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Research Article

A Method to Improve the Multiplicative Inconsistency Preserving the Preference Information of Every Element of an Intuitionistic Fuzzy Preference Relation

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Abstract

In general, almost intuitionistic fuzzy preference relations (IFPRs) provided by experts are multiplicutively inconsistent because of the complexity of a problem, lack of correct or sufficient knowledge about the problem domain, the ambiguity inherent in human thinking and so forth on. To solve this subject, we propose a method to improve the multiplicative inconsistency preserving the preference information of every element of an initial IFPR. For this, we formulate a formula that straightforwardly calculates the multiplicative consistent IFPR preserving the preference information of every element of the IFPR. Based on it, the necessary and sufficient results for the IFPR to be multiplicatively consistent are derived. By using the results, a consistency testing matrix and a consistency index that can select the most inconsistent elements in the IFPR are constructed and a method that revises them by a proper intuitionitic fuzzy numbers for improving inconsistency as well as preserving the initial preference information is proposed. Then, it is proved that the consistency index converges into zero. As a result, an acceptable consistent IFPR that preserves the preference information of every element and saves a lot of elements of the initial IFPR is constructed. In addition, this method needs a few calculations in comparison with previous methods to improve multiplicative inconsistency of IFPRs, because they calculate a multiplicative consisten IFPR by solving the optimal models constructed based on sufficient conditions for IFPRs to be mltiplicatively consistent. Finally, illustrative examples and comparative analysis are given to demonstrate the efficiency of the proposed method.

Keywords

Intuitionistic Fuzzy Preference Relation (IFPR), Multiplicative Consistency, Preference Information, Consistency Testing Matrix, Consistency Index

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1. Introduction

In decision making problems based on preperence relations, when comparing pairs of alternatives, it may be difficult or impossible for experts to accurately assess their preferences because of problem complexity, lack of correct or sufficient knowledge about the problem domain, pressure of time and the ambiguity inherent in human thinking. To overcome this issue, Szmidt and Kacprzyk [15] represented an intuitionistic preference relation as a combination of a fuzzy preference matrix and a hesitancy matrix. Later, Xu [24] gave the simple and useful notion of an IFPR based on Atanassov's intuitionistic fuzzy sets [1]. Every element of the IFPR as a matrix is compose of preference degree, non-preference degree and indeterminacy degree and can describes fuzzy and uncertain characteristics of considered alternatives more detail and comprehensively.

For application problems based on IFPRs, the primary task is to improve its inconsistency after checking up whether it is consistent or inconsistent. In general, process of improving inconsistency consists of a definition of consistency, consistency index and a method for inconsistency improvement. An intial IFPR is frequently inconsitent owing to the complexity of a problem, lack of correct or sufficient knowledge about the problem domain, the ambiguity inherent in human thinking and so forth on. Here, the consistency index shows which the IFPR is consistent with reality to some degree. In this paper, we only discuss consistency indices based on multiplicative consistency.

Various kinds of definition are proposed in connection with multiplicative consitency of IFPRs [4, 6, 8, 13, 18, 20-25]. These defintions are based on consistencies of multiplicative preference relations and fuzzy preference relations. Based on these definitions, several forms of consistency index are proposed [2-7, 10, 13, 22, 27]. Liao et al. [10] introduced a consitency index based on the distance deviation between the IFPR and its corresponding multiplicative consistent IFPR. Jin et al. [4] presented a consistency index on the log-deviation between the IFPR and the converted multiplicative consistent IFPR. Xu et al. [22] defined a consistency index by taking log-operation to the multiplicative consistency proposed by Liao and Xu [8]. Meng et al. [13] defined a consistency index as the sum of 0-1 indicators of $2^{\frac{n(n-1)}{2}-1}$ dual preferred IFPRs with respect to the IFPR, where n indicates the order of the IFPR as a matrix. Hyonil et al. [5] proposed a consistency index based on ratio deviation between a positive reciprocal matrix and its corresponding complete consistent positive reciprocal matrix after converting the IFPR into the positive reciprocal matrix by using the multiplicative consistency by Liao and Xu [8].

To date, a great deal of research has been conducted on inconsistency improvement problems, because unacceptable consistent IFPRs could lead to an irrational result in the applications based on ones [2, 3, 5-7, 9-14, 19, 21, 22, 26-28]. Liao et al. [10] obtained multiplicatively consistent IFPR through an optimal model based on a transformation formula to convert the normalized intuitionistic fuzzy priority weights into into a multiplicative consistent IFPR. Xu et al. [22] straightforwardly calculated an accptable consistent IFPR from an optimal model based on taking log-operation to the definition expression for the multiplicative consistency proposed in [8]. Hyonil et al. [5] proposed a method that revised the potential inconsistent elements by the consistent elements after converting the IFPR into a positive reciprocal matrix using the multiplicative consistency proposed in [8].

Based on reviewing previous methods to improve multiplicative inconsistency of IFPRs, we find that they have limitations in some aspects:

First, a lot of methods depends on intuitionistic fuzzy priority weights obtained through optimal models based on sufficient conditions for IFPRs to be multiplicatively consistent (Model 1, Theorem 1). When the number of alternatives is large, solutions to various models are very difficult to calculate in real time and solutions also include inherent errors.

Second, every element of an acceptable consistent IFPR is obtained by combining every element of a multiplicativel consistent IFPR with the corresponding element of the initial IFPR (Eq. (6)). Preference information of almost elements of the acceptable consistent IFPR constructed by these methods not only are differenr from preference information of the corresponding elements of the initial IFPR, but elements themselves are different. In fact, improving multiplicative inconsistency is to obtain the rational result in various applications based on IFPRs not that preference information between comparative alternatives are newly known. Therefore, the preference information of every element of the initial IFPR ought not change in improving process. In addition, an unacceptable consistency is caused by some elements not all.

In order to overcome these limitations, we formulate a formula that can strightforwardly calculates multiplicatively consistent IFPR preserving the preference information of every element of the initial IFPR without solving any optimal model and present a method that revises the most inconsitent elements by a proper intuitionitic fuzzy numbers for imroving inconsistency as well as preserving the initial preference information of every element of the IFPR.

The rest of the paper is set out as follows: Section 2 remembers multiplicative consistencies in fuzzy preference relations and IFPRs and reviews methods to imrove the multiplicative inconsistency. In section 3, we give a formula to strightforwardly calculate multiplicatively consistent IFPR preserving the preference information of every element of the initial IFPR and derive the necessary and sufficient results for the IFPR to be multiplicatively consistent. Section 4 constitutes a consistency-testing matrix and a consistency index and proves a theorem that the most inconsistent elements correspond to the most values of the consistency-testing matrix. Based on the theorem, we construct an algorithm to improve the multiplicative inconsistency of the IFPR and prove a theorem that the consistency index converges to zero. The paper ends conclusions in Section 5.

2. Preliminary

Let $X=\{x_1,x_2,...,x_n\}$ $(n\geq 3)$ be the set of feasible alternatives and $N=\{1,2,\cdots,n\}$. Then, experts compare each pair (x_i,x_j) of alternatives as to express their preferences.

