

Research Article

Average Fidelity of Two Qubits Entanglement Teleportation Mediated Via a Single Majorana Wire

Ngwa Engelbert Afuoti^{1,*} , Lukong Cornelius Fai² , Nguenang Jean Pierre³ , Jules Carsimir Ngana⁴ , Ateuafack Mathurin⁵ , Georges Collince Fouokeng² 

¹Department of Energy and Thermal Engineering, University of Douala, Douala, Cameroon

²Department of Physics, University of Dschang, Dschang, Cameroon

³Department of Physics, University of Douala, Douala, Cameroon

⁴Department of Physics, University of Buea, Buea, Cameroon

⁵Department of Electrical and Electronic Engineering, University of Buea, Buea, Cameroon

Abstract

We revisit the teleportation of two qubits entanglement using a single channel made of a single Majorana wire connecting two pairs of qubits each at the sending and receiving ends of the wire. The scheme is different from that use in our early work (Phys. Scr. 95 035803 (2020)) where the two qubit teleportation required two copies of the Majorana wires setup as required to implement the Lee and Kim teleportation scheme. The two pairs of qubit interact via the exchange coupling. It is in this setting that we investigate the role of the exchange coupling, the channel parameters, the spin chain parameters on the average fidelity of entanglement teleportation. We equally consider the fact that the geographically located two pairs of qubits interact differently with their respective spin chain environment. The dynamics of the average fidelity is shown to be enhanced when using a set-up with symmetric coupling between qubits and Majorana wire at both ends as compared to when using asymmetric coupling. Small values of the coupling between the qubits and spin environments should however be used. Large values of the exchange coupling between qubits in the two sub-systems should be used to have high fidelity of the teleported state. The dynamics of the average fidelity of entanglement teleportation is seen to depend on the size of the spin environment as compared to the teleportation protocol used in our earlier work where the dynamics is mildly sensitive to the size of the spin environment. Increasing the strength of the transverse magnetic field and large values of the anisotropy degree enhances the average fidelity of entanglement teleportation. The same observation is seen when considering the same values of the coupling between the two qubits sub-systems at the opposite ends of the Majorana wire and their respective spin environment. The dynamics of the system is however seen to observe almost a similar trend as that of the above cited work for the single Majorana wire setup.

Keywords

Spin Chain Environment, Qubits, Single Majorana Wire, Entanglement Teleportation, Average Fidelity

*Corresponding author: ngwaawah@yahoo.com (Ngwa Engelbert Afuoti)

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1. Introduction

This work reconsiders the two qubits entanglement teleportation via the Majorana bound states using the Lee and Kim teleportation scheme, where two copies of the setup was used to carry out the teleportation protocol [1]. This was to overcome the challenges faced by quantum information processing engineers using spin chain to route information between quantum registers and processors such as decoherence arising from quantum fluctuations due to quasi-particle excitations within the chain, Dzyaloshinsky-Moriya interactions, chain length, external degrees of freedom or finite precision control changes the coupling strengths or induces energy fluctuations of the spins thus affecting the transfer fidelity and again when the quantum code to be transmitted suffers inevitable interactions with its host environment. We recollect that in the context of the spin chain, the dynamics of a physical quantum channel connecting the sender and the receiver, the former encoding the information in a stationary qubit at its location, with the aim that the evolution of the quantum channel allows the information to be retrieved at the receiver's stationary qubit location. Spin $-1/2$ chains have been intensively investigated as faithful quantum channels for a variety of tasks [2-6]. Ref [7] show that the natural dynamics of a single spin chain is able to sustain the generation of two pairs of Bell states (possibly shared between a sender and a distant receiver) which can in turn enable two-qubit teleportation. This is via the use of a spin $1/2$ chain with XX interactions, connecting two pairs of spins located at its boundaries, playing the roles of sender and receiver. Appolaro T G J et al [8] present a modified version for the quantum channel for two qubit teleportation by exploiting Rabi-like oscillations due to excitations induced by the means of strong and localized magnetic fields. Analytic formulae for the fidelity of the quantum state transfer were derived showing high-quality transfer for general quantum states as well as for specific classes of states relevant for quantum information processing. Harshit V [9] develop a protocol for sharing entanglement between parties using the natural dynamics of multi-ferroic spin chains by introducing a novel kicking scheme of the electric field that thus enhances teleportation fidelity for appropriate choice of parameters. A comparison of the results to that of XXZ and XX models subject to a similar entanglement sharing protocol show that the helical multi-ferroic chain with the kicking scheme provides a better singlet fraction. They equally show that the kicking scheme in conjugation with the optimized parameters enhances the fidelity of teleportation even in the presence of impurities and/or decoherence. Cyclic quantum teleportation of two-qubit entangled states by using six-qubit cluster state and six-qubit entangled state is reported in [10]. The schemes requires at least the existence of three collaborators acting all as senders and receivers of quantum information, each one of them has an information to be transmitted to the next neighbour in a circular manner.

Apart from solving the dynamical behaviour of spin-chains and possible optimization techniques for enhancing its performances as a quantum channel by using the Majorana wire harbouring the Majorana bound states (MBSs), we seek to reduce the architectural requirements of using two copies of the setup for the two qubit teleportation protocol reported in our earlier work. Here the two qubit teleportation is carried out using four qubits; two each at the sending and receiving locations and in contact via the Majorana wire. The prize to be paid here is the fact that the two pairs of qubit interact via the exchange coupling. It is in this setting that we investigate the role of the exchange coupling, the channel parameters, the spin chain parameters on the average fidelity of entanglement teleportation. We equally consider the fact that the geographically located two pairs of qubits interact differently with their respective spin bath. It is instructive to recollect that the MBSs provide a natural resource for entanglement generation between the sending and receiving ends [11, 22]. Their robustness to local perturbations has been reported to be useful in quantum computation and information processing protocols [12-15]. As quasi-particles that have neither a fermionic nor a bosonic character, the MBSs or Majorana fermions in condensed matter Physics is a topological phase that occur in hybrid system when the system is rightly tuned [16-19]. In the quantum dot (QD) Majorana coupling setting, much work has been done over the recent years with emphasis on charge transport [11, 21] and associated teleportation phenomena [1, 20-24]. Ngwa et al [1] investigated the decoherence effects from a spin environment on two detached qubits coupled by a single Majorana wire (SMW) and double Majorana wires (DMWs) respectively when teleporting entanglement between the distinct qubits. The role of the composite system parameters on entanglement and entanglement teleportation were examined. Stephan Plugge et al [24] in what they called Majorana entanglement bridge, study the concurrence of entanglement between two quantum dots in contact to Majorana bound states on a floating superconducting island. They showed that the distance between the Majorana states, the charging energy of the island and average island charge to be decisive parameters for the efficiency of entanglement generation. Xin-Qi et al [20] revisited the non-locality of Majorana Zero mode and teleportation making effect to eliminate some inconsistencies between the Bogoliubov-de Gennes based treatment and the method using the associated regular fermion occupation number states within the second quantization setup. Comprehensive discussions and detailed comparison was presented for the zero-bias conductance limit and the proposed treatment is expected to be involved in analysing future experiment in this fast growing field.

