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Research Article

An Accurate Three-Step Hybrid Block Method Via Optimization Approach for Solving Mathematical Model of Continuous Fever

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Abstract

Emergence of novel infectious diseases and the resurgence of already known ones and its variants elicit significant concern in our contemporary world. Thus, it is very crucial to utilize all available resources to monitor and control their spread. Most of the epidemiological models developed to study and analyze the characteristics of diseases produced system of differential equations that are coupled in nature, which has become a challenge to researchers to find exact solutions. This work proposes an accurate three-step hybrid block method through optimization approach for solving mathematical models of continuous fever. The techniques of interpolation and collocation were applied to a power series polynomial for the derivation of the method using a three-parameter approximation of the hybrid points. The hybrid points were obtained by minimizing the local truncation error of the main method. The discrete schemes were produced as by-products of the continuous scheme and used to simultaneously solve mathematical models of continuous fever in block mode. The analysis of the basic properties of the method revealed that the schemes are self-starting, convergent, and A-stable. In addition, the analysis of the order of accuracy of the method showed that there is a gain of one order of accuracy in the main scheme where the optimization was carried out. Thereby, enhancing the accuracy of the whole method. The accuracy of the method was ascertained using three numerical examples. Comparison of the numerical results of the new method with those of the existing methods revealed that the newly developed method compares favorably with the existing hybrid block methods. Hence, the new method should be employed for the numerical solution of initial value problems of ordinary differential equations to obtain more accurate results.

Keywords

Local Truncation Error, Optimization, Initial Value Problems, Ordinary Differential Equations, Infectious Disease, Continuous Fever, Mathematical Model

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1. Introduction

Fever is a distinguishing characteristic of diseases from the ancient times. It is one of the prominent indicators of infections in human hosts. Fever often occurs in responses to infection, inflammation, trauma, and it represents a complex adaptive response of the host to various immune challenges [1]. Clinically, fever is a regulated rise in body temperature above normal daily fluctuations together with an elevated thermoregulatory set point [1]. The International Union of Physiological Science Commission for Thermal Physiology defined fever as a state of elevated core temperature, which is often, but not necessarily, part of the defensive responses of multicellular organisms (host) to the invasion of live (micro-organism) or inanimate matter recognized as pathogenic or alien by the host [1]. The mean oral temperature for a middle aged healthy adult is 36.8 $\pm$0.4 ℃ (98.2^{o}F) and body temperature has a maximum morning (at 6.00am) of 37.2 ℃ and maximum afternoon (at 4.00pm) temperature of 37.7 ℃. Thus, fever in healthy middle-aged adult may be defined as an early morning oral temperature of > 37.2 ℃ (> 99^{o}F) or a temperature of > 37.7 ℃ (100^{o}F) at any time during the day [1]. There are three major types of fever namely; continuous/sustained fever, intermittent fever, and remittent fever. This article focuses on continuous fever. Continuous or sustained fever is defined as fever that does not fluctuate more than about 1 ℃ (1.5^{o}F) during 24 hours, but at no time touches normal temperature. Continuous fevers are characteristics of Typhoid fever, Yellow fever, Dengue fever, Lassa fever, etc.

The emergency of novel infectious diseases, and the resurgence of old ones and their variants call for serious concern in our world today. As such, it is imperative to engage every tool available to check their spread. Mathematical models have been greatly employed in public health sector, in recent decades, to investigate the proliferation and control of diseases like Malaria fever, Typhoid fever, Lassa fever, etc.

To better understand the dynamics of most of the infectious diseases, experimental procedures are often required which may be financially demanding and/or physically challenging [2]. Hence, the importance of mathematical models in proffering solutions to real-life problems. These models often yield differential equations that are nonlinear, stiff or of fractional order which may have no exact solutions. Hence, the need for the development of accurate numerical methods which will ensure that appropriate inferences are drawn from the models.

The essence of this work is to investigate new advance and more accurate numerical methods for solving various mathematical models of continuous and intermittent fevers. The focus is on the general form of initial value problems (IVPs) of system of ODEs of the form

$$x' = f(t,x), x(t_0) = x_0 \quad (1)$$

is considered, where, $t \in [t_0, T], f: [t_0, T] \times \Re^n \to \Re^n$. It is assumed that equation (1) satisfies the conditions of the existence and uniqueness theorem for initial value problems [3].

Several methods, which include exact, approximate, and purely numerical, are employed to obtain solutions to epidemiological models. Some of the approximate/semi-analytical methods include Differential transform method [4-8], Multi-step differential transform method [7, 9], Laplace adomian decomposition method [5, 10], Revised adomian decomposition method [9], Variational iteration method [11], Multi-step homotopy analysis method [12].

Furthermore, purely numerical methods such as collocation [13], interpolation [14], integration [15], and interpolation polynomials [16, 17], have been thoroughly investigated in academic literature to construct continuous linear multistep methods (LMMs) for the direct solution of initial value problems in ordinary differential equations (see [14, 18] and the literature therein). Most of the traditional methods such as Runge-Kutta [19], multi-step Adams family [3], and higher-order multi-derivative types [20] did not yield desirable results in solving stiff differential equations because a large amount of computational effort was required or conditional stability was obtained. This necessitated the adoption of implicit block methods which possess the attribute of being self-starting, highly accurate, and absolutely stabile. One of such notable methods in this category are the hybrid block methods. Hybrid linear multi-step methods were introduced a few decades ago to overcome the first Dahlquist barrier on the step number and order of stable LMMs [14, 21].

The desire for improved accuracy of numerical methods has led to the development of new methods which are derived by minimization of the Local Truncation Errors (LTEs). Areo et al. proposed optimized hybrid block methods with high efficiency for the solution of first-order ODEs [14]. The derivation employed the interpolation and collocation techniques using a three-parameter approximation. The hybrid points were obtained by minimizing the local truncation error of the main method. The obtained schemes were reformulated to reduce the number of function evaluations. In their study, Singla et al. introduced an optimized hybrid block approach with distinct characteristics for numerically integrating initial value problems of ordinary differential systems [21]. The method successfully overcomes the first Dahlquist barrier on Linear Multi-Step Methods (LMMs) by incorporating both block and hybrid characteristics. The method of interpolation and collocation was employed by utilizing an approximate polynomial representation of the theoretical solution of the problem. Three intermediate points were added within a single block, with one point being fixed and the other two optimized to minimize the errors in the primary formula and an additional formula. The resulting scheme had a fifth order accuracy and possessed the attribute of A-stability. The study conducted by Ramos proposed a two-step method that involved the selection of two intermediate points through the optimization of the LTEs [22]. However, the most optimal formulation was attained through the process of reformulating the method in a manner that decreases the frequency of instances of the source term f. Ramos et al. employed an enhanced hybrid block technique in conjunction with a modified cubic B-spline method to solve non-linear partial differential equations [23]. No linearization was necessary in the approach, and the time step-size was optimized without compromising accuracy. Singla et al. devised a set of one-step hybrid block methods that incorporate two intra-step points [24]. These methods are designed to solve first-order initial value stiff differential systems. Within each family, there is an intra-step point that determines the sequence of the main technique, and a second point that governs the stability characteristics of the method. The approaches were also formulated as Runge-Kutta methods. Yakubu and Sibanda proposed a novel approach for solving first-order stiff initial value problems through the development of a one-step family of three optimized second-derivative hybrid block methods [25]. The optimization process was integrated into the derivation of the methods to achieve maximal accuracy. The analysis revealed that the methods exhibit convergence and A-stability. Some other recent and notable contributions on optimized hybrid block method may be found in [26] and the literature therein.

