system of equations is made up of the finite difference equation (18) at each internal nodal point on a specific i -level. According to Carnahan et al. [19], the Thomas method solves such a system of equations. For a given i at the $(k + 1)$ th time level, the values of T are therefore determined at each nodal point.

The values of u at $(k + 1)$ th time level are found similarly to the values of T at $(k + 1)$ th time level in eq. (13). Consequently, on a specific i -level, the values of T and u are known. Lastly, at each nodal point on a certain i -level at $(k + 1)$ th time level, the values of v are explicitly determined using the eq. (12). For different i -levels, this procedure is repeated. Thus, at the $(k + 1)$ th time level, the values of T , u , and v are known at

every grid point in the rectangular region.

They have been fixed at the level $\Delta x = 0.05$, $\Delta y = 0.25$, and time step $\Delta t = 0.01$ after a few sets of mesh sizes were taken into consideration. The results are compared after the spatial mesh size is reduced by 50% in one direction and subsequently in both directions. It has been noted that the results vary to the fourth decimal place when the mesh size is decreased by 50% in both the x and y directions. As a result, the sizes listed above have been deemed suitable for calculations.

Until the steady-state is achieved, calculations are made. When the absolute difference between the values of u and temperature T at two consecutive time steps is smaller than 10^{-5} at all grid points, the steady-state solution is said to have been reached.

The local truncation error is $O(\Delta t^2 + \Delta F^2 + \Delta Ax)$ and it $\rightarrow 0$ as Δt , Δx and $\Delta y \rightarrow 0$, which shows that the scheme is compatible. Additionally, it is demonstrated that the Crank-Nicolson type of implicit finite difference scheme is unconditionally stable for a natural convective flow, where the velocity u and v are always non-negative and non-positive, respectively. Therefore, the implicit finite difference scheme's convergence is guaranteed by compatibility and stability.

4. Results and Discussion

Heat conduction, sometimes referred to as diffusion, is the direct microscopic transfer of particle kinetic energy over a boundary between two systems. Water's high thermal capacity and low viscosity make it a great fluid for heat transmission. Oil has been a common substitute for water because it has a higher liquid temperature, which helps to avoid the issue of high pressure. Convection, radiation, and conduction are the three ways that heat moves from the Earth's surface to the atmosphere.

Convection is the process by which heat is transported when a heated fluid, like water or air, is compelled to move away from the heat source while carrying energy with it. When hot air expands, loses density, and rises above a heated surface, convection occurs. Even in a fully developed turbulent flow, heat transfer through molecular thermal conduction is significant in the flow core as well as the near-wall layer due to the low Prandtl's number of liquid metals.

The following ranges for λ , γ and Pr are considered in the present study are:

For air: $-0.7 \leq \lambda \leq 0$, $0 \leq \gamma \leq 6$, $Pr = 0.733$

For water: $0 \leq \lambda \leq 0.6$, $0 \leq \gamma \leq 0.12$, $2 \leq Pr \leq 7.00$

To assess the precision of our calculated values, we plot the curves calculated by G. palani, Kwang-Yong Kim, and Elbashbeshy & Ibrahim for different values of λ and γ for air ($Pr = 0.733$) in "Figure 2" and "Figure 3". Our results show excellent agreement with those of G. palani, Kwang-Yong Kim, and Elbashbeshy & Ibrahim at the steady state.

The body force hasn't had enough time to create the proper

motion in the fluid during the first phase of the subsequent step changes in the wall temperature. For short times t, the velocity components u and v are therefore insignificant. For constant viscosity and thermal conductivity, pure heat conduction dominates the heat transfer throughout this first transient period. The result of equation (10) is

$$\frac{\partial u}{\partial T} = \frac{1}{Pr} \frac{\partial^2 T}{\partial y^2}$$

For brief periods, it is seen that the temperature profile depends solely on time and the normal distance from the wall for a given Prandtl's number. Under the beginning and boundary conditions specified in the local Nusselt number, the solutions of equation (15) with $Pr = 1$ are

$$T = \text{erfc}\left(\frac{y}{2\sqrt{t}}\right) \quad (20)$$

"Figures 4 to 17" show that the variation of velocity and temperature at their transient, temporal maximum and steady state against the co-ordinate y at the leading edge of the plate viz., $x = 1.0$ for variable viscosity, thermal conductivity, heat conduction variation parameters, pressure work parameters and Prandtl's numbers. The fluid velocity increases and reached its maximum value at very near to the wall (i.e., $0 \leq y \leq 8$) and then decreases monotonically to zero as y becomes large for all time t. It is also observed that the velocity and temperature increases with time t, reaches a temporal maximum and consequently it reaches the steady state.