The rest of the work is organised as follows: In section 2 we provide the model describing the Majorana wire as a bridge to teleport bipartite entanglement between two pairs of decohered qubits initially prepared in an arbitrary superposition of state

that supports the Majorana end states. The evaluation of the average fidelity of entanglement teleportation is carried out in section 3. The interpretation of the numerical results and conclusion forms the basis of sections 4 and 5 respectively.

2. Model Hamiltonian

We study a hybrid structure shown in Figure 1 made up of a

semiconductor nanowire with strong spin-orbit coupling with the conspiracy of pronounced Zeeman magnetic field and proximity-induced pairing correlations inherited from an s-wave superconducting substrate will induce the Majorana end states that are tunnel-coupled to two QDs, respectively at the opposite ends of the wire which we name sub-systems A and B.

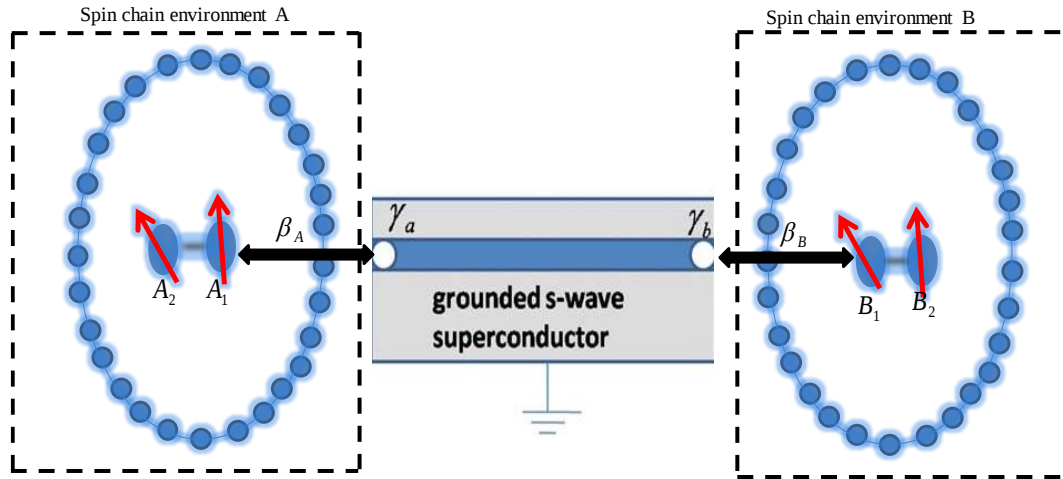


Figure 1. Schematic setup of the system to carry out the two qubit entanglement teleportation. The setup constitutes two sub-systems A and B each containing two coupled qubits submerge in spin chain environments A and B. The two sub-systems are in contact via the Majorana wire that harbours the Majorana bound states represented by white dots.

The Hamiltonian describing the system dynamics according to the setup is given as:

$$H = H_S + H_{E-A} + H_{E-B} + H_E \quad (1)$$

The system Hamiltonian, H_S describes the MBSs plus the single-level two pairs of QDs and their mutual coupling as follows [1, 22, 23]

$$H_S = \sum_{j=A_1, A_2} \varepsilon_j d_j^\dagger d_j + \beta_A (d_{A_1}^\dagger - d_{A_1}) \gamma_a + \sum_{i=B_1, B_2} \varepsilon_i d_i^\dagger d_i + \beta_B (d_{B_1}^\dagger - d_{B_1}) \gamma_b + H_{DQDA} + H_{DQDB} \quad (2)$$

H_{DQDA} and H_{DQDB} in equation (2) are the Hamiltonians for the interactions between the QDs of sub-system A and B respectively and are given as $H_{DQDA} = U_0 \sum_{j=A_1, A_2} \hat{n}_{j\downarrow} \hat{n}_{j\uparrow}$ and

$H_{DQDB} = U_0 \sum_{i=B_1, B_2} \hat{n}_{i\downarrow} \hat{n}_{i\uparrow}$. U_0 is Coulomb interaction

strength [25], $\hat{n}_{i(j)\sigma} = d_{i(j)\sigma}^\dagger d_{i(j)\sigma}$ and $\hat{n}_{i(j)} = \hat{n}_{i(j)\downarrow} + \hat{n}_{i(j)\uparrow}$ where $d_{i(j)\sigma}^\dagger$ ($d_{i(j)\sigma}$) are annihilation (creation) operators of

the single level QDs. γ_a and γ_b are the Majorana operators associated with the two MBSs at the ends of the nano-wire. β_A and β_B are respectively the coupling amplitudes between Majorana end states of QD A_1 and QD B_1 . In our studies we will consider symmetric and asymmetric coupling between the Majorana end modes and the dots. Following the transformations as reported in [1], and for simplicity considering only exchange coupling along the z-axis between the two pairs of QDs, the system Hamiltonian in the spin degree of freedom gives

$$H_S = \frac{\varepsilon}{2} \sigma_{A_1}^z + \frac{\varepsilon}{2} \sigma_{A_2}^z + J_z \sigma_{A_1}^z \sigma_{A_2}^z + \frac{\varepsilon}{2} \sigma_{B_1}^z + \frac{\varepsilon}{2} \sigma_{B_2}^z + J_z \sigma_{B_1}^z \sigma_{B_2}^z + \varepsilon_m \tau^z + \beta_A \sigma_{A_1}^y \tau^y + \beta_B \sigma_{B_1}^y \tau^y \quad (3)$$

