

Research Article

Local Time Stepping Scheme Using Structured Grids for Modelling of Shallow Water Flows

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Abstract

Numerical simulations of shallow water flows are widely used to predict global flows such as flood flows in the sea, river and reservoir flows, especially floods due to heavy rainfall and dam break. In particular, reducing the simulation time by applying the Local Time Stepping (LTS) scheme is one way to improve the practical efficiency of numerical simulation. In this paper, we proposed LTS scheme using a structured grid for the numerical simulation of shallow water system. When modeling any terrain, rectangular grid cells are used to facilitate grid generation. To estimate the momentum flux at the grid cell boundaries, we applied the second-order spatial accuracy Godunov finite volume algorithm with Roe approximation solver using the MUSCL method. The LTS scheme is applied to shallow water flow problems with tsunami reflection pattern and its accuracy and efficiency are compared with the traditional global time step (GTS) method. Results show that, with no loss of accuracy, the new LTS algorithm achieves 59-67% CPU time reduction when compared to the GTS method. The proposed LTS scheme accurately reflects time varying water regimes and reflected waves in shallow water flows and can be used for numerical simulations of the three-dimensional shallow water flows with arbitrary topography to reduce the simulation time significantly.

Keywords

Shallow Water, Godunov Finite Volume Method, Local Time Stepping

1. Introduction

Nowadays, numerical simulations are widely used to study shallow water flow. The explicit scheme for numerical simulation the shallow water system requires that the time step Δt satisfies the CFL condition. For example, in the case of a one-dimensional shallow water problem, this condition is expressed as follows.

$$\alpha = \frac{a\Delta t}{\Delta x} < 1 \quad (1)$$

Where, α is Courant number, Δx is grid cell size,

$a = |u| + \sqrt{g\eta}$ is the maximum wave velocity in shallow water flow, u is x direction component of the depth-averaged horizontal velocity, ' g ' is gravitational acceleration.

When the numerical solution is updated with the global time step (GTS) Δt , it is constrained by the maximum value of $a/\Delta x$ in the grid. It's maximum value is determined by the global largest a and smallest Δx . Since the Courant number α in many cells can be much smaller than 1, size of the grid cells is controlled by the parameters to improve the simulation efficiency. However, remeshing locally at each time step is not an efficient approach in numerical simulation,

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since it rather leads to computational complexity and requires many time steps [1-8, 11, 14, 15]. In [1] Amanda J. Crossley and Nigel G. Wright applied the LTS scheme to the numerical simulation of a system of shallow water equations by the triangular grid to evaluate the size of simulation time reduction and the accuracy of the results.

The LTS scheme takes a spatially variable time step which is selected according to a local CFL condition. Compared to the GTS method, the simulation efficiency increases because the cells with the Courant number much smaller than 1 are updated over an integer power of the global time step. A popular LTS scheme involves an hierarchy of L levels of cells where all cells on the same level, $\ell = 1, L$ use a common time step that is a power-of-two multiple of the base time step, i.e., $2^{\ell-1}\Delta t$ [12, 13, 16].

In [9] J. Rafael Cavalcanti and Michael Dumbser proposed an unstructured grid LTS scheme using the MUSCL-Hancock method to obtain second order of accuracy in space and time. This scheme has been applied to shallow water flows with complex wet and dry areas, confirmed mass conservation and observed improved computational efficiency compared to the standard second-order TVD scheme for scalar transport with global time step (GTS).

In [10] Walter Boscheri and Michael Dumbser proposed a cell-centered direct Arbitrary-Lagrangian-Eulerian (ALE) finite volume scheme on unstructured triangular grids that is high order accurate in space and time and that also allows for time-accurate local time stepping (LTS). The unstructured Lagrangian LTS scheme was applied to typical tests such as the shock-wave tube problem, the cylindrical explosion problem and it was verified that the computational efficiency was improved.

The aim of this paper is to find out the level number L of the hierarchy, the simulation time, and the correctness of the results when applying the LTS scheme to the numerical sim-

ulation of a shallow water equation system by the rectangular grid finite volume method.