"Figure 4" and "Figure 5" show that the variation of transient velocity and temperature profiles with area A. for a fixed value of $\gamma = 0.10$, $Pr = 0.73$, $Q = 0.50$, $N = 0.75$, $Jul = 0.80$ From "Figure 4" the velocity of the fluid increases with time till a transient maximum is reached and then a moderate reduction is mentioned till the alternate balanced condition is reached. It is mentioned that the time taken to go to the steady condition reduces hardly at all with a raising the viscosity variation parameter. From "Figure 4" it is obvious that velocity u at any vertical plane near to the plate increases as variable viscosity λ decrease. But after certain time the velocity profile twisted an opposite trend. And finally it meets with y axis asymptotically. From "Figure 5" it is mentioned that the temperature of the fluid reduces as λ raises. It is also related for different time.

The numerical values of the variation of transient velocity and temperature for the fixed value of $\lambda = 0.30$, $Q = 0.50$, $N = 0.75$, $Pr = 0.73$ and $Jul = 0.80$ with the variation of thermal conductivity parameter γ are shown graphically in "Figure 6" and "Figure 7" from these figures, it is observed that the velocity and temperature distribution in the fluid increases as γ increases for fixed value of λ , Q , N , Jul heating and Prandtl's number. It can also be noticed that with an increase in γ , the rise in the magnitude of the velocity and temperature is sig-

nificant, which implies that the volume flow rate increases with an increase in γ . The effects of variation of thermal conductivity on velocity and temperature is more important even in the initial transient timing. Also, it is observed that the time to reach the temporal maximum and steady state decreases with increasing thermal conductivity parameter γ .

The numerical values of variation velocity and temperatures are computed from eqs. (13) and (14) are depicted in the graphical form in the “Figure 8” and “Figure 9” for different values of N for fixed value of $Q = 0.50$, $Jul = 0.80$, $\gamma = 0.10$ in water ($Pr = 0.70$) and $\lambda = 0.30$. It is clearly mentioned that the taken time to reach the temporal extreme and balanced state reduces with decreasing the values of Jul . It can be represented from “Figure 8” it is obvious that the velocity u at any vertical plane near to the plate increases as parameter viscous dissipation increases. But after a certain time the velocity profile twisted an opposite trend. And finally it meets asymptotically.

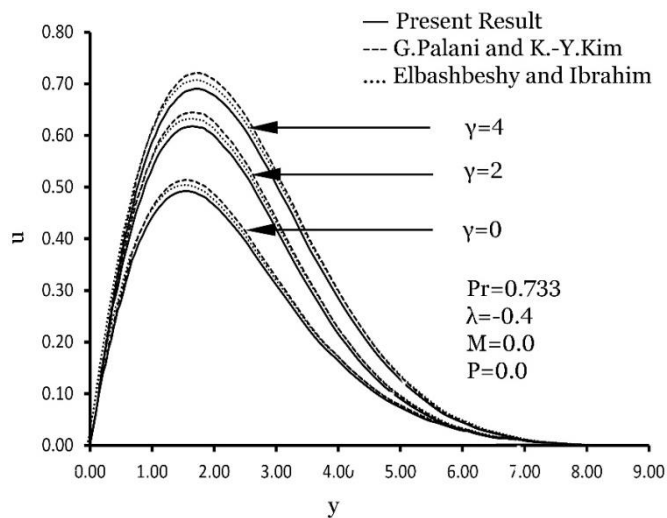


Figure 2. Comparison of Velocity profiles G. Palani and K.-Y. Kim. Elbashbeshy and Ibrahim for various values.