Here, ε is the single particle energy of each dot (qubit) and J_z the exchange coupling along the z-axis (Ising model) and describes the interaction between the qubits of the individual sub-systems A and B. $\varepsilon_m \approx e^{-L/\xi}$ is the strength of the two Majorana interaction that damps exponentially with the length of the nano-wire L, ξ is the superconducting coherent length. σ^x (σ^y) and τ^x (τ^y) represents the Pauli spin operators for the dots qubits and Majorana qubit respectively. In the fermionic representation of the systems Hamiltonian equation (2), we use the state basis, $|n_{A_2} n_{A_1} n_m n_{B_1} n_{B_2}\rangle$ describing the possible spin configuration of sub-system A, MBSs and sub-system B, where n_{A_2} (n_{A_1}) for sub-system A, n_{B_2} (n_{B_1}) for sub-system B and n_m MBSs, represents their respective spin 'up' or 'down' states. Synonymous to the subspace with odd charge parity, we consider the following subspaces with spin configurations: $|11111\rangle$, $|00100\rangle$, $|11100\rangle$, $|00111\rangle$, $|10011\rangle$, $|11010\rangle$, $|00001\rangle$, $|11001\rangle$, $|01000\rangle$, $|10000\rangle$, $|00010\rangle$. The Hamiltonians:

$$H_{E-A} = -\frac{g_A}{2} (\sigma_{A_1}^z + \sigma_{A_2}^z) \sum_{l=1}^N \sigma_l^z \quad (4)$$

and

$$H_{E-B} = -\frac{g_B}{2} (\sigma_{B_1}^z + \sigma_{B_2}^z) \sum_{l=1}^N \sigma_l^z \quad (5)$$

are interaction Hamiltonians between the two qubits of systems A and B with their respective environment, g_A (g_B) the strength of their respective interactions.

The Hamiltonian H_E represents that of the spin environment which we consider the same for the two separate pairs of qubits, is described by the one-dimensional XY model written in the unit where $\hbar = 1$, is given by [26-28]:

$$H_E = -\sum_{l=1}^N \left(\frac{1+\alpha}{2} \sigma_l^x \sigma_{l+1}^x + \frac{1-\alpha}{2} \sigma_l^y \sigma_{l+1}^y + \lambda \sigma_l^z \right) \quad (6)$$

The Pauli matrices σ_l^ν ($\nu = x, y, z$) describe the spin chain of the environment. The parameter λ characterizes the intensity of the transverse magnetic field, and α measures

the anisotropy in the in-plane interaction. The XY model described by equation (6) encompasses other two spin model; the Ising spin chain with $\alpha = 1$ and the XX spin chain with $\alpha = 0$. These different models shall be subject to our investigation. Following the result of the treatment in [26, 28], we can readily write for sub-system A

$$H_{E-A}^\lambda = H^\lambda + H_{E-A} \quad (7)$$

and sub-system B

$$H_{E-B}^\lambda = H^\lambda + H_{E-B} \quad (8)$$

Such that the diagonalized form of the Hamiltonian $H_{E-A(B)}^\lambda$ of the two qubits of sub-system A(B) dressed with the environment due to interaction is given by

$$H_{E-A}^{(\lambda_{j(m)})} = \sum_k \Omega_k^{\lambda_{j(m)}} \left(b_{k,\lambda_{j(m)}}^+ b_{k,\lambda_{j(m)}} - \frac{1}{2} \right) \quad (9)$$

The parameter $\lambda_{j(m)}$ correspond to the dressed eigen value due to interactions of sub-systems A(B) with their respective spin chain environment. $b_{k,\lambda_{j(m)}}^+$ ($b_{k,\lambda_{j(m)}}$) are the Bogoliubov-transformed annihilation (creation) fermion operators arising from the interactions between the two qubits and the environment, defined as

$$b_{k,\lambda_{j(m)}} = \cos \left(\frac{\theta_k^{\lambda_{j(m)}}}{2} \right) d_k - i \sin \left(\frac{\theta_k^{\lambda_{j(m)}}}{2} \right) d_{-k}^+ \quad (10)$$

The energy spectrum $\Omega_k^{\lambda_{j(m)}}$ is given by

$$\Omega_k^{\lambda_{j(m)}} = 2 \sqrt{\left(\varepsilon_k^{\lambda_{j(m)}} \right)^2 + \alpha^2 \sin^2 \frac{2\pi k}{N}}, \text{ with } \varepsilon_k^{\lambda_{j(m)}} = \lambda_{j(m)} - \cos \frac{2\pi k}{N} \quad \alpha_k^{\lambda_{j(m)}} = \left(\theta_k^{\lambda_{j(m)}} - \theta_k^\lambda \right) / 2 \quad \text{with angles } \theta_k^{\lambda_{j(m)}} \text{ satisfying, } \cos \theta_k^{\lambda_{j(m)}} = 2\varepsilon_k^{\lambda_{j(m)}} / \Omega_k^{\lambda_{j(m)}}. \text{ We}$$

note that the energy spectrum $\Omega_k^{\lambda_{j(m)}}$ carries information about all the possible excitations in the spin chain. Interaction between the spin chain and qubits induces overlap between these excitations and thus quantum fluctuations.

The time evolving density matrix of the composite system may be written as

$$\rho(t) = \exp \left(-i \left(H_S + H_{EA}^{\lambda_j} + H_{EB}^{\lambda_m} \right) t \right) \rho(0) \exp \left(i \left(H_S + H_{EA}^{\lambda_j} + H_{EB}^{\lambda_m} \right) t \right) \quad (11)$$

Considering the factorized form of the initial state preparation of the composite system,

$$\rho(0) = \rho_{A_1 A_2 B_1 B_2}(0) \otimes \rho_E(0) \otimes \rho_B(0).$$

$\rho_{A_1 A_2 B_1 B_2}(0) = |\Phi(0)\rangle\langle\Phi(0)|$ is the density matrix of

sub-system A, MBS and sub-system B in the initial state.

$|\Phi(0)\rangle$ is the initial prepared state of the system Hamiltonian assumed to be in the general form:

$$|\Phi(0)\rangle = \delta_1 |1111\rangle + \delta_2 |00100\rangle + \delta_3 |11100\rangle + \delta_4 |10011\rangle + \delta_5 |11010\rangle + \delta_6 |00001\rangle + \delta_7 |01000\rangle + \delta_8 |00111\rangle + \delta_9 |11001\rangle + \delta_{10} |00010\rangle + \delta_{11} |10000\rangle \quad (12)$$

Where the coefficients satisfy the normalization condition:

$$\delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2 + \delta_5^2 + \delta_6^2 + \delta_7^2 + \delta_8^2 + \delta_9^2 + \delta_{10}^2 + \delta_{11}^2 = 1 \quad (13)$$