2. Construction of the Local Time Stepping Scheme

The integral form of a two-dimensional conservative shallow water system neglecting bed slope and bed friction is expressed as follows.

$$\frac{\partial}{\partial t} \int_{\Omega} U dV + \int_{\Omega} \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right) dV = 0 \quad (2)$$

$$U = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, F = \begin{pmatrix} hu \\ hu^2 + gh^2/2 \\ huv \end{pmatrix}, G = \begin{pmatrix} hv \\ huv \\ hv^2 + gh^2/2 \end{pmatrix}$$

Where, h is depth, (u, v) are x and y direction components of the depth-averaged horizontal velocity, g is gravitational acceleration.

Applying Green's formula to the system of equations (2), we obtain Eq. (3)

$$\frac{\partial}{\partial t} \int_{\Omega} U dV + \int_{\Gamma} F_n d\Gamma = 0 \quad (3)$$

$$F_n = F \cdot n_x + G \cdot n_y$$

Where Γ is the boundary of the volume Ω and F_n is the flow rate across the boundary Γ . Applying Eq. (3) to a rectangular grid finite volume and time difference with respect to LTS and take the time difference into account LTS, we obtain the following LTS-type finite-volume equation.

$$U_{i,j}^{n+p} = U_{i,j}^n - \frac{p\Delta t}{\Omega_{i,j}} [(F_n^n \Gamma)_{i+1/2,j} - (F_n^n \Gamma)_{i-1/2,j} + (F_n^n \Gamma)_{i,j+1/2} - (F_n^n \Gamma)_{i,j-1/2}] \quad (4)$$

$$F_n^n = \frac{1}{2} (F_n(U_L^n) + F_n(U_R^n) - \hat{R} |\hat{A}| \Delta \hat{V}_{RL}^n) \quad (5)$$

Where, $\hat{R}$ and $|\hat{A}|$ are diagonal matrices consisting of the eigenvector matrix of the Roe mean matrix and the absolute values of the eigenvalues, and $\Delta \hat{V}_{RL}^n$ are characteristic variable difference matrices. $F_n(U_L^n)$ and $F_n(U_R^n)$ are fluxes calculated using MUSCL reconstructed data on the left and right sides of the cell boundary, respectively, and the label '^' indicates the quantities obtained from the Roe averages by reconstructed data.

The hierarchical LTS equation (4) was implemented with L levels resulting in an update cycle that repeats every $M = 2^{L-1}$ time steps. At each iteration cycle, care must be taken to ensure conservation of flow rate at the two neighboring cell interfaces, as the neighboring cell variables can

progress with different time steps. We have indicated the LTS level at each cell and each cell interface of a rectangular grid at the first step of each cycle, similar to the LTS method applied to the triangular grid finite volume method in [1]. The assignment of LTS levels to all cells, faces and vertices follows using a three-step process illustrated in Figure 1.

On the first time step of each LTS cycle, wave speeds $a = |u| + \sqrt{gh}$ is computed over interfaces of all cells. Then, applying formula (6) to the four boundaries of the cell, preliminary cell-based LTS level $\ell_{ci,j}$ is assigned to each cell (i, j) according to the local Courant number $\alpha_{i,j}$ obtained as the maximum in the cell.

$$\alpha_{i,j} = \frac{\Delta t}{\Omega_{i,j}} \max_{k=1,2,3,4} (a_k \Gamma_k)_{i,j} \quad (6)$$

$\ell_{ci,j}$ is given by the l from the range 1,..., L which satisfies

the following inequality, $\alpha_0/2^\ell < \alpha_{i,j} < \alpha_0/2^{\ell-1}$, with the exception of level 1 which is controlled by $\alpha_{i,j} < \alpha_0/2$ and level L which is controlled by $\alpha_{i,j} < \alpha_0/2^{L-1}$. The parameter α_0 represents a target Courant number, and by trial and error it was set to 0.8 or 20% below the theoretical stability limit. A value less than unity was selected because $\ell_{ci,j}$ is assigned once every M time steps, and $\alpha_{i,j}$ may increase during this sequence. Hence, $\alpha_0 = 0.8$ provides a degree of resilience

against rapidly changing flow conditions that may cause instability though it cannot completely preclude this possibility.

In step 2, LTS level of interfaces at each cell, $\ell_{fi,jk}$ is assigned as the minimum of the two neighbouring cell-based LTS levels.

In step 3, the final value of the LTS level of interfaces at each cell is given as the minimum of the four neighbouring face-based LTS levels.