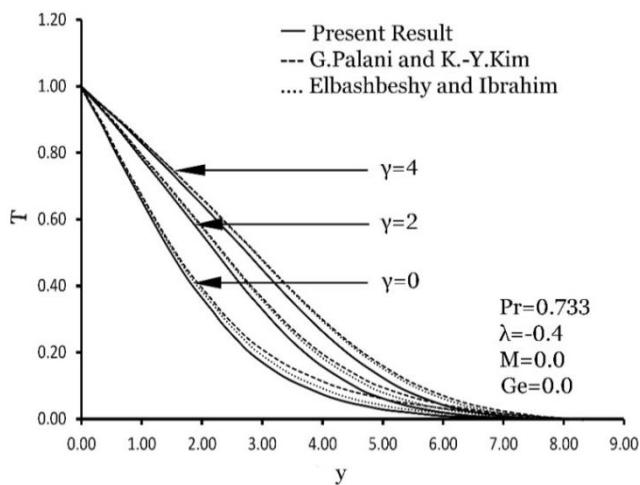


Figure 3. Comparison of Temperature profiles G. Palani and K.-Y. Kim. Elbashbeshy and Ibrahim for various values.

Figure 2 and Figure 3 Comparison of velocity profiles and temperature profiles G. Palani and K.-Y. Kim. Elbashbeshy and Ibrahim for various values of dependent thermal conductivity with fixed values.

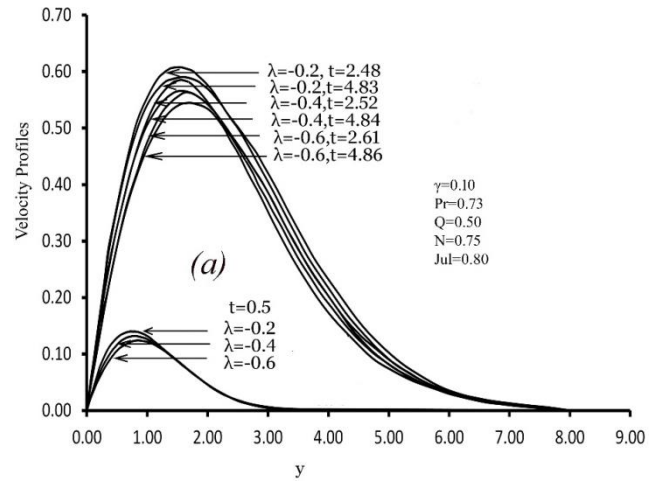


Figure 4. Variation of dimensionless Velocity profiles versus dimensionless y for various values of viscosity λ and steady state condition with $Q=0.5$, $N=0.75$, $Jul=0.8$, $\gamma=0.1$ and $Pr=0.73$.

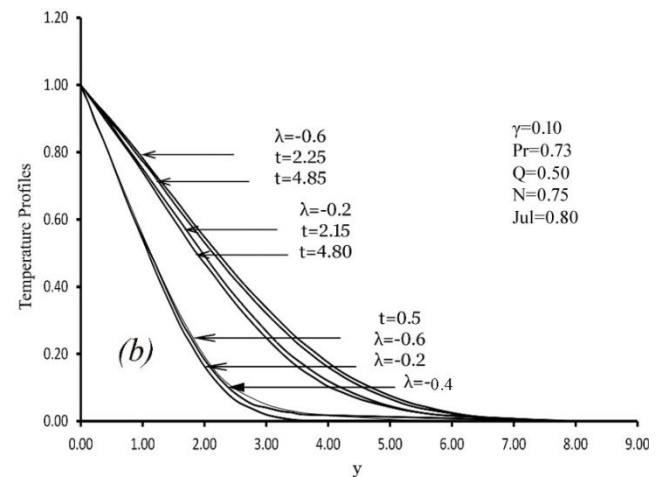


Figure 5. Variation of dimensionless Temperature profiles versus dimensionless y for various values of viscosity λ and steady state condition with $Q=0.5$, $N=0.75$, $Jul=0.8$, $\gamma=0.1$ and $Pr=0.73$.

Figure 4 and Figure 5 Variation of dimensionless velocity and temperature profiles versus dimensionless y for various values of viscosity λ and steady state condition with $Q=0.5$, $N=0.75$, $Jul=0.8$, $\gamma=0.1$ and $Pr=0.73$.