The density matrix, $\rho_E(0) = |\Psi(0)\rangle_E \langle\Psi(0)|$ describes the density matrix of the spin chain in the initial state $|\Psi(0)\rangle_E$. Tracing the environmental degrees of freedom leads to the following equation of the time evolving density matrix:

$$\rho(t) = \sum_{n,m,\mu,\nu=1}^{11} C_\mu^m C_\nu^{*n} |F_{mn}(t)|^2 |\phi_\mu\rangle\langle\phi_\nu| \quad (14)$$

where $C_\mu^m = \exp(-iE_\mu t) \langle\phi_\mu|\varphi_m\rangle \langle\varphi_m|\Phi(0)\rangle$ and $C_\nu^{*n} = \exp(iE_\nu t) \langle\phi_\nu|\varphi_n\rangle \langle\varphi_n|\Phi(0)\rangle$. E_μ and $|\phi_\mu\rangle$ are respectively the eigen value and eigen state of the system Hamiltonian equation (3). The absolute value of the decoherence factor $|F_{mn}(t)|$ written in the form as given in [26, 28]

$$|F_{mn}(t)| = \prod_{k>0} \left\{ \begin{aligned} &1 - \sin^2(2\alpha_k^{\lambda_m}) \sin^2(\Omega_k^{\lambda_m} t) - \sin^2(2\alpha_k^{\lambda_n}) \sin^2(\Omega_k^{\lambda_n} t) \\ &+ 2 \sin(2\alpha_k^{\lambda_m}) \sin(\alpha_k^{\lambda_n} t) \sin(\Omega_k^{\lambda_m} t) \sin(\Omega_k^{\lambda_n} t) \cos(\Omega_k^{\lambda_m} t - \Omega_k^{\lambda_n} t) \\ &- 4 \sin(2\alpha_k^{\lambda_m}) \sin(\alpha_k^{\lambda_n} t) \sin^2(\Omega_k^{\lambda_m} t) \sin^2(\Omega_k^{\lambda_n} t) \sin^2(\alpha_k^{\lambda_m} - \alpha_k^{\lambda_n}) \end{aligned} \right\}^{\frac{1}{2}} \quad (15)$$

The eigen states $|\varphi_m\rangle$ written in the basis: $|1111\rangle$, $|00100\rangle$, $|11100\rangle$, $|00111\rangle$, $|10011\rangle$, $|11010\rangle$, $|00001\rangle$, $|11001\rangle$, $|01000\rangle$, $|10000\rangle$, $|00010\rangle$ are those of the interaction Hamiltonians as discussed earlier of sub-system A and B interacting with their spin environment. The corresponding eigen values are:

$$\lambda_1 = \lambda - g_A - g_B, \quad \lambda_2 = \lambda + g_A + g_B, \quad \lambda_3 = \lambda - g_A + g_B, \\ \lambda_4 = \lambda - g_B, \quad \lambda_5 = \lambda - g_A, \quad \lambda_6 = \lambda + g_A,$$

$$\lambda_7 = \lambda + g_B, \quad \lambda_8 = \lambda + g_A - g_B, \quad \lambda_9 = -g_A + \lambda, \\ \lambda_{10} = \lambda + g_A, \quad \lambda_{11} = \lambda + g_B$$

We then eliminate the Majorana degrees of freedom and trace over the spin degree of freedom of sub-system A. The reduced density matrix of sub-system B written in the standard basis: $|00\rangle$, $|01\rangle$, $|10\rangle$, $|00\rangle$ is given as

$$\rho_B(t) = \begin{bmatrix} \rho_{11} & 0 & 0 & 0 \\ 0 & \rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{bmatrix} \quad (16)$$

This is in the block diagonal form, signalling the level of coherent evolution of the sub-system B. The elements of the reduced density matrix of equation (16) are given as

$$\rho_{11} = \frac{1}{11} + \frac{A_2^2}{11} (1+b_2^2) + \frac{A_{10}^2}{11} (b_4^2 + b_5^2) + \frac{A_{11}^2}{11} (1+b_6^2) + \frac{A_3^2}{11} + \frac{A_5^2}{11} a_4^2 - 2A_2 A_{11} (b_6 + b_2) Q_2 \cos((E_2 - E_{11})t) |F_{2,11}|^2 \\ - 2A_3 A_5 a_4 Q_3 \cos((E_3 - E_5)t) |F_{35}|^2 - 2A_2 A_{10} (b_4 + b_5) Q_7 \cos((E_2 - E_{10})t) |F_{2,10}|^2 \\ + 2A_{10} A_{11} (1+b_4 b_6) Q_8 \cos((E_{10} - E_{11})t) |F_{10,11}|^2$$

$$\begin{aligned}\rho_{22} &= \frac{A_1^2 a_1^2}{11} - \frac{A_4^2 a_6^2}{11} + \frac{A_6^2}{11} + \frac{A_8^2}{11} + \frac{A_9^2}{11} + 2A_1 A_9 a_1 Q_1 \cos((E_1 - E_9)t) |F_{1,9}|^2 + A_6 A_8 a_8 Q_6 \cos((E_6 - E_8)t) |F_{6,8}|^2 \\ \rho_{33} &= \frac{A_5^2}{11} a_5^2 + \frac{A_3^2}{11} a_3^2 + \frac{A_2^2}{11} b_3^2 + \frac{A_{10}^2}{11} - 2A_2 A_{11} b_3 b_7 Q_2 \cos((E_2 - E_{11})t) |F_{2,11}|^2 \\ &\quad - 2A_3 A_5 a_4 Q_3 \cos((E_3 - E_5)t) |F_{3,5}|^2 - 2A_2 A_{10} b_3 Q_7 \cos((E_2 - E_{10})t) |F_{2,10}|^2 \\ &\quad + 2A_{10} A_{11} b_7 Q_8 \cos((E_{10} - E_{11})t) |F_{10,11}|^2 \\ \rho_{44} &= \frac{A_4^2}{11} (1 + a_5^2) + \frac{A_8^2}{11} (1 + a_7^2) + \frac{A_6^2}{11} (b_1^2 + a_9^2) + \frac{A_1^2}{11} + \frac{A_9^2}{11} a_2^2 - 2A_4 A_6 (b_1 + a_5 a_9) Q_4 \cos((E_4 - E_6)t) |F_{4,6}|^2 \\ &\quad + 2A_6 A_8 (a_9 + b_1 a_7) Q_6 \cos((E_6 - E_8)t) |F_{6,8}|^2 + 2A_1 A_9 a_2 Q_1 \cos((E_1 - E_9)t) |F_{1,9}|^2\end{aligned}$$

where:

$$\begin{aligned}H_1 &= 2\varepsilon + 2J_z + \varepsilon_m, \quad H_2 = -2\varepsilon - 2J_z + \varepsilon_m, \quad H_3 = H_4 = \varepsilon_m, \quad H_5 = H_6 = H_8 = -\varepsilon_m + \varepsilon, \quad H_7 = H_9 = H_{10} = H_{11} = -\varepsilon_m - \varepsilon, \\ a_1 &= \frac{H_{11} - E_1}{\beta_B}, \quad A_1 = \frac{1}{\sqrt{a_1^2 + 1}}, \quad a_2 = \frac{H_9 - E_9}{\beta_B}, \quad A_9 = \frac{1}{\sqrt{a_9^2 + 1}}, \quad b_6 = -\frac{\beta_A (H_{10} - E_{11})}{\beta_B^2 - (H_{10} - E_{11})(H_2 - E_{11})}, \quad a_4 = \frac{H_5 - E_5}{\beta_B}, \\ A_5 &= \frac{1}{\sqrt{a_4^2 + 1}}, \quad a_3 = \frac{H_3 - E_3}{\beta_B}, \quad A_3 = \frac{1}{\sqrt{a_3^2 + 1}}, \quad a_5 = \frac{H_4 - E_4}{\beta_A}, \quad a_6 = \frac{(H_4 - E_4)\beta_B}{(H_6 - E_4)\beta_A}, \quad A_4 = \frac{1}{\sqrt{1 + a_6^2 + a_5^2}}, \quad a_6 = \frac{\beta_B}{H_4 - E_8}, \\ a_7 &= \frac{\beta_A}{H_4 - E_8}, \quad A_8 = \frac{1}{\sqrt{1 + a_7^2 + a_8^2}}, \quad a_9 = \frac{H_6 - E_6}{\beta_B}, \quad b_1 = \frac{(H_6 - E_6)\beta_A}{(H_4 - E_6)\beta_B}, \quad A_6 = \frac{1}{\sqrt{1 + b_1^2 + a_9^2}}, \\ b_7 &= -\frac{\beta_A}{\beta_B} \left(1 + \frac{\beta_A (H_{10} - E_{11})(H_2 - E_{11})}{\beta_B^2 - (H_{10} - E_{11})(H_2 - E_{11})} \right), \quad A_{11} = \frac{1}{\sqrt{1 + b_6^2 + b_7^2}}, \quad b_4 = \frac{H_{10} - E_{10}}{\beta_B}, \quad b_5 = \frac{(H_{10} - E_{10})\beta_A}{(H_{11} - E_{10})\beta_B}, \quad A_{11} = \frac{1}{\sqrt{1 + b_5^2 + b_4^2}}, \\ b_3 &= \frac{\beta_B}{H_{10} - E_2}, \quad b_2 = \frac{\beta_A}{H_{10} - E_2}, \quad A_2 = \frac{1}{\sqrt{1 + b_2^2 + b_3^2}}, \quad A_{11} = \frac{1}{\sqrt{1 + b_6^2 + b_7^2}}, \quad Q_1 = \frac{1}{11} a_1 a_2 A_1 A_9, \quad Q_2 = \frac{1}{11} b_2 b_6 A_2 A_{11}, \\ Q_3 &= \frac{1}{11} a_3 a_4 A_3 A_5, \quad Q_7 = \frac{1}{11} b_3 b_4 A_2 A_{10}, \quad Q_8 = \frac{1}{11} b_5 b_7 A_{11} A_{10}.\end{aligned}$$

3. Average Fidelity of Entanglement Teleportation

For the entanglement teleportation as discussed in [1 and refs. cited therein], we consider for example the two input entangled states: $|\psi_{in}\rangle = \cos\frac{\theta}{2}|10\rangle + e^{i\vartheta}\sin\frac{\theta}{2}|01\rangle$ and $|\psi_{in}\rangle = \cos\frac{\theta}{2}|00\rangle + e^{i\vartheta}\sin\frac{\theta}{2}|11\rangle$ with $0 \leq \theta \leq \pi$ and $0 \leq \vartheta \leq 2\pi$. θ represent the amplitude and ϑ the phase of the input state. The density matrix related to the input state is $\rho_{in} = |\psi_{in}\rangle\langle\psi_{in}|$. The output state can be obtained by applying joint measurements and local unitary transformations on the input state ρ_{in} . Following the teleportation protocol proposed by Lee and Kim the output state of the two qubit teleportation can be written as

$$\rho_{out} = \sum_{ij} P_{ij} (\sigma^i \otimes \sigma^j) \rho_{in} (\sigma^i \otimes \sigma^j), \quad \text{where } i, j = 0, 1, 2, 3$$

and $\sigma^0 = I$ is the identity matrix, $\sigma^1 = \sigma^x$, $\sigma^2 = \sigma^y$, $\sigma^3 = \sigma^z$ are the Pauli matrices. The probabilities $P_{ij} = P_i P_j$ satisfy the condition $\sum_{ij} P_{ij} = 1$, with $P_j = \text{Tr}\{E^j \rho_B\}$ and

ρ_B being the reduced density matrix of the channel corresponding to system B. The projective measurements E^j are defined from the four maximally entangle Bell states as $E^0 = |\psi^-\rangle\langle\psi^-|$, $E^1 = |\psi^+\rangle\langle\psi^+|$, $E^2 = |\phi^-\rangle\langle\phi^-|$, $E^3 = |\phi^+\rangle\langle\phi^+|$, where $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ and $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$.

The fidelity between the input state and the output state for our system is defined by

$F(\rho_{in}, \rho_{out}) = \left\{ \text{Tr} \left(\sqrt{(\rho_{in})^{\frac{1}{2}} \rho_{out} (\rho_{in})^{\frac{1}{2}}} \right) \right\}^2$. To efficiently

measure the quality of the protocol, we calculate the average fidelity between the input and output states by averaging over all possible input states. Under the influence of the environment and channel parameters, the maximally entangled component can vary with time. Therefore, the best quality of the teleportation can be obtained by the optimal estimation of the projective measurements. Thus the average fidelity will be a function of time. The average fidelity is defined from the

fidelity as $F_A = \frac{1}{4\pi} \int_0^{2\pi} d\vartheta \int_0^\pi F(\rho_{in}, \rho_{out}) \sin \theta d\theta$. From straight evaluation we obtain the fidelity and average fidelity

for the input state $|\psi_{in}\rangle = \cos \frac{\theta}{2} |10\rangle + e^{i\vartheta} \sin \frac{\theta}{2} |01\rangle$ respectively as:

$$F_1(\rho_{in}, \rho_{out}) = 4\rho_{22}^2 + \sin^2 \theta (\rho_{11} + \rho_{44})^2 \quad (17)$$

and

$$F_{A1} = 4\rho_{22}^2 + \frac{2}{3}(\rho_{11} + \rho_{44})^2. \quad (18)$$

Equally for the input state,

$|\psi_{in}\rangle = \cos \frac{\theta}{2} |00\rangle + e^{i\vartheta} \sin \frac{\theta}{2} |11\rangle$ we have the fidelity and average fidelity respectively as:

$$F_2(\rho_{in}, \rho_{out}) = (\rho_{22} + \rho_{33})^2 \left[\sin^{5/2} \theta + \sin^3 \theta \cos^{1/2} \theta + \cos \vartheta \sin 2\theta (\sin^{1/2} \theta + \cos^{1/2} \theta) \right] \\ + (\rho_{11} + \rho_{44})^2 \left[\frac{1}{2} (\cos^2 \theta \sin^{3/2} \theta + \sin \theta \cos^{5/2} \theta) + \sin^{5/2} \theta \cos \theta + \sin^2 \theta \cos^{3/2} \theta \right] \quad (19)$$

and

$$F_{A2} = \frac{843}{500} (\rho_{22} + \rho_{33})^2 + \frac{42}{200} (\rho_{11} + \rho_{44})^2 \quad (20)$$