In the paper, we describe fuzzy preference relations [16] and intuitionistic fuzzy preference relations [23].

Definition 1 [16]. A fuzzy preference relationon $B=(b_{ij})_{n\times n}$., $(b_{ij}\in[0,1])$ on the set X is a square matrix, that is additive reciprocal $b_{ij}=1-b_{ji}$, $i,j\in N=\{1,2,\cdots,n\}$, where b_{ij} denotes the preference degree of the alternative x_i over x_j.

Definition 2 [23]. An IFPR $R=(r_{ij})_{n\times n}$ $(r_{ij}=(\mu_{ij},\nu_{ij})$ on the set X is characterized by the following conditions.

$$\mu_{ij},\nu_{ij}\in[0,1],\ \mu_{ii}=\nu_{ii}=0.5,\ \mu_{ij}=\nu_{ji},$$
$$\mu_{ij}+\nu_{ij}\leq 1,\ i,j\in N,$$

where μ_{ij} indecates the preference degree to which alternative x_i is preferred to x_j and ν_{ij} denotes the non-preference degree to which the alternative x_i is not preferred to x_j. The sum of preference degree and non-preference degree, $\mu_{ij}+\nu_{ij}$ is called the preference information between alternatives x_i and x_j. In addition, $\pi_{ij}=1-(\mu_{ij}+\nu_{ij})$ is interpreted as the indeterminacy degree between alternative x_i and alternative x_j, i.e., the amount of the unknown information.

Definition 3 [16]. A fuzzy preference relation $B=(b_{ij})_{n\times n}$ is called multiplicatively consistent if it satisfies the multiplicative transitivity:

$$b_{ij}b_{jk}b_{ki}=b_{ik}b_{kj}b_{ji},\ i,j,k\in N \tag{1}$$

Meng et al. [13] presentet proporty that the fuzzy preference relation $B=(b_{ij})_{n\times n}$ is multiplicatively consistent if and only if the following condition is true:

$$b_{ij}=\frac{\prod_{s=1}^{n}(b_{is}b_{sj})^{\frac{1}{n}}}{\prod_{s=1}^{n}(b_{is}b_{sj})^{\frac{1}{n}}+\prod_{s=1}^{n}(b_{js}b_{si})^{\frac{1}{n}}},\quad i,j\in N \tag{2}$$

Liao and Xu [8] extended the multiplicative consistency of fuzzy preference relations to define the multiplicative consistency of IFPRs.

Definition 4 [8]. Let $R=(r_{ij})_{n\times n}(r_{ij}=(\mu_{ij},\nu_{ij}))$ be an IFPR. Then, it is called multiplicative consistency if IFPR R satisfies the condition:

$$\mu_{ij}\mu_{jk}\mu_{ki}=\nu_{ij}\nu_{jk}\nu_{ki},\ i,j,k\in N \tag{3}$$

Liao and Xu [8] proposed the transformation formula between the multiplicative consistent IFPR $R=(r_{ij})_{n\times n}(r_{ij}=(\mu_{ij},\nu_{ij}))$ and a normalized intuitionistic fuzzy priority weight's vector $w=(w_1,w_2,...,w_n)$ $(w_i=(w_i^{\mu},w_i^{\nu}))$:

$$r_{ij}=(\mu_{ij},\nu_{ij})=\begin{cases}(0.5,0.5) & i=j\\ \left(\dfrac{2w_i^{\mu}}{w_i^{\mu}-w_i^{\nu}+w_j^{\mu}-w_j^{\nu}+2},\dfrac{2w_j^{\mu}}{w_i^{\mu}-w_i^{\nu}+w_j^{\mu}-w_j^{\nu}+2}\right) & i\neq j\end{cases}, \tag{4}$$

where $w_i^{\mu},\ w_i^{\mu}\in(0,1),\ w_i^{\mu}+w_i^{\mu}\leq 1,\ \sum_{i=1,i\neq j}^{n}w_i^{\mu}\leq w_j^{\nu},w_j^{\mu}+n-2\geq\sum_{i=1,i\neq j}^{n}w_i^{\nu},\ i,j\in N$.

Then the following theorem is established as to Eq. (3).

Theorem 1. IFPR $R=(r_{ij})_{n\times n}(r_{ij}=(\mu_{ij},\nu_{ij}))$ in which the elements r_{ij} $(i,j\in N)$ are expressed as in Eq. (3) by a normalized intuitionistic fuzzy priority weight's vector $w=(w_1,w_2,...,w_n)$ $(w_i=(w_i^{\mu},w_i^{\nu}))$ is multiplicatively consistent.

Theorem 1 is nothing but it is sufficient condition for IFPR to be multiplicatively consistent. Based on the transformation formula (3), Liao and Xu [8] constructed a fractional programming model to derive an underlying normalized intuitionistic fuzzy priority weight's vector with respect to alternatives from IFPR $R=(r_{ij})_{n\times n}(r_{ij}=(\mu_{ij},\nu_{ij}))$:

Model 1. $Min\ Z^* = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\varepsilon_{ij}^+ + \varepsilon_{ij}^- + \eta_{ij}^+ + \eta_{ij}^-)$

$$s.t.\begin{cases} \frac{2w_i^\mu}{w_i^\mu - w_i^\nu + w_j^\mu - w_j^\nu + 2} - \mu_{ij} - \varepsilon_{ij}^+ + \varepsilon_{ij}^- = 0,\ i=1,2,\cdots,n-1,\ j=1,2,\cdots,n \\ \frac{2w_j^\mu}{w_i^\mu - w_i^\nu + w_j^\mu - w_j^\nu + 2} - \mu_{ij} - \eta_{ij}^+ + \eta_{ij}^- = 0,\ i=1,2,\cdots,n-1,\ j=1,2,\cdots,n \\ w_i^\mu,\ w_i^\nu \in [0,1],\ i=1,2,\cdots,n \\ \sum_{j=1, j\neq i}^{n} w_j^\mu \le w_i^\nu,\ w_i^\mu + n - 2 \ge \sum_{j=1, j\neq i}^{n} w_j^\nu,\ i=1,2,\cdots,n \\ \varepsilon_{ij}^+,\ \varepsilon_{ij}^- \ge 0,\ \xi_{ij}^+,\ \xi_{ij}^- \ge 0,\ i=1,2,\cdots,n-1;\ j=1,2,\cdots,n \end{cases}. \tag{5}$$

Liao and Xu [8] established the following theorem.

Theorem 2. IFPR $R=(r_{ij})_{n\times n}$ is multiplicatively consistent if and only if $Z^*=0$ where Z^* is the optimal value of the objective function.

In general, Theorem 2 does not hold to. If $Z^*=0$, IFPR $R=(r_{ij})_{n\times n}$ is multiplicatively consistent by means of Theorem 1. However, inverse does not true. Thus, even though IFPR $R=(r_{ij})_{n\times n}$ is multiplicatively consistent, elements of IFPR $R=(r_{ij})_{n\times n}$ can not to be expressed as in Eq. (3). In sequal, Z^* is not equal to zero.