The new Three Step Optimized Hybrid Block Method (THSOHBM) proposed in this research incorporates three hybrid points with a three-parameter approximation. The interval of integration is allowed to determine the optimal hybrid points through the optimization of the principal term of the LTE of the main method. Previous researches have not considered up to five intra-step points with three unknown parameters in an optimization technique of this nature.

This article is organized as follows: Derivation of the three step optimized hybrid block method is done in section 2, and analysis of the basic properties of the method is carried out in section 3. In section 4, numerical examples are solved to ascertain the performance of the new method, and discussion of the results is presented in section 5.

2. Materials and Methods

The theoretical solution $x(t)$ of equation (1) is approximated by the polynomial Q(t) of the form

$$Q(t) = \sum_{j=0}^{m} b_j t^j \tag{2}$$

where $b_j \in R$ are real unknown coefficients to be determined. $m = (C + I) - 1$, I and C denote the number of interpolation and collocation points respectively. The first derivative of (2) is obtained

$$Q'(t) = \sum_{j=0}^{m} j b_j t^{j-1}, \tag{3}$$

interpolating equation (2) at t_n collocating equation (3) at $t_{n+j}, j = 0, p, 1\, q, 2, r, 3$, where p, q, r are the hybrid points such that $0 < p < q < r < 3$. This yields a system of linear equations given in (4).

$$\begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 & t_n^6 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^4 & 7t_n^4 \\ 0 & 1 & 2t_{n+p} & 3t_{n+p}^2 & 4t_{n+p}^3 & 5t_{n+p}^4 & 6t_{n+p}^4 & 7t_{n+p}^4 \\ 0 & 1 & 2t_{n+1} & 3t_{n+1}^2 & 4t_{n+1}^3 & 5t_{n+1}^4 & 6t_{n+1}^4 & 7t_{n+1}^4 \\ 0 & 1 & 2t_{n+q} & 3t_{n+q}^2 & 4t_{n+q}^3 & 5t_{n+q}^4 & 6t_{n+q}^4 & 7t_{n+q}^4 \\ 0 & 1 & 2t_{n+2} & 3t_{n+2}^2 & 4t_{n+2}^3 & 5t_{n+2}^4 & 6t_{n+2}^4 & 7t_{n+2}^4 \\ 0 & 1 & 2t_{n+r} & 3t_{n+r}^2 & 4t_{n+r}^3 & 5t_{n+r}^4 & 6t_{n+r}^4 & 7t_{n+r}^4 \\ 0 & 1 & 2t_{n+3} & 3t_{n+3}^2 & 4t_{n+3}^3 & 5t_{n+3}^4 & 6t_{n+3}^4 & 7t_{n+3}^4 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{pmatrix} = \begin{pmatrix} x_n \\ f_n \\ f_{n+p} \\ f_{n+1} \\ f_{n+q} \\ f_{n+2} \\ f_{n+r} \\ f_{n+3} \end{pmatrix} \tag{4}$$

Solving the system in (4) by Gaussian Elimination method to obtain the coefficients b_j 's, $j = 0, 1, \ldots, 7$ and putting back into equation (2) to obtain the implicit scheme of the form

$$x(t) = \alpha_0(t)x_n + h(\beta_0(t)f_n + \beta_p(t)f_{n+p} + \beta_1(t)f_{n+1} + \beta_q(t)f_{n+q} + \beta_2(t)f_{n+2} + \beta_r(t)f_{n+r} + \beta_3(t)f_{n+3}. \tag{5}$$

where, $\alpha_0(t)$, and $\beta_j(t), j = 0, p, 1\, q, 2, r, 3$ are continuous coefficients.

Evaluating equation (5) at the points $t = t_{n+p}, t_{n+1}, t_{n+q}, t_{n+2}, t_{n+r}, t_{n+3}$, yield the respective formulas for $x_{n+p}, x_{n+1}, x_{n+q}, x_{n+2}, x_{n+r}, x_{n+3}$. Expanding the main formula $x(t_{n+3})$ in the Taylor series around t_n yield after some simplification the following local truncation error.

$$\mathcal{L}(x(t_{n+3}); h) = \frac{1}{156800}(28pqr - 42\text{pq} - 42\text{pr} + 108p - 42\text{qr} + 108q + 108r - 297)h^8 x^{(8)}(t_n) + O(h^9). \tag{6}$$

Setting the principal term of the LTE in (6) to zero yields the following equation in three unknowns:

$$28pqr - 42\text{pq} - 42\text{pr} + 108p - 42\text{q}r + 108q + 108r - 297 = 0. \tag{7}$$

There are infinite number of solutions for p, q, r since there are more unknowns than equations. q is optimized when p and r are treated as free parameters yielding:

$$\text{q} = \frac{3(14\text{pr}-36\text{p}-36\text{r}+90)}{2(14\text{pr}-21\text{p}-21\text{r}+54)} \tag{8}$$

while the other two parameters are given as

$$p = \frac{1}{14}\left(7 - \sqrt{35}\right);\ r = \frac{1}{14}\left(7 + \sqrt{35}\right) \tag{9}$$

Substituting equation (8) into (9) yields $q = \frac{3}{3}$. Furthermore, substituting the values of p, q, r into the local truncation error formulae (6) gives

$$\mathcal{L}(x(t_{n+3}); h) = \frac{81}{8780800} h^9 x^{(9)}(t_n) + O(h^{10}) \tag{10}$$