Figure 6 and Figure 7 Variation of dimensionless velocity and temperature profiles versus dimensionless y for various values of thermal conductivity γ and steady state condition with $Q=0.5$, $N=0.75$, $Jul=0.8$, $\lambda=0.30$ and $Pr=0.73$.

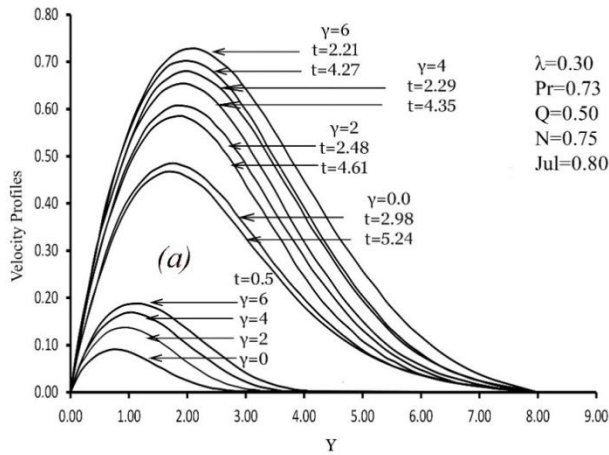


Figure 6. Variation of dimensionless Velocity profiles versus dimensionless y for various values of thermal conductivity γ and steady state condition with $Q=0.5$, $N=0.75$, $Jul=0.8$, $\lambda = 0.30$ and $Pr=0.73$.

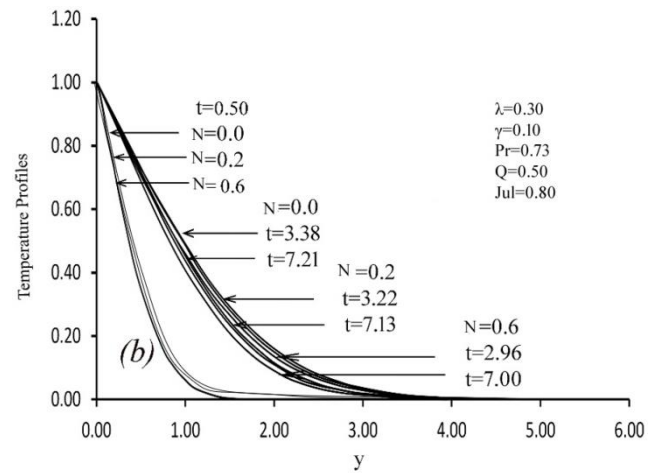


Figure 9. Variation of dimensionless velocity and temperature profiles versus dimensionless y for different values of viscous dissipation N and steady state condition with $Q=0.5$, $\gamma = 0.10$, $Jul=0.8$, $\lambda = 0.30$ and $Pr=0.73$.

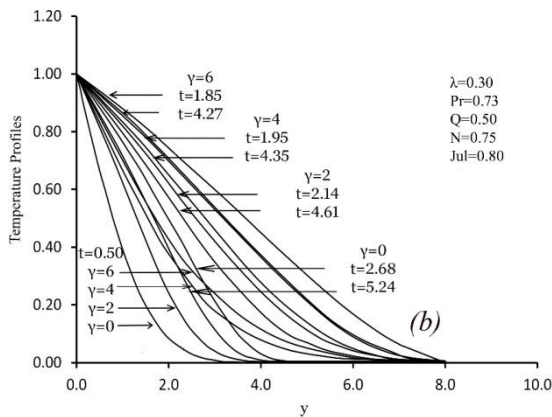


Figure 7. Variation of dimensionless velocity and temperature profiles versus dimensionless y for various values of thermal conductivity γ and steady state condition with $Q=0.5$, $N=0.75$, $Jul=0.8$, $\lambda = 0.30$ and $Pr=0.73$.