Construction of a multiplicativele consistent IFPR is nearly impossible due to the complexity of a problem, lack of correct or sufficient knowledge about the problem domain, the ambiguity inherent in human thinking and so forth on. Hence, Liao et al. [10] introduced the concept of acceptably multiplicative consistent IFPR.

Definition 5 [10]. IFPR $R=(r_{ij})_{n\times n}(r_{ij}=(\mu_{ij},\nu_{ij}))$ is called acceptably multiplicative consistent if it satisfies the following condition:

$$CI(R,\bar{R}) = \frac{1}{(n-1)(n-2)}\sum_{1\le i<j\le n}^{n}(|\mu_{ij}-\bar{\mu}_{ij}| + |\nu_{ij}-\bar{\nu}_{ij}| + |\pi_{ij}-\bar{\pi}_{ij}|) \le CI_0 \tag{6}$$

where $\bar{R}=(\bar{r}_{ij})_{n\times n}(\bar{r}_{ij}=(\bar{\mu}_{ij},\bar{\nu}_{ij}))$ is multiplicatively consistent IFPR corresponding to $R=(r_{ij})_{n\times n}$ and CI_0 is a prescribed consistency threshold.

If IFPR $R=(r_{ij})_{n\times n}(r_{ij}=(\mu_{ij},\nu_{ij}))$ is unacceptable, then acceptably multiplicative consistent IFPR $\hat{R}=(\hat{r}_{ij})_{n\times n}$ $(\hat{r}_{ij}=(\hat{\mu}_{ij},\hat{\nu}_{ij}))$ is calculated through the following formula [8, 9].

$$\hat{\mu}_{ij} = \mu_{ij}^{1-p\delta}\bar{\mu}_{ij}^{p\delta},\ \hat{\nu}_{ij} = \nu_{ij}^{1-p\delta}\bar{\nu}_{ij}^{p\delta},\ i,\ j\in N, \tag{7}$$

where p indicates the iteration number and $\delta\in(0,1)$ is the controlling parameter determined by an expert.

Xu et al. [22] defined a log-consistency index only depending on a given IFPR based on the multiplicative consistency Eq. (3).

Definition 6 [22]. Let $R=(r_{ij})_{n\times n}(r_{ij}=(\mu_{ij},\nu_{ij}))$ be an IFPR. Then the log-consistency index is defined as follows:

$$CI(R) = \frac{1}{C_n^3}\sum_{1\le i<j<k\le n}\left|(\log_2\mu_{ij}+\log_2\mu_{jk}+\log_2\mu_{ki}) - (\log_2\nu_{ij}+\log_2\nu_{jk}+\log_2\nu_{ki})\right|. \tag{8}$$

If $CI(R)\le CI_0$, then R is acceptably multiplicative consistent IFPR. Otherwise, it is unacceptable.

Xu et al. [22] have built the following optimal model to obtain an acceptable multiplicative consistent IFPR $\tilde{R}=(\tilde{r}_{ij})_{n\times n}(\tilde{r}_{ij}=(\tilde{\mu}_{ij},\tilde{\nu}_{ij}))$:

Model 2.

$$Min\ d(R,\ \tilde{R}) = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\log_2\mu_{ij}-\log_2\tilde{\mu}_{ij}\right| + \left|\log_2\nu_{ij}-\log_2\tilde{\nu}_{ij}\right|\right)$$

$$s.t.\begin{cases} CI(\tilde{R})\le CI_0 \\ \tilde{\mu}_{ij}=\tilde{\nu}_{ji},\ \tilde{\mu}_{ii}=\tilde{\nu}_{ii},\ i,j\in N \\ 0<\tilde{\mu}_{ij},\tilde{\nu}_{ij}\le 1,\ \tilde{\mu}_{ij}+\tilde{\nu}_{ij}\le 1,\ i,j\in N \end{cases}. \tag{9}$$

Remark 1. Preference information of almost elements of the acceptable multiplicative consistent IFPR calculated through one of the above Models are different from preference information of corresponding elements of the initial IFPR and the elements themselves are different.

3. The Necessary and Sufficient Results for IFPRs to Be Multiplicatively Consistent

In this section, we formulate a formula to to straightforwardly calculate multiplicatively consistent IFPR preserving the preference information of every element of the initial IFPR and prove the necessary and sufficient results for IFPRs to be multiplicatively consistent. Therefore, if the IFPR degenerates to a fuzzy preference relation, the results for one to be multiplicatively consistent are obtained.

By using the reciprocal property $\nu_{ij} = \mu_{ji}$, $i, j \in N$, Eq. (3) can be expressed as:

$$\mu_{ij}\mu_{jk}\mu_{ki} = \mu_{ik}\mu_{kj}\mu_{ji}, \ i, j, k \in N. \tag{10}$$

Then IFPR $R = (r_{ij})_{n\times n}(r_{ij} = (\mu_{ij}, \nu_{ij}))_{n\times n}$ can be represented equivalently as $R = (\mu_{ij})_{n\times n}$ by Theorem 1 of [17].

Let us calculate the following matrix $\tilde{R} = (\tilde{\mu}_{ij})_{n\times n}$ associated with IFPR $R = (\mu_{ij})_{n\times n}$:

$$\tilde{\mu}_{ij} = \begin{cases} 0.5: & i = j \\ \dfrac{\prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}}}{\prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}} + \prod_{s=1}^{n}(\mu_{js}\mu_{si})^{\frac{1}{n}}}(\mu_{ij} + \mu_{ji}): & i \neq j \end{cases}. \tag{11}$$

Theorem 3. Matrix $\tilde{R} = (\tilde{\mu}_{ij})_{n\times n}$ satisfies the following conditions:

(1) $0 \le \tilde{\mu}_{ij} + \tilde{\mu}_{ji} = \mu_{ij} + \mu_{ji} \le 1$, $i, j \in N$,

(2) $\tilde{\mu}_{ij}\tilde{\mu}_{jk}\tilde{\mu}_{ki} = \tilde{\mu}_{ik}\tilde{\mu}_{kj}\tilde{\mu}_{ji}$, $i, j, k \in N$.

Thus, the matrix $\tilde{R} = (\tilde{\mu}_{ij})_{n\times n}$ is multiplicatively consistent IFPR preserving the preference information of every element of the initial IFPR $R = (\mu_{ij})_{n\times n}$.