Lastly, putting the values of the parameters p, q, r into the equations for $x_{n+p}, x_{n+1}, x_{n+q}, x_{n+2}, x_{n+r}, x_{n+3}$, yield the following three-step optimized hybrid block method:

$$\left.\begin{array}{c}
x_{n+p} = x_n + \frac{h}{7299040}\begin{pmatrix} 817\left(539+50\sqrt{35}\right)f_n + \left(1882384-130928\sqrt{35}\right)f_{n+p} + \left(3441123-619650\sqrt{35}\right)f_{n+1} + 36480\sqrt{35}f_{n+q} \\ +\left(3560193-619650\sqrt{35}\right)f_{n+2} + \left(1882384-312032\sqrt{35}\right)f_{n+r} + 19\left(-13573+2150\sqrt{35}\right)f_{n+3} \end{pmatrix}, \\
x_{n+1} = x_n + \frac{h}{61560}\left(323f_n + \left(16072-3136\sqrt{35}\right)f_{n+p} + 39393f_{n+1} + 19456f_{n+q} + 8073f_{n+2} + \left(16072-3136\sqrt{35}\right)f_{n+r} + 1083f_{n+3}\right), \\
x_{n+q} = x_n + \frac{h}{3040}\left(38f_n + \left(784+147\sqrt{35}\right)f_{n+p} + 2673f_{n+1} + 243f_{n+2} + \left(784-147\sqrt{35}\right)f_{n+r} + 38f_{n+3}\right), \\
x_{n+2} = x_n + \frac{h}{7695}\left(57f_n + \left(1960-392\sqrt{35}\right)f_{n+p} + 6372f_{n+1} + 2432f_{n+q} + 2457f_{n+2} + \left(1960-392\sqrt{35}\right)f_{n+r} + 152f_{n+3}\right), \\
x_{n+r} = x_n + \frac{h}{7299040}\begin{pmatrix} -817\left(50\sqrt{35}-539\right)f_n + \left(1882384+130928\sqrt{35}\right)f_{n+p} + \left(3441123+619650\sqrt{35}\right)f_{n+1} - 36480\sqrt{35}f_{n+q} \\ +\left(3560193+619650\sqrt{35}\right)f_{n+2} + \left(1882384+130928\sqrt{35}\right)f_{n+r} - 19\left(13573+2150\sqrt{35}\right)f_{n+3} \end{pmatrix}, \\
x_{n+3} = x_n + \frac{h}{760}\left(19f_n + 392f_{n+p} + 729f_{n+1} + 729f_{n+2} + 392f_{n+r} + 19f_{n+3}\right),
\end{array}\right\} \tag{11}$$

3. Basic Properties of the THSOHBM

This section examines the basic properties of the THSOHBM (11) namely; accuracy, consistency, zero-stability, convergence, linear stability, and *A*-stability are investigated.

3.1. Order of Accuracy and Consistency

Rewriting the THSOHBM (11) in the matrix difference form yields

$$A_1 X_n = A_0 X_{n-3} + h(B_0 F_{n-3} + B_1 F_n), \tag{12}$$

Where A_0, A_1, B_0, and B_1 are 6×6 matrices given by

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix};\ A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix};\ B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{817(539+50\sqrt{35})}{7299040} \\ 0 & 0 & 0 & 0 & 0 & \frac{323}{61560} \\ 0 & 0 & 0 & 0 & 0 & \frac{38}{3040} \\ 0 & 0 & 0 & 0 & 0 & \frac{57}{7695} \\ 0 & 0 & 0 & 0 & 0 & \frac{-817(50\sqrt{35}-539)}{7299040} \\ 0 & 0 & 0 & 0 & 0 & \frac{19}{760} \end{bmatrix} \tag{13}$$

$$B_1 = \begin{bmatrix} \frac{1882384-130928\sqrt{35}}{7299040} & \frac{3441123-619650\sqrt{35}}{7299040} & \frac{36480\sqrt{35}}{7299040} & \frac{3560193-619650\sqrt{35}}{7299040} & \frac{1882384-130928\sqrt{35}}{7299040} & \frac{19(-13573+2150\sqrt{35})}{7299040} \\ \frac{16072-3136\sqrt{35}}{61560} & \frac{39393}{61560} & \frac{19456}{61560} & \frac{8073}{61560} & \frac{16072-3136\sqrt{35}}{61560} & \frac{1083}{61560} \\ \frac{784+147\sqrt{35}}{3040} & \frac{2673}{3040} & 0 & \frac{243}{3040} & \frac{784-147\sqrt{35}}{3040} & \frac{38}{3040} \\ \frac{3920-784\sqrt{35}}{7695} & \frac{12744}{7695} & \frac{4864}{7695} & \frac{4914}{7695} & \frac{3920-784\sqrt{35}}{7695} & \frac{304}{7695} \\ \frac{1882384+130928\sqrt{35}}{7299040} & \frac{3441123+619650\sqrt{35}}{7299040} & \frac{-36480\sqrt{35}}{7299040} & \frac{3560193+619650\sqrt{35}}{7299040} & \frac{1882384+130928\sqrt{35}}{7299040} & \frac{19(-13573-2150\sqrt{35})}{7299040} \\ \frac{392}{760} & \frac{729}{760} & 0 & \frac{729}{760} & \frac{392}{760} & \frac{19}{760} \end{bmatrix} \quad (14)$$

$$X_n = (x_{n+p}, x_{n+1}, x_{n+q}, x_{n+2}, x_{n+r}, x_{n+3})^T, X_{n-3} = (x_{n-3+p}, x_{n-2}, x_{n-3+q}, x_{n-1}, x_{n-3+r}, x_n)^T, F_n = (f_{n+p}, f_{n+1}, f_{n+q}, f_{n+2}, f_{n+r}, f_{n+3})^T, F_{n-3} = (f_{n-3+p}, f_{n-2}, f_{n-3+q}, f_{n-1}, f_{n-3+r}, f_n)^T. \quad (15)$$

In line with [5, 27], for a sufficiently differentiable test function $\varphi(t)$ in the interval $[0, T]$, let the difference operator $\bar{D}$ be defined as

$$\bar{D}(x(\mathrm{t}); h) = \sum_{j=\omega}\left[\mathrm{u}(t)(\mathrm{t}+jh) - h\mathrm{v}_j(t)\varphi'(\mathrm{t}+jh)\right], \omega = 0, p, 1\, q, 2, r, 3. \quad (16)$$

Where, u_j and v_j are column vectors of the matrices A_0 and A_1, respectively. The Taylor series expansion about t for $x(t+jh)$ and $x'(t+jh)$ yield

$$\bar{L}(\varphi(\mathrm{t}); h) = c_0 x(\mathrm{t}) + c_1 h x'(\mathrm{t}) + c_2 h^2 x^{(2)}(\mathrm{t}) + \cdots + c_\rho h^\rho x^{(\rho)}(\mathrm{t}) + \cdots \quad (17)$$

where $c_\rho, i = 0,1,2, \ldots$ are vectors obtained as

$$c_0 = \sum_{j=0}^{m} u_j,$$

$$c_1 = \sum_{j=1}^{m} j u_j - \sum_{j=0}^{n} v_j\,,$$

$$c_2 = \frac{1}{2!}\sum_{j=1}^{m} j^2 u_j - 2\sum_{j=1}^{n} j v_j,$$

$$\vdots$$

$$c_\rho = \frac{1}{\rho!}\left(\sum_{j=1}^{m} j^\rho u_j - \frac{1}{(\rho-1)!}\sum_{j=1}^{n} j^{\rho-1} v_j,\right), \rho = 3,4, \ldots$$