4. Results and Discussions

We first study the trend in the dynamics of the average fidelities considering two different entangled input states:

$$|\psi_{in}\rangle = \cos \frac{\theta}{2} |00\rangle + e^{i\vartheta} \sin \frac{\theta}{2} |11\rangle \quad \text{and}$$

$$|\psi_{in}\rangle = \cos \frac{\theta}{2} |10\rangle + e^{i\vartheta} \sin \frac{\theta}{2} |01\rangle \quad \text{with corresponding average}$$

fidelities F_{A1} and F_{A2} given in equations (18) and (20) respectively, in Figure 2. The same trend in the dynamics is observed in both cases however with enhancement when the entangled input state, $|\psi_{in}\rangle = \cos \frac{\theta}{2} |10\rangle + e^{i\vartheta} \sin \frac{\theta}{2} |01\rangle$ is used. Figure 2(b) is the scenario when the values of the coupling between the spin chain and the two qubits sub-systems

are interchanged.

To observe the effect of symmetry in the coupling between the Majorana bound states and the qubits at the opposite ends of the wire, in Figure 3 is the plot of the time evolution of the average fidelity of entanglement teleportation for the two different input states. For both input states, it is observed that when the coupling between the Majorana bound states and the qubits at the opposite ends of the wire are symmetric, there is enhancement in the average fidelity as compare to when the coupling are asymmetric. Considering the two different Bell states for entanglement teleportation, enables us to test the validity of our findings as a generalisation. For the rest of the figures we shall consider the input state $|\psi_{in}\rangle = \cos \frac{\theta}{2} |10\rangle + e^{i\vartheta} \sin \frac{\theta}{2} |01\rangle$ with average fidelity given by F_{A2} .

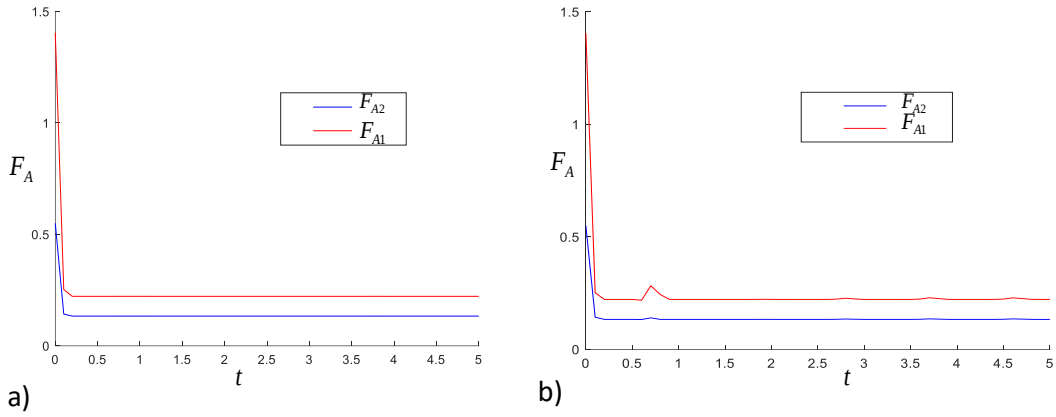


Figure 2. Plot of the dynamics of the average fidelity for two different input states: $|\psi_{in}\rangle = \cos\frac{\theta}{2}|00\rangle + e^{i\theta}\sin\frac{\theta}{2}|11\rangle$ (F_{A1} red curve) and $|\psi_{in}\rangle = \cos\frac{\theta}{2}|10\rangle + e^{i\theta}\sin\frac{\theta}{2}|01\rangle$ (F_{A2} blue curve), a) $g_A = 0.3, g_B = 2.1$ b) $g_B = 0.3, g_A = 2.1$ other parameters are set to: $\varepsilon = 0.1, \varepsilon_m = 0.31, \beta_A = \beta_B = 0.5, \lambda = 0.02, \gamma = 0.08, J_Z = 10.01, N = 150$.

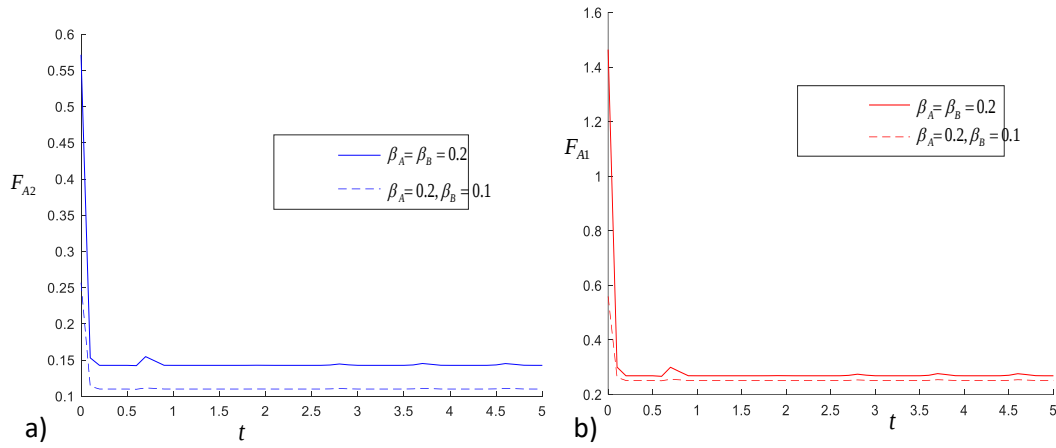


Figure 3. Plot of the dynamics of the average fidelity for two different input states: a) $|\psi_{in}\rangle = \cos\frac{\theta}{2}|10\rangle + e^{i\theta}\sin\frac{\theta}{2}|01\rangle$ with average fidelity F_{A2} (blue solid curve symmetric coupling, blue dash curve asymmetric coupling) b) $|\psi_{in}\rangle = \cos\frac{\theta}{2}|00\rangle + e^{i\theta}\sin\frac{\theta}{2}|11\rangle$ with average fidelity F_{A1} (red solid curve symmetric coupling, red dash curve asymmetric coupling) other parameters are set to: $g_A = 0.3, g_B = 2.1, \varepsilon = 0.1, \varepsilon_m = 0.31, \lambda = 0.02, \alpha = 0.08, J_Z = 10.01, N = 150$.