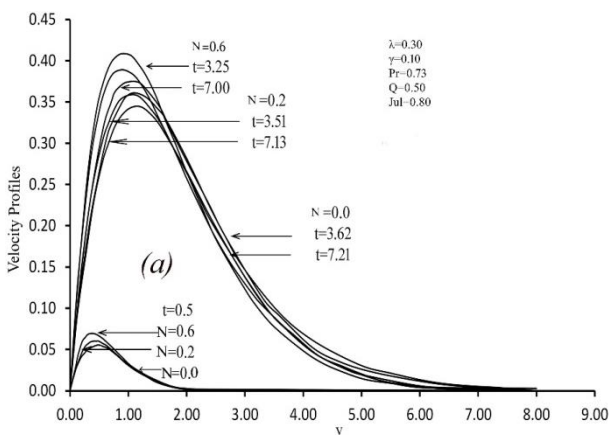


Figure 8. Variation of dimensionless velocity and temperature profiles versus dimensionless y for different values of viscous dissipation N and steady state condition with $Q=0.5$, $\gamma = 0.10$, $Jul=0.8$, $\lambda = 0.30$ and $Pr=0.73$.

Figure 8 and Figure 9 Variation of dimensionless velocity and temperature profiles versus dimensionless y for different values of viscous dissipation N and steady state condition with $Q=0.5$, $\gamma = 0.10$, $Jul=0.8$, $\lambda = 0.30$ and $Pr=0.73$.

Figure 10 and Figure 11 Variation of dimensionless velocity and temperature profiles versus dimensionless y for different values of heat generation Q and steady state condition with $N=0.75$, $\gamma = 0.10$, $Jul=0.8$, $\lambda = 0.30$ and $Pr=0.73$.

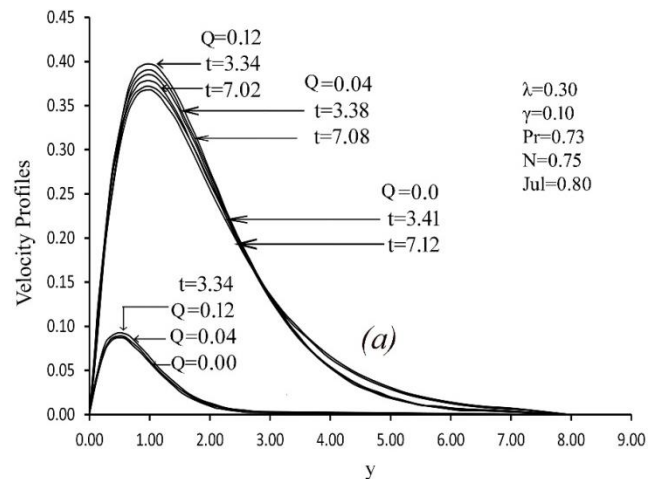


Figure 10. Variation of dimensionless Velocity profiles versus dimensionless y for different values of heat generation Q and steady state condition with $N=0.75$, $\gamma = 0.10$, $Jul=0.8$, $\lambda = 0.30$ and $Pr=0.73$.

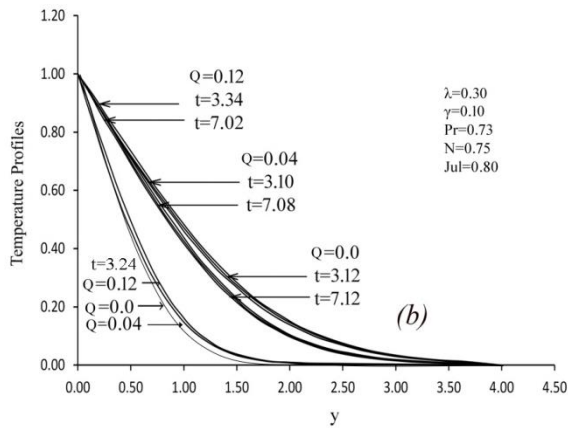


Figure 11. Variation of dimensionless velocity and temperature profiles versus dimensionless y for different values of heat generation Q and steady state condition with $N=0.75$, $\gamma=0.10$, $Jul=0.8$, $\lambda=0.30$ and $Pr=0.73$.