The LMM in equation (5) is said to be of order ρ if

$$\bar{D}(x(\mathrm{t}); h) = O(h^{\rho+1}), c_0 = c_1 = \cdots = c_\rho = 0, c_{\rho+1} \neq 0 \quad (18)$$

Hence, $c_{\rho+1}$ is the error constant and $c_{\rho+1} h^{\rho+1} x^{(\rho+1)}(t_n)$ is the principal local truncation error at the point t_n. The computation of the order and error constant shows that the order of the THSOHBM (11) is $\rho = (7,7,7,7,7,8)^T$ with the corresponding error constants

$$c_{p+1} = \frac{1269}{172103680}, \frac{-1}{21168}, \frac{-297}{8028160}, \frac{-1}{21168}, \frac{1269}{172103680}, \frac{81}{8780800} \quad (19)$$

showing that the THSOHBM has at least seventh order accuracy.

Since $p \geq 1$, then the block method THSOHBM (11) is consistent (see [19]).

3.2. Zero-stability and Convergence

The zero-stability pertains to the stability of the difference system in (12) in the limit as $h \to 0$. As $h \to 0$, (12) becomes

$$A_1 X_n - A_0 X_{n-3} = 0 \quad (20)$$

The first characteristic polynomial $\rho(\sigma) = det\,(\sigma A_1 - A_0) = \sigma^5(\sigma - 1) = 0$. Thus, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = 0, \sigma_6 = 1$. Hence the block method (11) is zero-stable.

Since the THSOHBM satisfy the properties of consistency and zero-stability, then the method is convergent according to [5, 28].

3.3. Linear Stability

Consider the linearized test problem

$$x'(t) = \sigma x(t), Re(\sigma) < 0 \quad (21)$$

Applying the proposed block method to the trial problem (19), we obtain the recurrence relation

$$X_n = H(\hbar)X_{n-1}, \hbar = \sigma h \quad (22)$$

where the matrix $H(\hbar)$ is given by $(A_1 - rB_0)^{-1}(A_0 - rB_0)$. The stability property of this matrix's eigenvalues, which governs how the numerical solution behaves, is the spectral radius, $H(\hbar)$, which is used in the method to define the region of absolute stability S. The method is A-stable if

$$S = \{\hbar \in C: |\rho[H(\hbar)]| < 1\} \quad (23)$$

After various computations and simplifications, it becomes evident that the dominant eigenvalue can be expressed as a rational function.

$$\rho[H(\hbar)] = \frac{27\hbar^6+387\hbar^5+2502\hbar^4+9780\hbar^3+24160\hbar^2+35280\hbar+23520}{27\hbar^6-387\hbar^5+2502\hbar^4-9780\hbar^3+24160\hbar^2-35280\hbar+23520} \quad (24)$$

which has a modulus of less than one in C− (see Figure 1). The plot of the order star in Figure 1 which reveal that there are no poles in the left-half complex plane (according to [28]) validates the result obtained from the region of absolute stability. Hence, the THSOHBM (11) is A-stable.

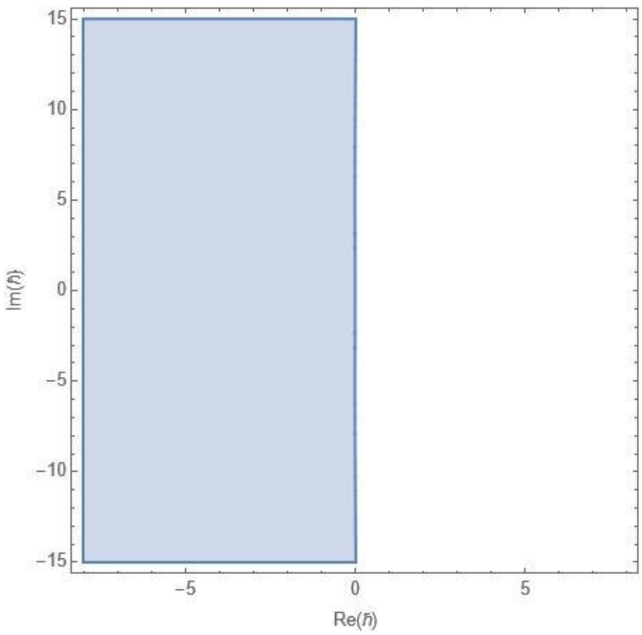

Figure 1. *Region of absolute stability of the THSOHBM.*

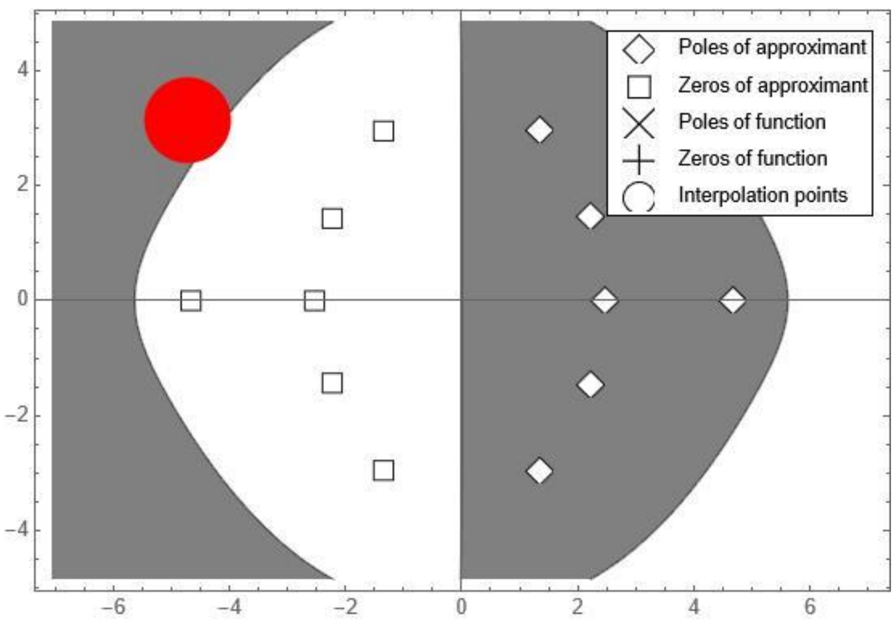

Figure 2. *Order star of the THSOHBM.*

4. Results

The accuracy of the THSOHBM is demonstrated by applying the method to solve three popular applied problems of the form (1). Comparison of the performance of the THSOHBM is made with method (3.2), and method (3.4) in [29]. The results are presented Tables 1, 2, and 3. Afterwards, the THSOHBM is implemented to solve a real-life problem: the mathematical model of Typhoid fever.