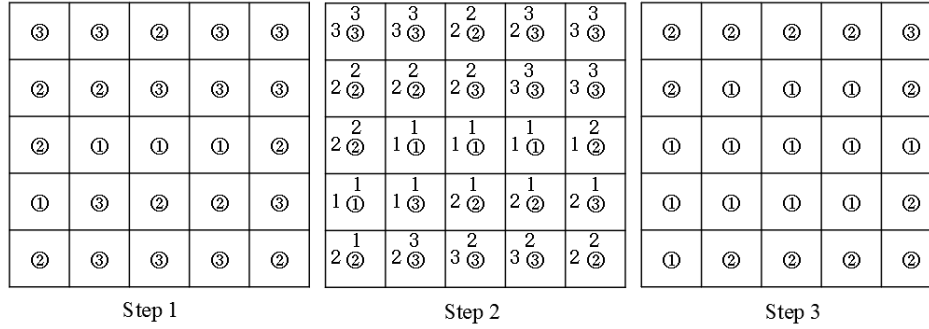


Figure 1. LTS level setting of cell and cell interfaces.

①,②,③- cell level, 1, 2, 3- level of cell interfaces

Once LTS levels are assigned, the model carries out a complete LTS cycle of M time loops, i.e., $m = 1, \dots, M$. During each sweep, data are only operated on when the LTS level is less than or equal to a threshold ℓ_0 . The selection of ℓ_0 depends on whether m is odd or even. If m is odd, $\ell_0 = 1$. If m is even, ℓ_0 further depends on m. For example, the values of ℓ_0 correspond to $L = 3$, $m = 1, \dots, 4$ are 1, 2, 1, 3, and the values of ℓ_0 correspond to $L = 4$, $m = 1, \dots, 8$ are 1, 2, 1, 3, 1, 2, 1, 4.

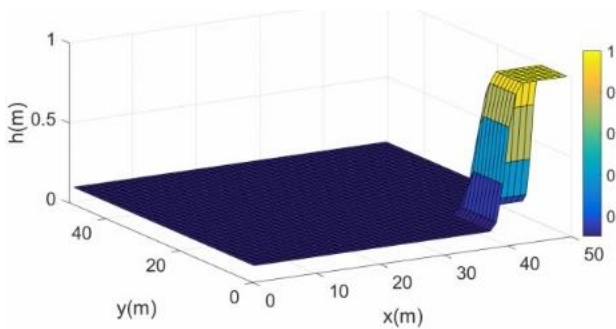


Figure 2. Computational domain and initial water surface.

3. Numerical Example

The numerical experiment is conducted in a square region with a length of 50 m and a width of 50 m surrounded by a vertical wall boundary. Figure 2 shows the computational domain and initial water level. At the initial moment, the bottom is filled with water at a depth of 0.1 m, and a water column with a height of 1 m is placed in right corner. We simulated the flow in which this water column collapses. The basic time step is taken as $\Delta t = 0.15s$.

Figure 3 shows the development of the water column with time. We presented simulation results for hierarchical structures of L from 1 to 5. Many numerical experiments show that the LTS schemes for hierarchical structures above $L = 6$ diverge and the numerical fails. As shown in Figure 3, the initial water column gradually collapses over time, and at time = 18 s, a reflected wave is formed by colliding with the opposite side. This reflected wave propagates upstream again and begins to interfere with each other. At time = 30 s, the highest water column is formed on the opposite side of the initial water column, and the water column collapses and the movement of the water surface takes place repeatedly. Through simulation, it can be seen that the LTS scheme accurately reflects the propagation process of the water surface without divergence during the numerical simulation.