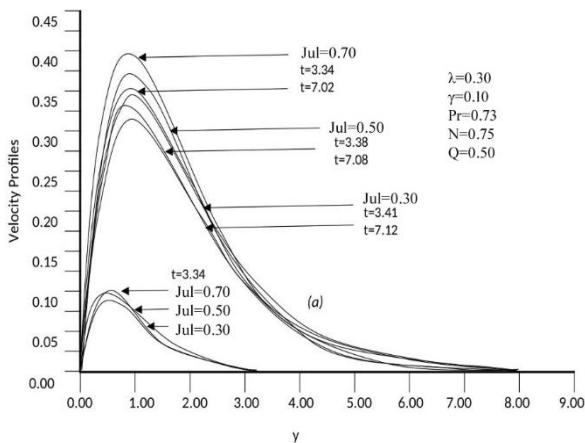


Figure 12. Variation of dimensionless Velocity profiles versus dimensionless y for different values of Joule and steady state condition with $N=0.75$, $\gamma=0.10$, $Q=0.50$, $\lambda=0.30$ and $Pr=0.73$.

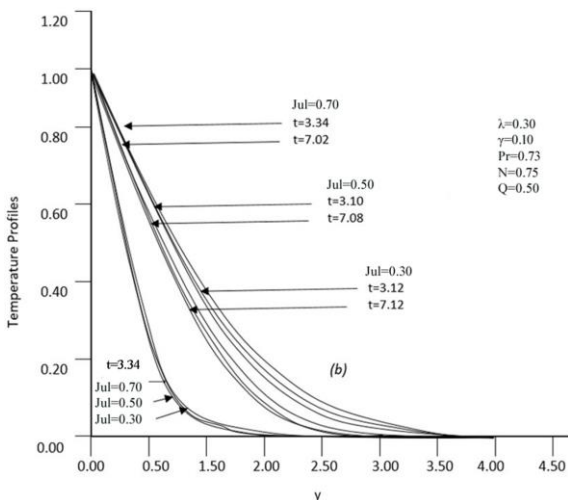


Figure 13. Variation of dimensionless Temperature profiles versus dimensionless y for different values of Joule and steady state condition with $N=0.75$, $\gamma=0.10$, $Q=0.50$, $\lambda=0.30$ and $Pr=0.73$.

Figure 12 and Figure 13 Variation of dimensionless velocity and temperature profiles versus dimensionless y for different values of Joule and steady state condition with $N=0.75$, $\gamma=0.10$, $Q=0.50$, $\lambda=0.30$ and $Pr=0.73$.

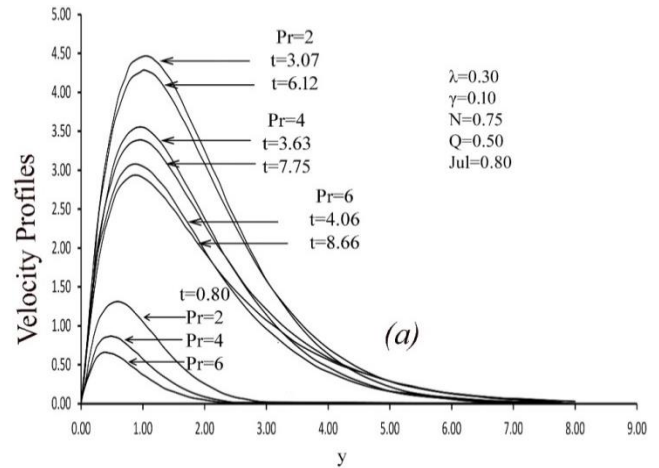


Figure 14. Variation of dimensionless Velocity profiles against dimensionless y for different values of prandtl's number Pr and steady state condition with $N=0.75$, $\gamma=0.10$, $Q=0.5$, $\lambda=0.30$ and $Jul=0.80$.