Example 1

Consider the stiff nonlinear system of IVPs of ODEs also known as Kaps problem which has appeared in [18, 29]:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1002x_1(t) + 1000x_2(t)^2 \\ x_1(t) - x_2(t)\big(1 + x_2(t)\big) \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (25)$$

The exact solution is $x(t) = \sin x$.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \exp(-2t) \\ \exp(-t) \end{bmatrix}$$

The problem is solved in the interval [0,10] for $t = 5, 50, 150, 250, ,500$. The computed values and errors for this example are shown in Table 1 while Figure 3 shows the solution plot.

Example 2

Given the linear stiff problem investigated by [18, 29]:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -10x_1 + 21x_2 \\ -21x_1 - 10x_2 \\ 10x_1 \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_1(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (26)$$

with exact solution

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_1(t) \end{bmatrix} = \begin{bmatrix} \exp(-10t)(\cos 21t + \sin 21t) \\ \exp(-10t)(\cos 21t - \sin 21t) \\ \exp(-10t) \end{bmatrix}$$

The problem is solved in the interval [0,0.1] for $t = 5, 50, 150, 250, 500$. The computed values and errors for this example are shown in Table 2 while Figure 4 shows the solution plot.

Example 3

Consider the first order stiff initial value problem has appeared in [18]:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -21x_1 + 19x_2 - 20x_1 \\ 19x_1 - 21x_2 + 20x_1 \\ 40x_1 - 40x_2 - 40x_1 \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_1(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad (27)$$

The exact solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_1(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\exp(-2t) + \exp(-40t)\,(\cos 40t + \sin 40t)) \\ \frac{1}{2}(\exp(-10t) - \exp(-40t)\,(\cos 40t + \sin 40t)) \\ \exp(-40t)\,(\cos 40t + \sin 40t) \end{bmatrix}.$$

The problem is solved in the interval [0,1] for $t = 5, 50, 250, 500$. The computed values and errors for this example are shown in Table 3 while Figure 5 shows the solution plot.

Example 4

We consider the mathematical model of typhoid fever with optimal control proposed by [30]:

$$\begin{cases} \frac{dV}{dt} = \alpha\Lambda + \nu\,\mu_2 S - \left(\sigma + \frac{(1-u_1)\rho_\varepsilon B_C}{(k+B_C)} + \mu\right)V \\ \frac{dS}{dt} = (1-\alpha)\Lambda + \sigma V + \Psi T - \left(\nu + \frac{(1-u_1)\rho B_C}{(k+B_C)} + \mu\right)S \\ \frac{dE}{dt} = \frac{(1-u_1)\rho B_C}{(k+B_C)}S + \frac{(1-u_1)\rho_\varepsilon B_C}{(k+B_C)}V - (q_1\Upsilon + (1+q_1)\Upsilon + \mu)E \\ \frac{dI_A}{dt} = q_1\Upsilon E - (\nu\phi_1 u_3 + (1-\nu)\phi_1 u_3 + \sigma_1 + \mu_1 + \mu)I_A \\ \frac{dI_S}{dt} = (1-q_1)\Upsilon E - (\omega\phi_2 u_3 + (1-\omega)\phi_2 u_3 + \sigma_2 + \mu_0 + \mu)I_S \\ \frac{dR_S}{dt} = (1-\nu)\phi_1 u_3 I_A + (1-\omega)\phi_2 u_3 I_S - (\theta u_4 + \mu_2 + \mu)R_S \\ \frac{dT}{dt} = \nu\phi_1 u_3 I_A + \omega)\phi_2 u_3 I_S + \theta u_4 R_S - (\Psi + \mu)T \\ \frac{dB_C}{dt} = \sigma_1 I_A + \sigma_2 I_S - \delta u_5 B_C \end{cases} \quad (28)$$

with initial conditions:

$$\begin{cases} V(0) = V_0 \geq 0, S(0) = S_0 \geq 0, E(0) = E_0 \geq 0, I_A = I_{A0} \geq 0, \\ I_S = I_{S0} \geq 0, R_S = R_{S0} \geq 0, T(0) = T_0 \geq 0, B_C(0) = B_{C0} \geq 0. \end{cases}$$

where, $V(t)$ is the number of vaccinated human in the population at time t, $S(t)$ is the number of susceptible humans in the population at time t, $E(t)$ Number of infectious human in latent period at time t, $I_A(t)$ is the number of asymptomatic infectious human in the population at time t, $I_S(t)$ is the number of symptomatic infectious human in the population at time t, $R_S(t)$ is the number of human who are resistant to treatment at time t, $T(t)$ is the number of treated human(recovered) in the population at time t, and $B_C(t)$ is the total number of bacteria population at time t. Figures 6, 7, 8, 9, 10, and 11 are the solution plots of exposed individual, asymptotic individual, symptomatic individual, resistant individual, bacteria population, and all variables respectively for the mathematical model of Typhoid fever. The red line represents the solution for implementing all control strategies ($u_1 \neq 0, u_2 \neq 0, u_3 \neq 0, u_4 \neq 0, u_5 \neq 0,$), while the blue line represents the solution when no control strategy was implemented $u_1 = 0, u_2 = 0, u_3 = 0, u_4 = 0, u_5 = 0,$