Proof. From Eq. (11), we have:

$$\begin{aligned}\tilde{\mu}_{ij} + \tilde{\nu}_{ij} &= \frac{\prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}}}{\prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}} + \prod_{s=1}^{n}(\mu_{js}\mu_{si})^{\frac{1}{n}}}(\mu_{ij} + \mu_{ji}) + \frac{\prod_{s=1}^{n}(\mu_{js}\mu_{si})^{\frac{1}{n}}}{\prod_{s=1}^{n}(\mu_{js}\mu_{si})^{\frac{1}{n}} + \prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}}}(\mu_{ji} + \mu_{ij}) \\ &= \mu_{ij} + \mu_{ji} = \mu_{ij} + \nu_{ij} \le 1.\end{aligned}$$

Therefor, the matrix $\tilde{R} = (\tilde{\mu}_{ij})_{n\times n}$ is an IFPR and preserves the preference information. In addition, we have:

$$\begin{aligned}\tilde{\mu}_{ij}\tilde{\mu}_{jk}\tilde{\mu}_{ki} &= \frac{\prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}}}{\prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}} + \prod_{s=1}^{n}(\mu_{js}\mu_{si})^{\frac{1}{n}}}(\mu_{ij} + \mu_{ji}) \times \frac{\prod_{s=1}^{n}(\mu_{js}\mu_{sk})^{\frac{1}{n}}}{\prod_{s=1}^{n}(\mu_{js}\mu_{sk})^{\frac{1}{n}} + \prod_{s=1}^{n}(\mu_{ks}\mu_{sj})^{\frac{1}{n}}}(\mu_{jk} + \mu_{kj}) \\ &\times \frac{\prod_{s=1}^{n}(\mu_{ks}\mu_{si})^{\frac{1}{n}}}{\prod_{s=1}^{n}(\mu_{ks}\mu_{si})^{\frac{1}{n}} + \prod_{s=1}^{n}(\mu_{is}\mu_{sk})^{\frac{1}{n}}}(\mu_{ki} + \mu_{ik}) = \frac{\prod_{s=1}^{n}(\mu_{js}\mu_{si})^{\frac{1}{n}}}{\prod_{s=1}^{n}(\mu_{js}\mu_{si})^{\frac{1}{n}} + \prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}}}(\mu_{ji} + \mu_{ij}) \\ &\times \frac{\prod_{s=1}^{n}(\mu_{is}\mu_{sk})^{\frac{1}{n}}}{\prod_{s=1}^{n}(\mu_{is}\mu_{sk})^{\frac{1}{n}} + \prod_{s=1}^{n}(\mu_{ks}\mu_{si})^{\frac{1}{n}}}(\mu_{ik} + \mu_{ki}) \times \frac{\prod_{s=1}^{n}(\mu_{ks}\mu_{sj})^{\frac{1}{n}}}{\prod_{s=1}^{n}(\mu_{ks}\mu_{sj})^{\frac{1}{n}} + \prod_{s=1}^{n}(\mu_{js}\mu_{sk})^{\frac{1}{n}}}(\mu_{kj} + \mu_{jk}) \\ &= \tilde{\mu}_{ji}\tilde{\mu}_{ik}\tilde{\mu}_{ki}.\end{aligned}$$

As a result, IFPR $\tilde{R}=(\tilde{\mu}_{ij})_{n\times n}$ is multiplicatively consistent and Theorem 3 is proved.

Eq. (11) is called a formula to straightforwardly calculate multiplicatively consistent IFPR preserving the preference information of every element of the initial IFPR.

Example 1 [8]. Consider the IFPR R concerning the appropriate selection of a flexible manufacturing system (FMS):

$$R=\begin{pmatrix}(0.5,0.5) & (0.2,0.6) & (0.6,0.4)\\(0.6,0.2) & (0.5,0.5) & (0.7,0.1)\\(0.4,0.6) & (0.1,0.7) & (0.5,0.5)\end{pmatrix}.$$

By using the reciprocal property $v_{ij}=\mu_{ji}$, $i,j\in N$, IFPR R can be expressed as:

$$R=\begin{pmatrix}0.5 & 0.2 & 0.6\\0.6 & 0.5 & 0.7\\0.4 & 0.1 & 0.5\end{pmatrix}.$$

We calculate multiplicatively consistent IFPR $\tilde{R}$ using Eq. (10):

$$.\tilde{R}=\begin{pmatrix}0.50 & 0.18 & 0.63\\0.62 & 0.50 & 0.69\\0.37 & 0.11 & 0.50\end{pmatrix}..$$

Then, the deviation between R and $\tilde{R}$ is 0.12.

Next, multiplicatively consistent IFPR $\bar{R}$ is constructed using Model 1. Then, the optimal model is as:

Model 3.

$$Min\ Z=(\varepsilon_{12}^{+}+\varepsilon_{12}^{-}+\xi_{12}^{+}+\xi_{12}^{-})+(\varepsilon_{13}^{+}+\varepsilon_{13}^{-}+\xi_{13}^{+}+\xi_{13}^{-})+(\varepsilon_{23}^{+}+\varepsilon_{23}^{-}+\xi_{23}^{+}+\xi_{23}^{-})$$

$$s,t\begin{cases}\dfrac{2\omega_1^{\mu}}{\omega_1^{\mu}-\omega_1^{v}+\omega_2^{\mu}-\omega_2^{v}+2}-0.2-\varepsilon_{12}^{+}+\varepsilon_{12}^{-}=0\\ \dfrac{2\omega_1^{\mu}}{\omega_1^{\mu}-\omega_1^{v}+\omega_3^{\mu}-\omega_3^{v}+2}-0.6-\varepsilon_{13}^{+}+\varepsilon_{13}^{-}=0\\ \dfrac{2\omega_2^{\mu}}{\omega_2^{\mu}-\omega_2^{v}+\omega_3^{\mu}-\omega_3^{v}+2}-0.7-\varepsilon_{23}^{+}+\varepsilon_{23}^{-}=0\\ \dfrac{2\omega_2^{\mu}}{\omega_1^{\mu}-\omega_1^{v}+\omega_2^{\mu}-\omega_2^{v}+2}-0.6-\xi_{12}^{+}+\xi_{12}^{-}=0\\ \dfrac{2\omega_3^{\mu}}{\omega_1^{\mu}-\omega_1^{v}+\omega_3^{\mu}-\omega_3^{v}+2}-0.4-\xi_{13}^{+}+\xi_{13}^{-}=0\\ \dfrac{2\omega_3^{\mu}}{\omega_2^{\mu}-\omega_2^{v}+\omega_3^{\mu}-\omega_3^{v}+2}-0.1-\xi_{23}^{+}+\xi_{23}^{-}=0\\ 0\le\omega_1^{\mu},\ \omega_2^{\mu},\ \omega_3^{\mu}\le 1,\ 0\le\omega_1^{v},\ \omega_2^{v},\ \omega_3^{v}\le 1\\ \omega_1^{\mu}+\omega_1^{v}\le 1,\ \omega_2^{\mu}+\omega_2^{v}\le 1,\ \omega_3^{\mu}+\omega_3^{v}\le 1\\ \omega_2^{\mu}+\omega_3^{\mu}\le\omega_1^{v},\ \omega_1^{\mu}+\omega_3^{\mu}\le\omega_2^{v},\ \omega_1^{\mu}+\omega_2^{\mu}\le\omega_3^{v}\\ \omega_1^{\mu}+1\ge\omega_2^{v}+\omega_3^{v},\ \omega_2^{\mu}+1\ge\omega_1^{v}+\omega_3^{v},\ \omega_3^{\mu}+1\ge\omega_1^{v}+\omega_2^{v}\\ \varepsilon_{12}^{+},\ \varepsilon_{12}^{-},\ \xi_{12}^{+},\ \xi_{12}^{-}\ge 0,\ \varepsilon_{12}^{+}\cdot\varepsilon_{12}^{-}=0,\ \xi_{12}^{+}\cdot\xi_{12}^{-}=0\\ \varepsilon_{13}^{+},\ \varepsilon_{13}^{-},\ \xi_{13}^{+},\ \xi_{13}^{-}\ge 0,\ \varepsilon_{13}^{+}\cdot\varepsilon_{13}^{-}=0,\ \xi_{13}^{+}\cdot\xi_{13}^{-}=0\\ \varepsilon_{23}^{+},\ \varepsilon_{23}^{-},\ \xi_{23}^{+},\ \xi_{23}^{-}\ge 0,\ \varepsilon_{23}^{+}\cdot\varepsilon_{23}^{-}=0,\ \xi_{23}^{+}\cdot\xi_{23}^{-}=0\end{cases}.$$

Solving Model 3, it follows that the objective function value is $Z^{*}=0.29$ and the optimal intuitionistic fuzzy weights are $w_1=(0.2384,0.7178)$, $w_2=(0.5178,0.3123)$, $w_3=(0.0740,0.8000)$.