The influence of the size of the spin environment on the dynamics of the average fidelity is presented in Figure 4(a). We see that when the size of the spin environment increases, the dynamics of the average fidelity of the entanglement teleportation dies out quickly. This scenario is similar to that observed in [26, 27] that for an exchange coupled spin configuration the dynamics of quantum correlations decreases with the number of spin atoms. We attribute this to the possible fact that the two qubits in both sub-space A and B are exchange coupled. In the opposite end, using the teleportation

protocol in our finding in ref [1], it was shown that the dynamics of entanglement teleportation is mildly influence by the number of spin atom. While less computational resources are involved in the present teleportation scenario as compared to that used in the previous work, a better engineering of the exchange interaction would be necessary. In Figure 4(b) is a plot of the dynamics of the average fidelity for different values of the exchange coupling. The trend observed here is that, teleportation is enhanced when the values of the exchange coupling between the two qubits sub-space increases.

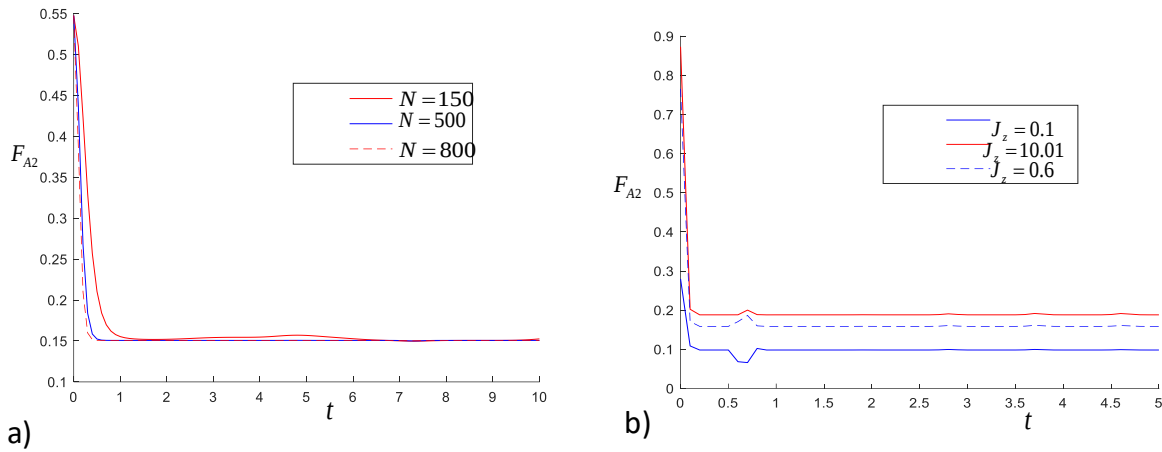


Figure 4. Plot of the dynamics of the average fidelity (a) for different numbers of atoms N setting $J_Z = 10.01$ b) for different exchange coupling strength J_Z , setting $N = 150$. The other parameters are set to: $g_A = 2.1, g_B = 0.3, \varepsilon = 0.1, \varepsilon_m = 0.31, \lambda = 0.02, \alpha = 0.08, \beta_A = .2, \beta_B = 0.3, N = 150$.

In Figure 5(a), it is observed that weak and symmetric coupling between spin environment and the qubits leads to better average fidelity teleportation as compared to increasing the coupling between the qubits and spin environment. Strong magnetic field is seen enhance the fidelity of entanglement teleportation, Figure 5(b). Strong anisotropy degree also enhances the fidelity of entanglement teleportation, Figure 6(a) after a short period of time. The average fidelity of entanglement teleportation is shown to depend on the spin model of the environment Figure 6(b) with the Ising spin model $\alpha = 1$, showing enhancement in the dynamics of teleportation after a

short period of time. This scenario has already been reported in [27] whereby using the same type of spin environment at low temperatures, the decoherence factor increases with the decrease in the degree of anisotropy factor α . Large values of the anisotropy parameter strengthens the interactions between the atomic spin environment and coherent excitations having less decoherence effect prevails in the spin environment, as compared to small values of the anisotropy parameter that makes the spin environment prone to quantum fluctuations and increasing decoherence effects on the dynamics of the teleported states.

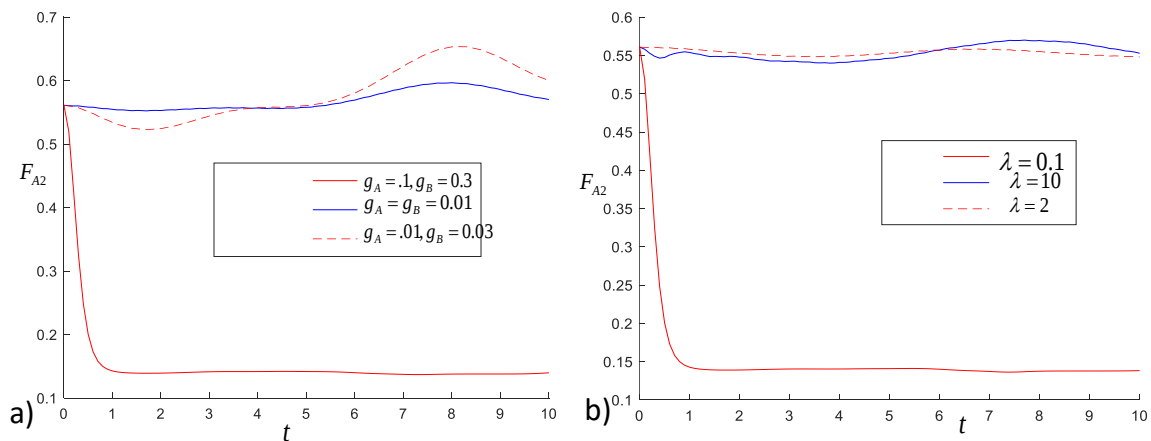


Figure 5. Plot of the dynamics of the average fidelity (a) for different coupling strength between spin chain environment and qubits $g_{A(B)}$ setting $\lambda = 0.02, \gamma = 0.08$ b) for different transverse magnetic field strength λ , setting $g_A = 0.1, g_B = 0.2, \alpha = 0.08$. The other parameters are set to: $\varepsilon = 0.1, \varepsilon_m = 0.31, \beta_A = \beta_B = 0.3, N = 150, J_Z = 10.01$.