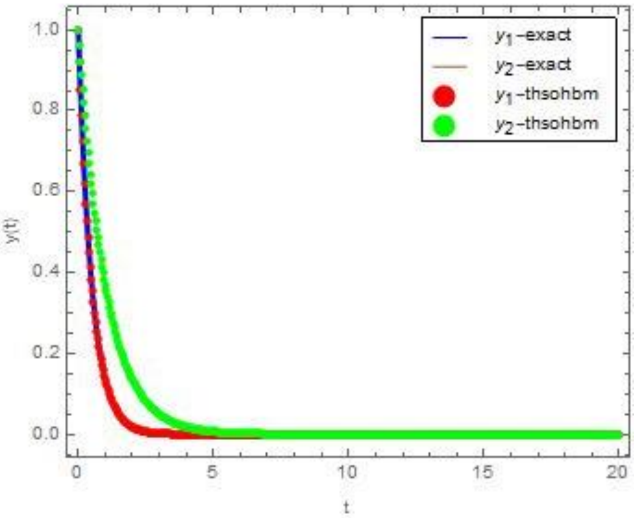

Figure 3. *Solution plot for example 1.*

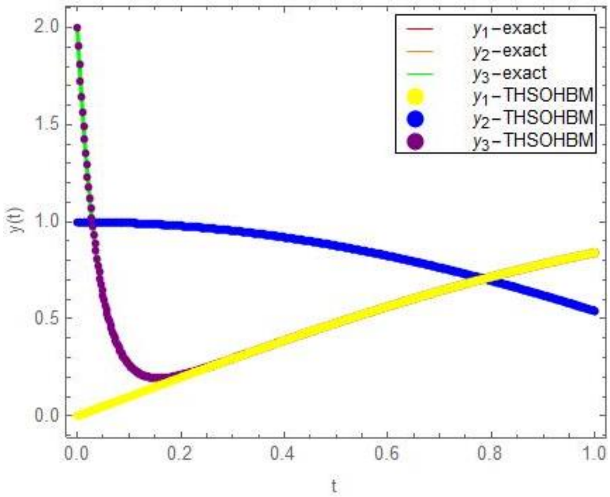

Figure 4. *Solution plot for Example 2.*

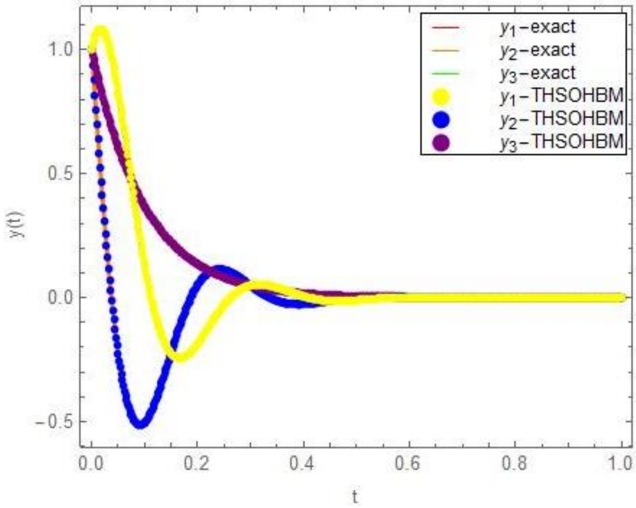

Figure 5. *Solution plot for Example 3.*

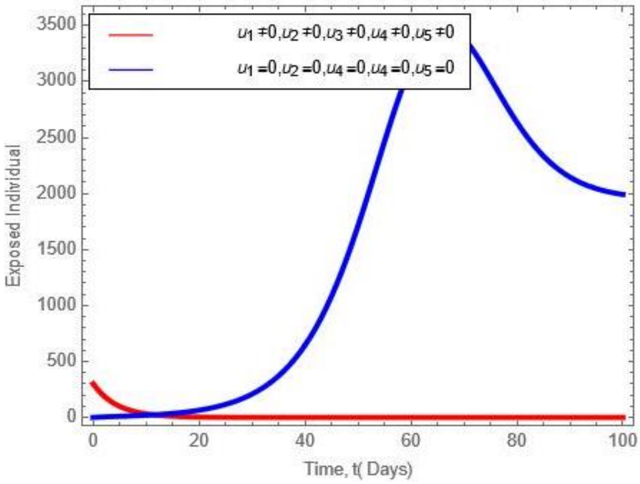

Figure 6. *Solution plots for exposed individual.*

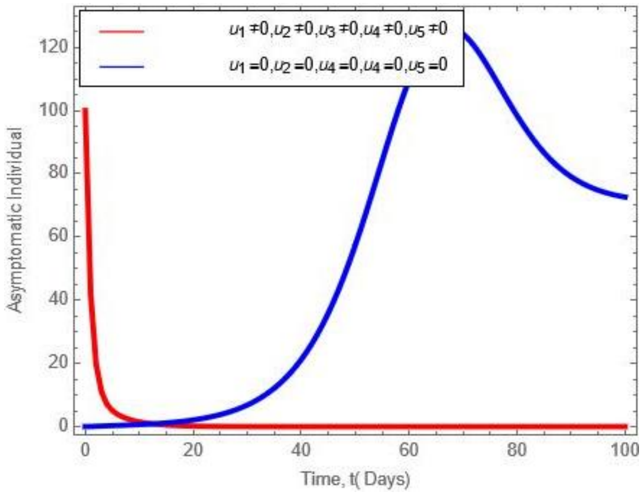

Figure 7. *Solution plot for asymptomatic individual.*

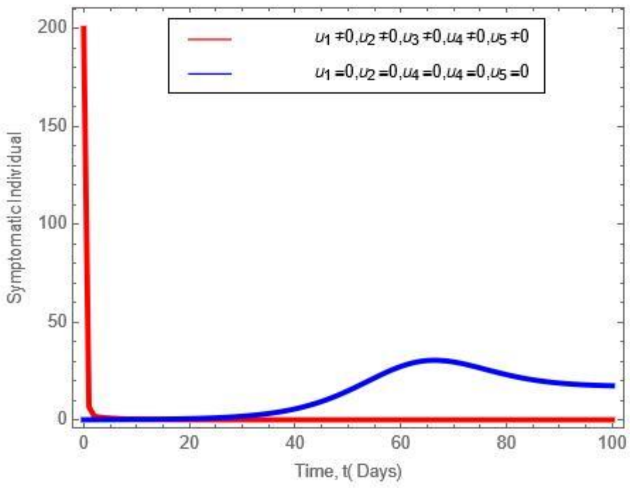

Figure 8. *Solution plot for symptomatic individual.*

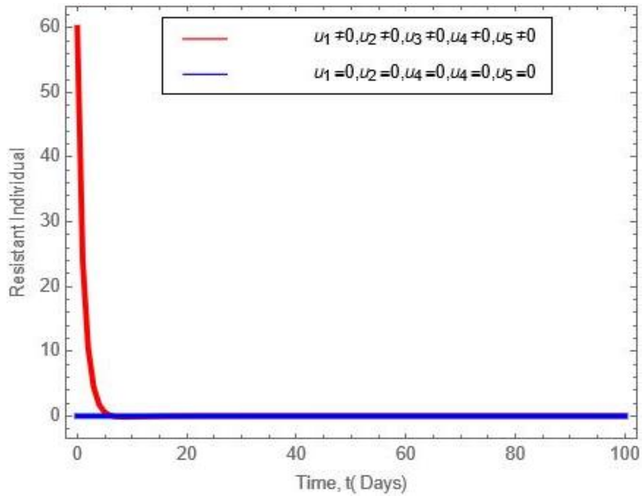

Figure 9. *Solution plot for resistant individual.*

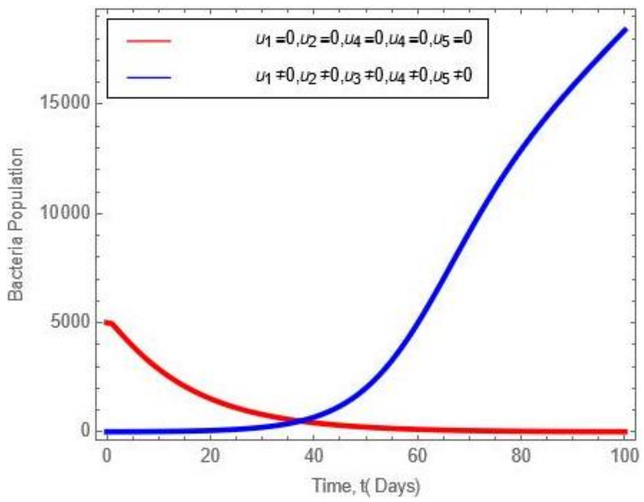

Figure 10. *Solution plot for bacteria population.*

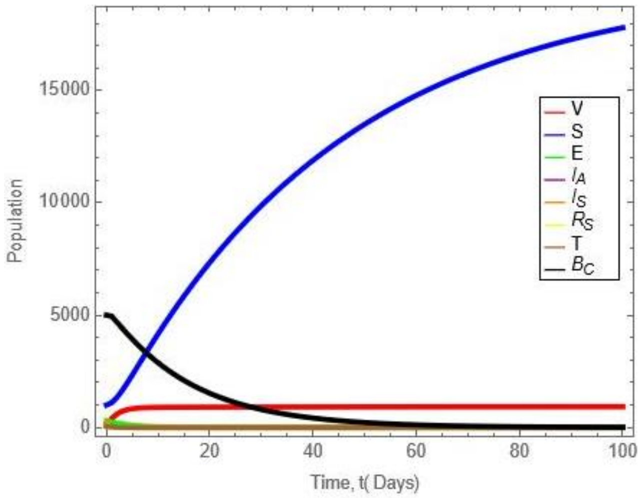

Figure 11. *Solution plot for all variables.*

Figures 6, 7, 8, 9, 10, and 11 are the solution plots of exposed individual, asymptomatic individual, symptomatic individual, resistant individual, bacteria population, and all variables respectively for the mathematical model of Typhoid fever.

5. Discussion

Table 1 shows the comparison of absolute errors in THSOHBM and Method (3.4) in [29]. The first, second, third, fourth and fifth columns indicate number of iterations, dependent variable, exact solution, computed solution, error in Method (3.4), and error in THSOHBM respectively. The result indicates that even though the order of Method (3.4) (p =14) is twice that of the THSOHBM (p = 7), the error in the THSOHBM is approximately half of that in Method (3.4). This shows that the accuracy of THSOHBM is approximately twice that of Method (3.4).

Table 2 shows the comparison of absolute errors in THSOHBM and Method (3.4) in [29]. The first, second, third, fourth and fifth columns indicate number of iterations, dependent variable, exact solution, computed solution, error in Method (3.4), and error in THSOHBM respectively. The results indicate that the THSOHBM compares favorably with Method (3.4). More specifically, it can be observed from the Table that the accuracy of the THSOHBM improves as the number of iteration increases thereby establishing the good performance of the method.