By using Eq. (4), multiplicatively consistent IFPR $\bar{R}$ is constructed:

$$\bar{R}=\begin{pmatrix}(0.50,0.50) & (0.28,0.60) & (0.60,0.19)\\(0.60,0.28) & (0.50,0.50) & (0.70,0.10)\\(0.19,0.60) & (0.10,0.70) & (0.50,0.50)\end{pmatrix}.$$

In calculation of Example 1, the preference information of every element of multiplicatively consistent IFPR $\tilde{R}$ calculated by Eq. (11) is equal with the preference information of corresponding element of the initial IFPR R and the deviation between R and $\tilde{R}$ is 0.12. However, the preference information of every element of multiplicatively consistent IFPR $\bar{R}$ calculated by Model 3 greatly differs from the preference information of corresponding element of the one except digonal elements and the deviation between R and $\bar{R}$ is 0.29. This shows that the calculation of multiplicatively consistent IFPR by formula (11) is more correct and effective than the calculation of multiplicatively consistent IFPR by Model 3.

Theorem 4. IFPR $R=(\mu_{ij})_{n\times n}$ is multiplicatively consistent if and only if $R=\tilde{R}$.

Proof. If $R=\tilde{R}$, IFPR R is multiplicatively consistent by Thorem 3.

Inversely, let IFPR R be multiplicatively consistent. By Eq. (10), we have:

$$\frac{\mu_{ij}}{\mu_{ji}}=\frac{\mu_{ik}}{\mu_{ki}}\frac{\mu_{kj}}{\mu_{jk}}, \quad i,j,k\in N. \qquad (12)$$

Multiplying all the equations according to $k\in N$ in Eq. (12), we have:

$$\frac{\mu_{ij}}{\mu_{ji}}=\left(\prod_{k=1}^{n}\frac{\mu_{ik}}{\mu_{ki}}\frac{\mu_{kj}}{\mu_{jk}}\right)^{\frac{1}{n}}.$$

From Eq. (11), we have:

$$\tilde{\mu}_{ij}=\frac{\prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}}}{\prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}}+\prod_{s=1}^{n}(\mu_{js}\mu_{si})^{\frac{1}{n}}}(\mu_{ij}+\mu_{ji})$$

$$=\frac{\dfrac{\prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}}}{\prod_{s=1}^{n}(\mu_{is}\mu_{sj}\mu_{js}\mu_{si})^{\frac{1}{2n}}}}{\dfrac{\prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}}}{\prod_{s=1}^{n}(\mu_{is}\mu_{sj}\mu_{js}\mu_{si})^{\frac{1}{2n}}}+\dfrac{\prod_{s=1}^{n}(\mu_{js}\mu_{si})^{\frac{1}{n}}}{\prod_{s=1}^{n}(\mu_{is}\mu_{sj}\mu_{js}\mu_{si})^{\frac{1}{2n}}}}(\mu_{ij}+\mu_{ji})$$

$$=\frac{\prod_{s=1}^{n}\left(\frac{\mu_{is}\mu_{sj}}{\mu_{js}\mu_{si}}\right)^{\frac{1}{2n}}}{\prod_{s=1}^{n}\left(\frac{\mu_{is}\mu_{sj}}{\mu_{js}\mu_{si}}\right)^{\frac{1}{2n}}+\prod_{s=1}^{n}\left(\frac{\mu_{js}\mu_{si}}{\mu_{is}\mu_{sj}}\right)^{\frac{1}{2n}}}(\mu_{ij}+\mu_{ji}).$$

By using Eq. (12), we have:

$$\tilde{\mu}_{ij}=\frac{\prod_{s=1}^{n}\left(\frac{\mu_{is}\mu_{sj}}{\mu_{js}\mu_{si}}\right)^{\frac{1}{2n}}}{\prod_{s=1}^{n}\left(\frac{\mu_{is}\mu_{sj}}{\mu_{js}\mu_{si}}\right)^{\frac{1}{2n}}+\prod_{s=1}^{n}\left(\frac{\mu_{js}\mu_{si}}{\mu_{is}\mu_{sj}}\right)^{\frac{1}{2n}}}(\mu_{ij}+\mu_{ji})$$

$$=\frac{\left(\frac{\mu_{ij}}{\mu_{ji}}\right)^{\frac{1}{2}}}{\left(\frac{\mu_{ij}}{\mu_{ji}}\right)^{\frac{1}{2}}+\left(\frac{\mu_{ji}}{\mu_{ij}}\right)^{\frac{1}{2}}}(\mu_{ij}+\mu_{ji})$$

$$=\frac{\left(\frac{\mu_{ij}}{\mu_{ji}}\right)^{\frac{1}{2}}}{\left(\frac{\mu_{ij}}{\mu_{ji}}\right)^{\frac{1}{2}}+\left(\frac{\mu_{ji}}{\mu_{ij}}\right)^{\frac{1}{2}}}(\mu_{ij}\mu_{ji})^{\frac{1}{2}}\left(\left(\frac{\mu_{ij}}{\mu_{ji}}\right)^{\frac{1}{2}}+\left(\frac{\mu_{ji}}{\mu_{ij}}\right)^{\frac{1}{2}}\right)=\mu_{ij},$$

$$i,j\in N.$$

As a result, $R=\tilde{R}$ and Theorem 4 is proved.

If IFPR $R=(\mu_{ij})_{n\times n}$ is degenerated to a fuzzy preference relation, i.e., $\mu_{ij}+\mu_{ji}=1$, then Eq. (11) is expressed as:

$$k\in N\ \tilde{\mu}_{ij}=\frac{\prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}}}{\prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}}+\prod_{s=1}^{n}(\mu_{js}\mu_{si})^{\frac{1}{n}}}. \qquad (13)$$

Eq. (13) is equivalent to well known Eq. (2) in fuzzy preference relations.