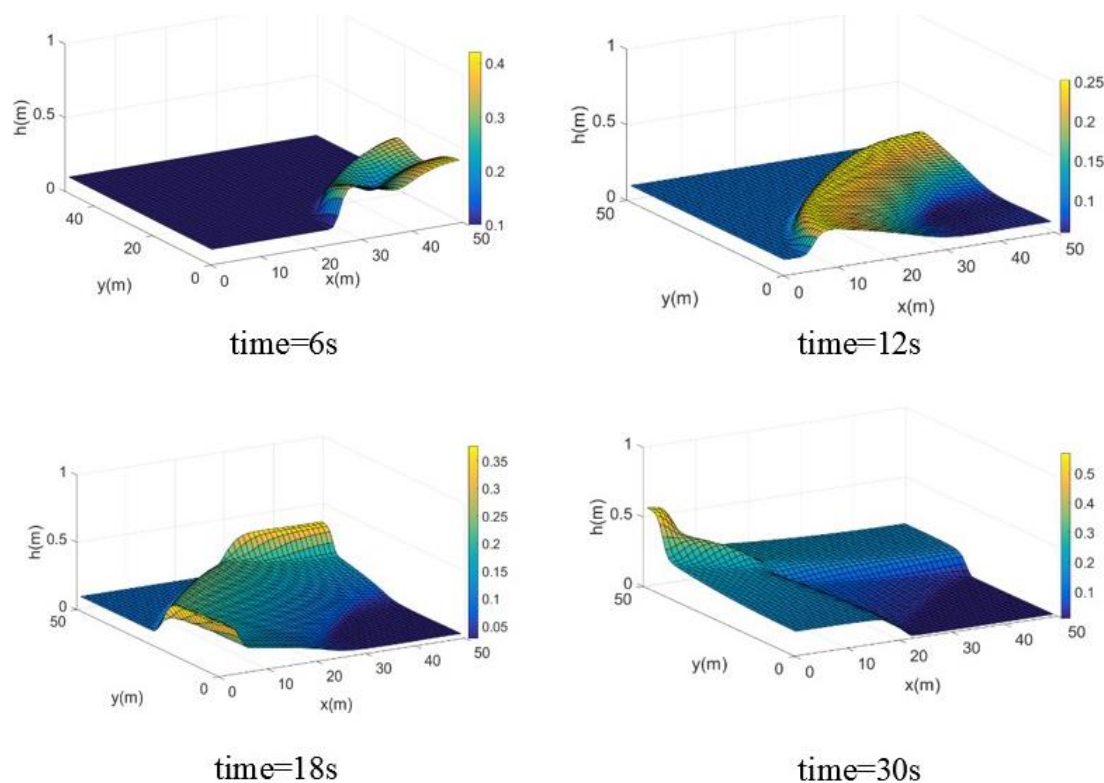


Figure 3. The development of water column change with time.

The numerical results for hierarchies L from 1 to 5 are in good agreement. Figure 4 shows the free surface height along the x -axis at the $y = 25$ m section at time = 30 s for each of the hierarchical structures. As shown in Figure 4, it is concluded that the differences of results are almost negligible.

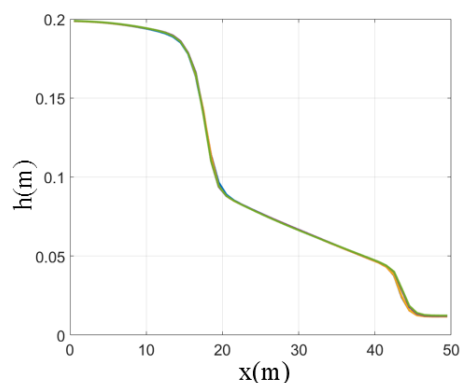


Figure 4. water level variation along the x -axis in the $y = 25$ m section at time = 30 s.

However, as shown in Table 1, the difference in computation time is noticeable. For the results with the same accuracy, the LTS scheme with $L = 2$ reduces the computation time by 67% compared to the GTS scheme and the computation time by 58.9% with $L = 5$, respectively. The rate of decrease

of computational time decreases rapidly as L increases.

Table 1. comparison of CPU computation time with level.

L	CPU computation time, s	Computation time percent, %
1	425	100
2	285	67
3	271	63.7
4	256	60.2
5	249	58.9

4. Conclusions

In the paper, we investigated the relationship between the number of setting level L and CPU computation time when applying the LTS scheme to the numerical simulation of a homogeneous shallow water by the rectangular grid finite volume method. As setting level L increases, the simulation time decreases significantly and there is little change in the accuracy of the simulation results. However, there is a limit in the set value of L , which diverges beyond that threshold, causing simulation failure. The limit is 5 in our computational experiments.

Abbreviations

MUSCL	Monotone Upwind Scheme for Conservation Laws
CFL	Courant-Friedrichs-Lewy
ALE	Arbitrary-Lagrangian-Eulerian

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Conflicts of Interest

The authors declare no conflicts of interest.

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