Table 3 shows the comparison of absolute errors in THSOHBM and Method (3.2) in [29]. The first, second, third, fourth and fifth columns indicate number of iterations, dependent variable, exact solution, computed solution, error in Method (3.2), and error in THSOHBM respectively. The results indicate that the THSOHBM performs better than Method (3.2). More specifically, it can be observed from the Table that the accuracy of the THSOHBM improves as the number of iteration increases thereby establishing the good performance of the new method.

Table 1. Comparative analysis of absolute errors in the numerical integration of Example 1.

t	x	Exact value	Computed value	Error in Method (3.4) [29] p=14	Error in THSOHBM p=7
5	x_1	$4.2483542552916 \times 10^{-18}$	$2.791593776051 \times 10^{-7}$	$5.82586126945793 \times 10^{-2}$	$4.8023872845655 \times 10^{-4}$
	x_2	$2.0611536224386 \times 10^{-9}$	$1.3578674743237 \times 10^{-8}$	$3.22595741549568 \times 10^{-2}$	$8.5689159364364 \times 10^{-3}$
50	x_1	$4.2483542552917 \times 10^{-18}$	$4.1535846176355 \times 10^{-18}$	$6.73587600368532 \times 10^{-3}$	$2.8426854425945 \times 10^{-8}$
	x_2	$2.0611536224386 \times 10^{-9}$	$2.0611559603033 \times 10^{-9}$	$2.61818804334955 \times 10^{-2}$	$2.2718916192765 \times 10^{-8}$
150	x_1	$4.2483542552919 \times 10^{-18}$	$4.2479274780906 \times 10^{-18}$	$2.46861111282455 \times 10^{-6}$	$1.7232548721324 \times 10^{-11}$
	x_2	$2.0611536224386 \times 10^{-9}$	$2.0611536237596 \times 10^{-9}$	$5.36087903326521 \times 10^{-4}$	$1.3339218618569 \times 10^{-11}$
250	x_1	$4.2483542552922 \times 10^{-18}$	$4.2483855405243 \times 10^{-18}$	$8.16360724925787 \times 10^{-10}$	$5.0254245209658 \times 10^{-13}$
	x_2	$2.0611536224387 \times 10^{-9}$	$2.0611536224767 \times 10^{-9}$	$9.75974730914864 \times 10^{-6}$	$3.9185321654145 \times 10^{-13}$
500	x_1	$4.2483542552937 \times 10^{-18}$	$4.2484036600869 \times 10^{-18}$	$1.61658927943642 \times 10^{-18}$	$2.8865798640254 \times 10^{-15}$
	x_2	$2.0611536224391 \times 10^{-9}$	$2.0611536224385 \times 10^{-9}$	$4.34316552414621 \times 10^{-10}$	$2.2759572004816 \times 10^{-15}$

Table 2. Comparative analysis of absolute errors in the numerical integration of Example 2.