Thorem 5. IFPR $R=(\mu_{ij})_{n\times n}$ is multiplicatively consistent if and only if there exist a proper intuitionistic fuzzy priority weight vector $w=(w_1,w_2,\ldots,w_n)$ $w_i=(w_i^{\mu},w_i^{\nu})$, $w_i^{\mu},w_i^{\nu}\in[0,1]$, $w_i^{\mu}+w_i^{\nu}\le 1$, $i\in N$ and $0<t_{ij}\le 1$, $t_{ij}=t_{ji}$, $t_{ii}=1$, $i,\ j\in N$ such that

$$\mu_{ij}=\frac{w_i^{\mu}w_j^{\nu}}{w_i^{\mu}w_j^{\nu}+w_i^{\nu}w_j^{\mu}}t_{ij},\quad i,\ j\in N. \qquad (14)$$

Proof. Let IFPR $R=(\mu_{ij})_{n\times n}$ be multiplicatively consistent. By Thorem 4, we have:

$$\mu_{ij}=\frac{\prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}}}{\prod_{s=1}^{n}(\mu_{is}\mu_{sj})^{\frac{1}{n}}+\prod_{s=1}^{n}(\mu_{js}\mu_{si})^{\frac{1}{n}}}(\mu_{ij}+\mu_{ji}),\quad i,\ j\in N.$$

We introduce the following expressions:

$$w_i^\mu = \left(\prod_{s=1}^n \mu_{is}\right)^{\frac{1}{n}}, \quad w_i^\nu = \left(\prod_{s=1}^n \nu_{is}\right)^{\frac{1}{n}} = \left(\prod_{s=1}^n \mu_{si}\right)^{\frac{1}{n}}, \ i,\ j \in N.$$

From the limitation condition of IFPR $R=(\mu_{ij})_{n\times n}$, $0 \le \mu_{ij} + \nu_{ij} = \mu_{ij} + \mu_{ji} \le 1$, we have:

$$0 \le w_i^\mu + w_i^\nu = \left(\prod_{s=1}^n \mu_{is}\right)^{\frac{1}{n}} + \left(\prod_{s=1}^n \mu_{si}\right)^{\frac{1}{n}} \le \frac{1}{n}\sum_{s=1}^n \mu_{is} + \frac{1}{n}\sum_{s=1}^n \mu_{si} = \frac{1}{n}\sum_{s=1}^n (\mu_{is} + \mu_{is}) \le 1, \ i \in N.$$

In addition, denoting $t_{ij} = \mu_{ij} + \mu_{ji}$, we have:

$$\mu_{ij} = \frac{\sqrt[n]{\prod_{s=1}^n \mu_{is}\mu_{sj}}}{\sqrt[n]{\prod_{s=1}^n \mu_{is}\mu_{sj}} + \sqrt[n]{\prod_{s=1}^n \mu_{js}\mu_{si}}}(\mu_{ij} + \mu_{ji}) = \frac{w_i^\mu w_j^\nu}{w_i^\mu w_j^\nu + w_i^\nu w_j^\mu} t_{ij}, \quad i,\ j \in N,$$

Here $0 < t_{ij} \le 1$, $t_{ij} = t_{ji}$, $t_{ii} = 1$, $i,\ j \in N$.

Conversely, if Eq. (14) holds to, then we have:

$$\mu_{ij}\mu_{jk}\mu_{ki} = \frac{w_i^\mu w_j^\nu}{w_i^\mu w_j^\nu + w_i^\nu w_j^\mu} t_{ij} \times \frac{w_j^\mu w_k^\nu}{w_j^\mu w_k^\nu + w_j^\nu w_k^\mu} t_{jk} \times \frac{w_k^\mu w_i^\nu}{w_k^\mu w_i^\nu + w_k^\nu w_i^\mu} t_{ki}$$

$$= \frac{w_i^\mu w_k^\nu}{w_i^\mu w_k^\nu + w_i^\nu w_k^\mu} t_{ik} \times \frac{w_k^\mu w_j^\nu}{w_k^\mu w_j^\nu + w_k^\nu w_j^\mu} t_{kj} \times \frac{w_j^\mu w_i^\nu}{w_j^\mu w_i^\nu + w_j^\nu w_i^\mu} t_{ji} = \mu_{ik}\mu_{kj}\mu_{ji}.$$

Therefore, IFPR $R=(\mu_{ij})_{n\times n}$ is multiplicatively consistent and Theorem 5 is proved.

If IFPR $R=(\mu_{ij})_{n\times n}$ is degenerated to a fuzzy preference relation, i.e., $\mu_{ij}+\mu_{ji}=1$, equation $\mu_{ij} = \frac{w_i^\mu w_j^\nu}{w_i^\mu w_j^\nu + w_i^\nu w_j^\mu} t_{ij}$ is converted into $r_{ij} = \frac{w_i^\mu w_j^\nu}{w_i^\mu w_j^\nu + w_i^\nu w_j^\mu}$. Then,

$$r_{ij} = \frac{\frac{w_i^\mu w_j^\nu}{w_i^\nu w_j^\nu}}{\frac{w_i^\mu w_j^\nu}{w_i^\nu w_j^\nu} + \frac{w_i^\nu w_j^\mu}{w_i^\nu w_j^\nu}} = \frac{\frac{w_i^\mu}{w_i^\nu}}{\frac{w_i^\mu}{w_i^\nu} + \frac{w_j^\mu}{w_j^\nu}}. \tag{15}$$

If there exist a proper constant $M > 0$ to satisfy conditions $0 < u_i = M\frac{w_i^\mu}{w_i^\nu} \le 1$, $i \in N$, Eq. (15) can be expressed as $r_{ij} = \frac{u_i}{u_i + u_j}$, $i, j \in N$. Hence, the famous relationship between multiplicatively consistent fuzzy preference relation $R=(\mu_{ij})_{n\times n}$ and priority weight vector is coming into.

Corollary 1. A fuuzzy preference relation $R=(r_{ij})_{n\times n}$ is multiplicatively consistent if and only if there exist a fuzzy priority weight vector $u=(u_1,u_2,\cdots,u_n)$, $0 \le u_i \le 1$, $i \in N$ such that

$$r_{ij} = \frac{u_i}{u_i + u_j}, \quad i, j \in N \tag{16}$$

4. A method to Improve the Multiplicative Inconsistency of an IFPR

In this section, we define a consistency index of IFPR and establish the result that the greater the ratio-deviation between every element of IFPR and the correspondin element of multiplicatively consistent IFPR calculated by formula (11) is, the greater is its associated index, and vice versa. Based on th result, we construct an algorithm to improve the multiplicative inconsistency preserving the preference information of every element of the initial IFPR and prove that the consistency index converges into zero.

In general, IFPRs provided by experts are very nearly inconsistent because of various causes and the results obtained based on it are not rational. For this reason, the primary task has to improve its inconsistency.

Given an IFPR $R=(\mu_{ij})_{n\times n}$, we denote multiplicatively consistent IFPR as $\tilde{R}=(\tilde{\mu}_{ij})_{n\times n}$. The following expressions are introduced:

$$e_{ij} = \max\{\frac{\mu_{ij}}{\tilde{\mu}_{ij}}, \frac{\tilde{\mu}_{ij}}{\mu_{ij}}\}, \ i, j \in N. \tag{17}$$

Then $e_{ij} \ge 1$ and $0 < \frac{1}{e_{ij}} = \min\{\frac{\mu_{ij}}{\tilde{\mu}_{ij}}, \frac{\tilde{\mu}_{ij}}{\mu_{ij}}\} \le 1$. If IFPR $R=(\mu_{ij})_{n\times n}$ is multiplicatively inconsistent, ineqialties $e_{lm} > 1$, $0 < \frac{1}{e_{lm}} < 1$ are holded for some $l, m \in N$.