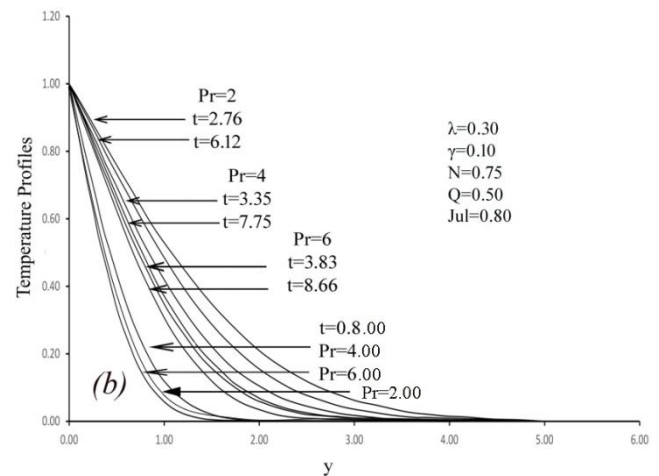


Figure 15. Variation of dimensionless Temperature profiles against dimensionless y for different values of prandtl's number Pr and steady state condition with $N=0.75$, $\gamma=0.10$, $Q=0.5$, $\lambda=0.30$ and $Jul=0.80$.

Figure 14 and Figure 15 Variation of dimensionless velocity and temperature profiles against dimensionless y for different values of prandtl's number Pr and steady state condition with $N=0.75$, $\gamma=0.10$, $Q=0.5$, $\lambda=0.30$ and $Jul=0.80$.

Figure 16 and Figure 17 Variation of dimensionless local skin friction and local Nusselt number versus dimensionless distance x for different values of Q , λ , γ , N , Jul and Pr at steady state condition.

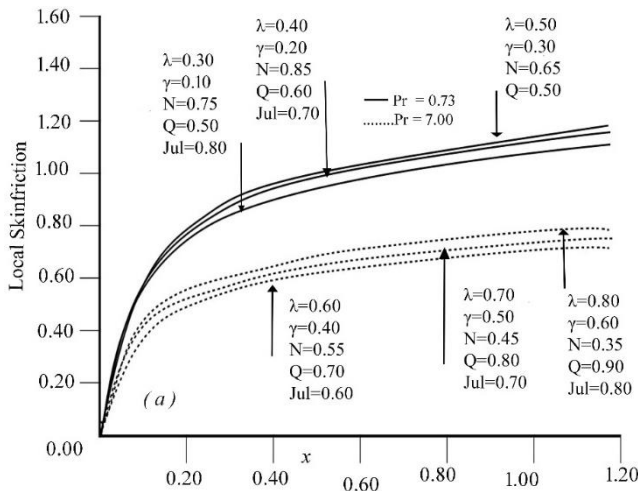


Figure 16. Variation of dimensionless Local Skin Friction versus dimensionless distance x for different values of Q , λ , γ , N , Jul and Pr at steady state condition.

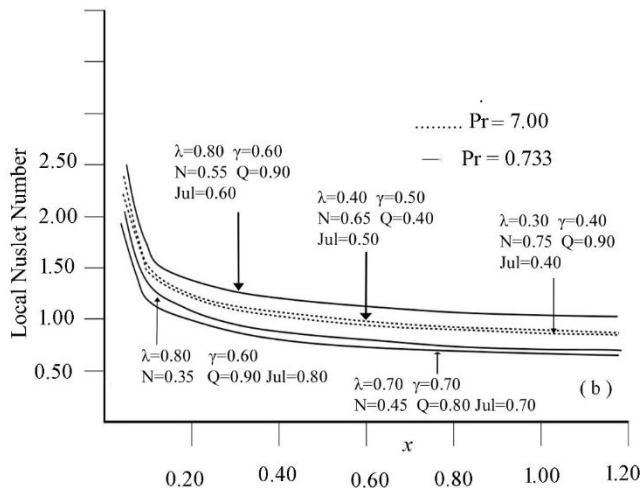


Figure 17. Variation of dimensionless Local Nusselt number versus dimensionless distance x for different values of Q , λ , γ , N , Jul and Pr at steady state condition.

From “Figure 9”, it is mentioned that the temperature of the fluid decreases as the parameter of viscous dissipation increases. It is also related for different time.

“Figure 10” and “Figure 11” show that the variation of velocity and temperature for various values of heat generation Q for fixed value of $\lambda = 0.30$, $\gamma = 0.10$, $N = 0.75$, $Jul = 0.80$ in air ($Pr = 0.73$). It is clearly mentioned that the taken time to reach the temporal extreme and balanced state reduces with decreasing the values of heat generation Q . It can be represented from “Figure 10” it is obvious that the velocity u at any vertical plane near to the plate increases as parameter heat generation Q increases. But after a certain time the velocity profile twisted an opposite trend. And finally it meets asymptotically. From “Figure 11” it is mentioned that the temperature of the fluid decreases as the parameter heat generation Q increases. It is also related for different time.