t	x	x-exact	x-computed	Method (3.4) [29] p=14	THSOHBM
5	x_1	$1.31172372 \times 10^{-5}$	$2.1495192784443 \times 10^{-4}$	$6.66133814775094 \times 10^{-16}$	$2.0183469056255 \times 10^{-4}$
	x_2	$-6.28509771 \times 10^{-5}$	$1.9044017586245 \times 10^{-4}$	$2.88657986402541 \times 10^{-15}$	$2.53291153030411 \times 10^{-4}$
	x_3	$4.53999297 \times 10^{-5}$	$4.6155716022475 \times 10^{-5}$	$8.88178419700125 \times 10^{-16}$	$7.5578625999027 \times 10^{-7}$
50	x_1	$1.31172372 \times 10^{-5}$	$1.3117381060627 \times 10^{-5}$	$7.35522753814166 \times 10^{-15}$	$1.43778743393801 \times 10^{-10}$
	x_2	$-6.28509771 \times 10^{-5}$	$-6.2850977167963 \times 10^{-5}$	$2.22044604925031 \times 10^{-15}$	$2.37564439298297 \times 10^{-10}$
	x_3	$4.53999297 \times 10^{-5}$	$4.5399929995543 \times 10^{-5}$	$9.99200722162641 \times 10^{-16}$	$2.33057869606767 \times 10^{-13}$
150	x_1	$1.31172372 \times 10^{-5}$	$1.3117237327539 \times 10^{-5}$	$1.74166236988071 \times 10^{-15}$	$4.56586181487922 \times 10^{-14}$
	x_2	$-6.28509771 \times 10^{-5}$	$-6.2850977031803 \times 10^{-5}$	$1.78329573330416 \times 10^{-15}$	$1.36159340096212 \times 10^{-13}$
	x_3	$4.53999297 \times 10^{-5}$	$4.5399929762603 \times 10^{-5}$	$1.08940634291343 \times 10^{-15}$	$1.19363882927076 \times 10^{-16}$
250	x_1	$1.31172372 \times 10^{-5}$	$1.3117237283012 \times 10^{-5}$	$1.49186218934005 \times 10^{-16}$	$1.12886621979422 \times 10^{-15}$
	x_2	$-6.28509771 \times 10^{-5}$	$-6.2850977164005 \times 10^{-5}$	$5.73759789679329 \times 10^{-16}$	$3.95844923679889 \times 10^{-15}$
	x_3	$4.53999297 \times 10^{-5}$	$4.5399929762487 \times 10^{-5}$	$2.26381413614973 \times 10^{-16}$	$2.77149180341607 \times 10^{-18}$
500	x_1	$1.31172372 \times 10^{-5}$	$1.3117237281892 \times 10^{-5}$	$7.63854311833928 \times 10^{-18}$	$8.43306002286381 \times 10^{-18}$
	x_2	$-6.28509771 \times 10^{-5}$	$-6.2850977167928 \times 10^{-5}$	$1.32814766129474 \times 10^{-18}$	$3.46944695195361 \times 10^{-17}$
	x_3	$4.53999297 \times 10^{-5}$	$4.5399929762483 \times 10^{-5}$	$3.59819595993627 \times 10^{-18}$	$1.64663204946236 \times 10^{-18}$

Table 3. *Comparative analysis of absolute errors in the numerical integration of Example 3.*

t	x	x-exact	x-computed	Method (3.2) [29] p=10	THSOHBM p=7
5	x_1	0.0676676416	0.0676676416	$5.66046680552379 \times 10^{-5}$	$2.56824169098113 \times 10^{-7}$
	x_2	0.0676676416	0.0676676416	$5.66087578904514 \times 10^{-5}$	$2.57013017979091 \times 10^{-7}$
	x_3	$5.9988938182 \times 10^{-18}$	$1.7479720641 \times 10^{-5}$	$1.61778352103514 \times 10^{-5}$	$1.97327652635001 \times 10^{-6}$
50	x_1	0.0676676416	0.0676676416	$4.19604912917926 \times 10^{-5}$	$1.93178806284777 \times 10^{-14}$
	x_2	0.0676676416	0.0676676416	$4.19147028993261 \times 10^{-5}$	$2.11497486191092 \times 10^{-14}$
	x_3	$5.9988938182 \times 10^{-18}$	$-2.7043560586 \times 10^{-20}$	$2.15757151141333 \times 10^{-4}$	$3.20620197745928 \times 10^{-14}$
250	x_1	0.0676676416	0.0676676416	$2.49393655449293 \times 10^{-7}$	$4.46864767411626 \times 10^{-15}$
	x_2	0.0676676416	0.0676676416	$9.34237112670822 \times 10^{-8}$	$4.6074255521944 \times 10^{-15}$
	x_3	$5.9988938182 \times 10^{-18}$	$8.111854085 \times 10^{-18}$	$1.94078737508479 \times 10^{-7}$	$8.85356452508551 \times 10^{-18}$
500	x_1	0.0676676416	0.0676676416	$9.47822290653377 \times 10^{-8}$	$8.88178419700125 \times 10^{-15}$
	x_2	0.0676676416	0.0676676416	$9.47988235966424 \times 10^{-8}$	$8.88178419700125 \times 10^{-15}$
	x_3	$5.9988938182 \times 10^{-18}$	$4.6572523055 \times 10^{-20}$	$3.16824322296340 \times 10^{-11}$	$7.34713644616456 \times 10^{-17}$

6. Conclusions

This work has presented an accurate three-step optimized hybrid block method for the solution of continuous fever. The method incorporated three hybrid points with a three-parameter approximation. The technique was designed such that the interval of integration determines the best hybrid points through the optimization of the principal term of the local truncation error of the main method. Consequently, the accuracy of the resulting numerical scheme was greatly enhanced as demonstrated in the numerical results obtained when the method was implemented to solve some well-known initial value problems of ordinary differential equations. The scheme was then implemented to solve the mathematical model of Typhoid fever. It was also established through an in-depth analysis that the method is consistent, convergent, zero-stable and efficient for solving first-order ordinary differential equations. Hence, the newly developed method is highly recommended for the solution differential systems.

Abbreviations

THSOHBM	Three Step Optimized Hybrid Block Method
LTE	Local Truncation Error
IVPs	Initial Value Problems

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Author Contributions

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Temitayo Emmanuel Olaosebikan: Methodology, Writing - review & editing

Opeyemi Vincent Omole: Formal Analysis, Writing - review & editing

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Data Availability Statement

The data supporting the research is included in the article.

Conflicts of Interest

The authors declare no conflicts of interest.

Appendix

Table A1. *Typhoid fever model parameters notation and values.*

Parameters	Description	Value
$\alpha\Lambda$	Recruitment rate into the vaccinated human compartment	467 human/day
μ	Natural death rate of human	0.02347
$(1-\alpha)\Lambda$	Recruitment rate into the susceptible human compartment	0.3
ν	Vaccination wine rate	0.0009041
σ	Vaccination rate	0.5
ρ	Rate of bacteria ingestion	0.9
k	Concentration of Salmonella bacteria in foods and waters	50,000
ε	Modification parameter for $V(t)$	0.35
q_1	Proportion of humans in latent that become asymptomatic infectious	0.3
γ	Rate of progression to symptomatic and asymptomatic respectively	0.125
ϕ_1	Rate of treatment of asymptomatic infectious humans	0.000315/day
v	Proportion of asymptomatic humans that become treated	9.041*E-04
μ_1	Disease-induced death rate of asymptomatic infectious humans	0.01/year
σ_1	Rate of bacteria excretion from asymptomatic infectious humans	1
ϕ_2	Rate of treatment of symptomatic infectious humans	0.0657/day
ω	Proportion of symptomatic humans that become treated	6/year
μ_0	Disease-induced death rate of symptomatic infectious humans	0.012
σ_2	Rate of bacteria excretion from symptomatic infectious humans	10
θ	Rate of treatment of resistant humans	0.75
μ_2	Disease-induced death rate of resistant humans	0.01
ψ	Rate of treated humans that loosed immunity and become susceptible	0.05
δ	Death rate of typhoid-causing bacteria	0.0645/day

Source: [30]

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