By using Eq. (17), we introduce the ratio-deviation between every element of IFPR $R=(\mu_{ij})_{n\times n}$ and the corresponding element of IFPR $\tilde{R}=(\tilde{\mu}_{ij})_{n\times n}$ as:

$$d_{ij}=e_{ij}+\frac{1}{e_{ij}}-2=\frac{\mu_{ij}}{\tilde{\mu}_{ij}}+\frac{\tilde{\mu}_{ij}}{\mu_{ij}}-2=\frac{(\mu_{ij}-\tilde{\mu}_{ij})^2}{\mu_{ij}\tilde{\mu}_{ij}}\ge 0,\ i,j\in N. \quad (18)$$

Definition 7. The consistency index of IFPR $R=(\mu_{ij})_{n\times n}$, $CI_3(R)$ is defines as:

$$CI_3(R)=\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\ne j}}^{n}d_{ij}. \quad (19)$$

Theorem 6. The greater e_{ij} is, the greate is d_{ij} and vice versa.

Proof. If $e_{ij}>e_{st}\ge 1$, we have:

$$d_{ij}-d_{st}=e_{ij}+\frac{1}{e_{ij}}-2-\left(e_{st}+\frac{1}{e_{st}}-2\right)$$

$$=(e_{ij}-e_{st})-\left(\frac{1}{e_{st}}-\frac{1}{e_{ij}}\right)=(e_{ij}-e_{st})-\frac{e_{ij}-e_{st}}{e_{ij}e_{st}}>0.$$

Thus, inequality $d_{ij}>d_{st}$ is true.

Conversely, if $d_{ij}>d_{st}$, we have:

$$d_{ij}-d_{st}=(e_{ij}-e_{st})-\frac{e_{ij}-e_{st}}{e_{ij}e_{st}}>0.$$

If there were inequality $e_{ij}<e_{st}$, we should have from $e_{ij}\ge 1,\ e_{st}>1$:

$$e_{ij}-e_{st}\le\frac{e_{ij}-e_{st}}{e_{ij}e_{st}}\Leftrightarrow(e_{ij}-e_{st})-\frac{e_{ij}-e_{st}}{e_{ij}e_{st}}\le 0$$

$$\Leftrightarrow(e_{ij}+\frac{1}{e_{ij}}-2)-(e_{st}+\frac{1}{e_{st}}-2)=d_{ij}-d_{st}\le 0.$$

This contradicts the condition $d_{ij}>d_{st}$. As a result, Theorem 6 is proved.

From Theorem 6. the following corollary follows.

Corollary 2. The ratio-deviation between μ_{ij} and $\tilde{\mu}_{ij}$ corresponding to the largest d_{ij} is the greatest, and vice versa.

By using Theorem 6, we construct an algorithm to improve multiplicative inconsistency of IFPR $R=(\mu_{ij})_{n\times n}$.

Algorithm 1.

Input: IFPR $R=(\mu_{ij})_{n\times n}$, the prescribed threshold CI_0 and the maximum number of iterations $z_{\max}$

Output: The final improved IFPR.

Step 1: Calculate multiplicatively consistent IFPR $\tilde{R}=(\tilde{\mu}_{ij})_{n\times n}$ corresponding to IFPR $R=(\mu_{ij})_{n\times n}$ using fomular (11).

Step 2: Let $z=1$ and $R^{(z)}=R$.

Step 3: Construct consistency testing matrix $D(R^{(z)})=(d_{ij}^{(z)})_{n\times n}$ and calculate consistency index $CI_3(R^{(z)})$ using Eqs.(18) and (19). If $CI_3(R^{(z)})\le CI_0$, go to Step 6. Otherwise, go to next step.

Step 4: Select the largest element $d_{st}^{(z)}$ in $D(R^{(z)})$. Replace $\mu_{st}^{(z)}$ and $\mu_{ts}^{(z)}$ of IFPR $R^{(z)}$ with $\hat{\mu}_{st}=(1-p\alpha^{(z)})\mu_{st}+p\alpha^{(z)}\tilde{\mu}_{st}$ and $\hat{\mu}_{ts}=(1-p\alpha^{(z)})\mu_{ts}+p\alpha^{(z)}\tilde{\mu}_{ts}$ where $p\ge 1$ is the number of iteration and $\frac{1}{p}>\alpha^{(z)}>0$ is the controlling parameter.

Step 5: Denote $z=z+1$ and go back Step 3.

Step 6: Output IFPR $R^{(z)}=(\mu_{ij}^{(z)})_{n\times n}$.

Remark 2. In order to preserve the preference information of every element of the revised IFPR $R^{(z)}=(\mu_{ij}^{(z)})_{n\times n}$, we replace $\mu_{ts}^{(z)}$ as well as $\mu_{st}^{(z)}$ in $R^{(z)}=(\mu_{ij}^{(z)})_{n\times n}$ and the controlling parameter $\alpha^{(z)}$ is also changed in the process of iteration.

The following theorem holds in connection with Algorithm 1.

Theorem 7. Algorithm 1 preserve the preference information of every element of the intial IFPR and the consistency index converges into zero.

Proof. By Step 4 of Algorithm 1, $\mu_{st}^{(z)}$ and $\mu_{ts}^{(z)}$ are replced with $\mu_{st}^{(z+1)}=(1-p\alpha^{(z)})\mu_{st}^{(z)}+p\alpha^{(z)}\tilde{\mu}_{st}$ and $\mu_{ts}^{(z+1)}=(1-p\alpha^{(z)})\mu_{ts}^{(z)}+p\alpha^{(z)}\tilde{\mu}_{ts}$ respectively. Then, we have:

$$\mu_{st}^{(z+1)}+\mu_{ts}^{(z+1)}=((1-p\alpha^{(z)})\mu_{st}^{(z)}+p\alpha^{(z)}\tilde{\mu}_{st})+((1-p\alpha^{(z)})\mu_{ts}^{(z)}+p\alpha^{(z)}\tilde{\mu}_{ts})$$

$$=(1-p\alpha^{(z)})(\mu_{st}^{(z)}+\mu_{ts}^{(z)})+\frac{p\alpha^{(z)}(\mu_{st}^{(z)}+\mu_{ts}^{(z)})(a_{st}^{(z)}+a_{ts}^{(z)})}{a_{st}^{(z)}+a_{ts}^{(z)}}$$

$$=(1-p\alpha^{(z)})(\mu_{st}^{(z)}+\mu_{ts}^{(z)})+p\alpha^{(z)}(\mu_{st}^{(z)}+\mu_{ts}^{(z)})=\mu_{st}^{(z)}+\mu_{ts}^{(z)},$$

where $a_{st}^{(z)}=\sqrt[n]{\prod_{l=1}^{n}\mu_{sl}^{(z)}\mu_{lt}^{(z)}}$, $a_{ts}^{(z)}=\sqrt[n]{\prod_{l=1}^{n}\mu_{tl}^{(z)}\mu_{ls}^{(z)}}$. Therefore, the preference information of every element of the initial IFPR is preserved.