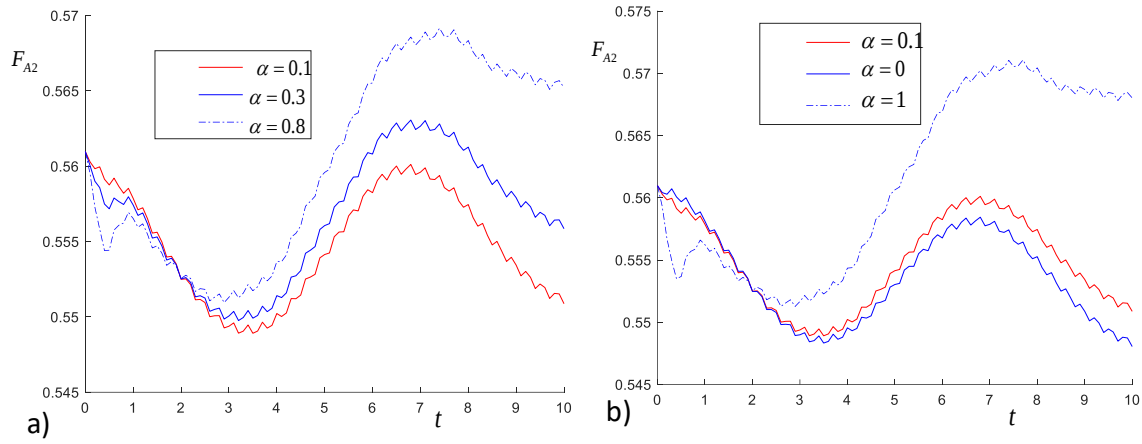


Figure 6. Plot of the dynamics of the average fidelity (a) for different anisotropy values α b) for different spin model; $\alpha = 0$, XX spin model (blue solid curve), $\alpha = 0.1$, arbitrary XY spin model (red solid curve), $\alpha = 1$, Ising spin model (blue dash curve), other parameters are set to: $\varepsilon = 0.1$, $\varepsilon_m = 0.3$, $\beta_A = \beta_B = 0.3$, $N = 150$, $J_Z = 10.01$, $\lambda = 2$, $g_A = g_B = 0.05$.

In Figure 7, we study the time evolution of entanglement teleportation for different values of the coupling between the Majorana bound states (MBSs) ε_m . We see that as the values of the coupling between the MBSs state increases, the average fidelity of teleportation decreases. Large values of ε_m corresponds to very small wire length that lead to overlap between the Majorana end modes which may intend lifting of the topological phase that harbour's the Majorana bound state.

In the opposite scenario, small values imply long wire length and enhancement in the teleportation state. This owes explanation from non-locality of the MBSs. We recollect that in [20] it is observed that in the limit $\varepsilon_m \rightarrow 0$ (very long wire length), a remarkable teleportation feature known as superluminal holds in electron transport.

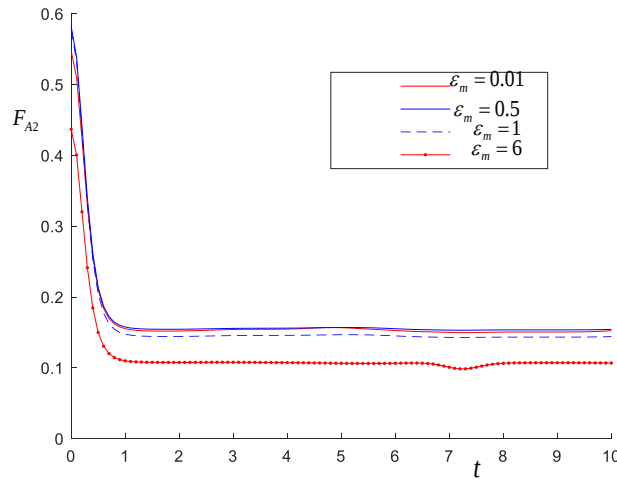


Figure 7. Plot of the dynamics of the average fidelity for different values of the coupling between Majorana bound states, ε_m setting other parameters to: $g_A = 0.1$, $g_B = 0.3$, $\varepsilon = 0.1$, $\beta_A = \beta_B = 0.3$, $N = 150$, $J_Z = 10.01$, $\lambda = 0.02$, $\alpha = 0.08$.

5. Conclusion

Seeking to reduce the architectural cost using two copies of the setup in the Kim and Lee two qubits entanglement teleportation scheme as reported in our earlier work, here we consider the two qubit entanglement teleportation where the

teleportation protocol involves two sub-systems A and B with each sub-system containing two exchange coupled qubits that suffers decoherence from respective spin environments. The two qubits sub-systems are correlated via the Majorana wire to mediate the entanglement teleportation. Using the average fidelity to measure the quality of the teleported state, we first observe a similar trend in the dynamics of the average

fidelity using two different entangled Bell states. The dynamics of the average fidelity is shown to be enhanced when using a set-up with symmetric coupling between qubits and Majorana wire at both ends as compared to when using asymmetric coupling. The same observation is seen when considering the same values of the coupling between the two qubits sub-systems at the opposite ends of the Majorana wire and their respective spin environment. Small values of the coupling between the qubits and spin environments should however be used. Large values of the exchange coupling between qubits in the two sub-systems should be used to have high fidelity of the teleported state. The dynamics of the average fidelity of entanglement teleportation is seen to depend on the size of the spin environment as compared to the teleportation protocol used in our earlier work where the dynamics is mildly sensitive to the size of the spin environment. Increasing the strength of the transverse magnetic field and large values of the anisotropy degree enhances the average fidelity of entanglement teleportation. The Ising spin model of the environment is shown to enrich the fidelity of the teleported state. The non-local nature of the MBSs and thus enhancement in teleportation is observed when the coupling between the Majorana bound states takes small values. We have equally notice that the dynamics of the system is however seen to observe almost a similar trend as that of the previous investigation for the single Majorana wire setup. We look forward to consider the effect of the Dzyloshinski-Moriya interaction between the two qubits sub-space and also the effect of transverse coupling on the fidelity of entanglement teleportation. This work shall also be extended to the case of the double Majorana wire network setting.

Abbreviations

MBSs	Majorana Bound States
QD	Quantum Dot
SMW	Single Majorana Wire
DMWs	Double Majorana Wires

Conflicts of Interest

The authors declare no conflicts of interest.

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