The variation of transient velocity and temperature with Joule heating for fixed values $Pr = 0.73$, $\lambda = 0.30$, $Q = 0.50$, $N = 0.75$ and $\gamma = 0.10$ are shown in “Figure 12” and “Figure 13”. It is observed that the time taken to reach the temporal maximum and steady state increases with the increasing value of Joule heating parameter Jul . From the numerical results, we observe that the velocity profile increases with the increasing value of Joule heating parameter Jul .

The variation of transient velocity and temperature with Prandtl number for fixed values $Jul = 0.73$, $\lambda = 0.30$, $Q = 0.50$, $N = 0.75$ and $\gamma = 0.10$ are shown in “Figure 14” and “Figure 15”. It is observed that the time taken to reach the temporal maximum and steady state increases with the increasing value of Prandtl number parameter Pr of the fluid. From the numerical results, we observe that the velocity profile increases with the increasing value of Prandtl’s number parameter Pr .

The assessment of derivatives using a five-point approximation formula is included in eqs. (12), (14), (15), and (16), followed by the evaluation of integrals using the Newton-Cotes closed integration formula. The local skin-friction values are calculated using eq. (13) and plotted as a function of the axial coordinate λ as well as selected values of the variation parameters λ , γ , Q , N , and Jul in “Figure 16”. The local skin friction increases as the temperature rises. Local skin friction appears to diminish as the value of the viscous variation parameter λ increases. It’s also worth mentioning that the local wall shear stress increases as the value of the heat conductivity parameter γ rises.

“Figure 17” shows a dimensionless steady state local heat transfer rate for various values of variance parameters. As the viscosity, thermal conductivity properties rise, the rate of local heat transfer increases. As Pr rises, the rate of local heat transfer rises as well.

5. Conclusions

With a Joule heating parameter of Jul , this paper analyses the effects of varying viscosity and thermal conductivity on heat generation in a laminar natural convection boundary-layer vertical plate. The fluid viscosity is predicted to fluctuate as an exponential function of temperature, while the thermal conductivity is considered to be a linear function of temperature. Dimensionless governing equations are solved using an implicit Crank-Nicolson type finite difference approach. The current numerical results and previously published research are graphically compared. The two sides’ agreement is regarded as excellent. This analysis has revealed that:

- As the fluid temperature decreases and the viscosity parameter λ increases, the dimensionless fluid velocity increases. When the viscosity variation parameter λ is significant, a position close to the wall experiences greater velocity, which leads to a larger Nusselt number and less skin friction.

- ii. As the thermal conductivity parameter γ rises, so do the fluid temperature, fluid velocity, dimensionless wall velocity gradient, and dimensionless rate of heat transfer from the plate to the fluid.
- iii. It has been found that there would be major errors if the viscosity and thermal conductivity differences were ignored. Therefore, we propose that in order to predict more accurate results, the effects of altering viscosity and heat conductivity should be taken into consideration.
- iv. When the Joule heating parameter Jul is increased the velocity profiles are somewhat increased. Furthermore, when the Joule heating parameter rises, the temperature profile rises. When the Heat formation parameter Q is increased, the velocity and temperature profiles are significantly increased.
- v. The variation of heat formation parameter Q , Pressure work parameter, variable viscosity parameter and temperature dependent parameter, the local skin friction coefficient, the local Nusselt number and the velocity distribution over the whole boundary layer decreases, but the temperature distribution increases.
- vi. When the Heat formation parameter Q is increased, the velocity and temperature profiles are significantly increased.
- vii. The variation of heat formation parameter Q , Joule heating parameter Jul , variable viscosity parameter λ and temperature dependent parameter γ , the local skin friction coefficient, the local Nusselt number and the velocity distribution over the whole boundary layer decreases, but the temperature distribution increase.

Abbreviations

Md Mohammad

Conflicts of Interest

The authors declare no conflicts of interest.

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