In addition, $\mu_{st}^{(z+1)} = (1-p\alpha^{(z)})\mu_{st}^{(z)} + p\alpha^{(z)}\tilde{\mu}_{st}$ and $\mu_{st}^{(z+1)} = (1-p\alpha^{(z)})\mu_{st}^{(z)} + p\alpha^{(z)}\tilde{\mu}_{st}$ is only changed in $(z+1)-th$ execution of Algorithm 1. By configuration of Algorithm 1, the differences between $z-th$ execution of consistency index $CI_3(R^{(z)})$ and $(z+1)-th$ execution of consistency index $CI_3(R^{(z+1)})$ are the difference between $d_{st}^{(z)} = \frac{(\mu_{st}^{(z)} - \tilde{\mu}_{st})^2}{\mu_{st}^{(z)}\tilde{\mu}_{st}}$ the and $d_{st}^{(z+1)} = \frac{(\mu_{st}^{(z+1)} - \tilde{\mu}_{st})^2}{\mu_{st}^{(z+1)}\tilde{\mu}_{st}}$ and the difference between $d_{ts}^{(z)} = \frac{(\mu_{ts}^{(z)} - \tilde{\mu}_{ts})^2}{\mu_{ts}^{(z)}\tilde{\mu}_{ts}}$ and $d_{ts}^{(z+1)} = \frac{(\mu_{ts}^{(z+1)} - \tilde{\mu}_{ts})^2}{\mu_{ts}^{(z+1)}\tilde{\mu}_{ts}}$.

First, Let us compaire $d_{st}^{(z)} = \frac{(\mu_{st}^{(z)} - \tilde{\mu}_{st})^2}{\mu_{st}^{(z)}\tilde{\mu}_{st}}$ with $d_{st}^{(z)} = \frac{(\mu_{st}^{(z)} - \tilde{\mu}_{st})^2}{\mu_{st}^{(z)}\tilde{\mu}_{st}} = \frac{(1-p\alpha)^2(\mu_{st}^{(z)} - \tilde{\mu}_{st})^2}{((1-p\alpha)\mu_{st}^{(z)} + p\alpha\tilde{\mu}_{st})\tilde{\mu}_{st}}$.

If $d_{st}^{(z+1)} = \frac{(1-p\alpha)^2(\mu_{st}^{(z)} - \tilde{\mu}_{st})^2}{((1-p\alpha)\mu_{st}^{(z)} + p\alpha\tilde{\mu}_{st})\tilde{\mu}_{st}}$ is larger than $d_{st}^{(z)} = \frac{(\mu_{st}^{(z)} - \tilde{\mu}_{st})^2}{\mu_{st}^{(z)}\tilde{\mu}_{st}}$, inequality $\frac{(1-p\alpha)^2}{(1-p\alpha)\mu_{st}^{(z)} + p\alpha\tilde{\mu}_{sr}} > \frac{1}{\mu_{st}^{(z)}}$ should hold from the definition expressions. Hence, we have:

$$(1-p\alpha^{(z)})^2\mu_{st}^{(z)} > (1-p\alpha^{(z)})\mu_{st}^{(z)} + p\alpha^{(z)}\tilde{\mu}_{st} \Leftrightarrow (-p\alpha^{(z)} + (p\alpha^{(z)})^2)\mu_{st}^{(z)} > p\alpha^{(z)}\tilde{\mu}_{st} \Leftrightarrow (-1+p\alpha^{(z)})\mu_{st}^{(z)} > \tilde{\mu}_{st} \quad (20)$$

From the limitation condition of $\alpha^{(z)}$ and $0 \le \mu_{st}^{(z)}, \tilde{\mu}_{st} < 1$, inequalities $-1+p\alpha^{(z)} \le 0$ and $(-1+p\alpha^{(z)})\mu_{st}^{(z)} < \tilde{\mu}_{st}$ should hold. This shows tha Eq. (20) is wrong, and therefore $d_{st}^{(z)}$ is larger than $d_{st}^{(z+1)}$. We can prove in the same way that $d_{ts}^{(z)}$ is larger than $d_{ts}^{(z+1)}$. Accordingly, there exists a sequence $\{R^{(z)}\}$ satisfying inequalities

$$CI_3(R^{(1)}) > CI_3(R^{(2)}) > \cdots > CI_3(R^{(z)}) > CI_3(R^{(z+1)})\cdots.$$

As a result, inequality $CI_3(R^{(m)}) \le CI_0$ is satisfied in $m-th$ execution for some $m \ge 1$ and Theorem 7 is proved.

Example 2 [22]. Let IFPR $R = (r_{ij})_{n\times n}(r_{ij} = (\mu_{ij}, \nu_{ij}))$ be defined as follows:

$$R = \begin{pmatrix} (0.50,0.50) & (0.50,0.30) & (0.80,0.10) & (0.60,0.20) \\ (0.30,0.50) & (0.50,0.50) & (0.50,0.10) & (0.30,0.15) \\ (0.10,0.80) & (0.10,0.50) & (0.50,0.50) & (0.20,0.70) \\ (0.20,0.60) & (0.15,0.30) & (0.70,0.20) & (0.50,0.50) \end{pmatrix}.$$

Using of reciprocal property $\nu_{ij} = \mu_{ji}$, $i,j \in N$ repsents IFPR R as:

$$R = \begin{pmatrix} 0.50 & 0.50 & 0.80 & 0.60 \\ 0.30 & 0.50 & 0.50 & 0.30 \\ 0.10 & 0.10 & 0.50 & 0.20 \\ 0.20 & 0.15 & 0.70 & 0.50 \end{pmatrix}.$$

Step 1: Construct multiplicatively consistent IFPR $\tilde{R}$ using Eq. (11):

$$\tilde{R} = \begin{pmatrix} 0.50 & 0.59 & 0.81 & 0.59 \\ 0.21 & 0.50 & 0.51 & 0.29 \\ 0.09 & 0.09 & 0.50 & 0.22 \\ 0.21 & 0.16 & 0.68 & 0.50 \end{pmatrix}.$$

Step 3: Construct consistency matrix $D(R^{(1)})$ and calculate consistency index $CI_3(R^{(1)})$ using Eqs. (18) and (19):

$$D(R^{(1)}) = \begin{pmatrix} 0.0000 & 0.2745 & 0.0001 & 0.0002 \\ 0.1285 & 0.0000 & 0.0003 & 0.0011 \\ 0.0111 & 0.0111 & 0.0000 & 0.0090 \\ 0.0023 & 0.0041 & 0.0008 & 0.0000 \end{pmatrix},$$

$$CI_3(R^{(1)}) = 0.03692 < CI_0.$$

Step 6: Output IFPR $R^{(1)}$.

Next, Let us improve the multiplicative inconsistency of the same IFPR R usin Model 2. By Eq. (8), consistency index is $CI_2(R) = 0.277 > CI_0$ and IFPR R is an unacceptable. Calculate the improved IFPR $\hat{R}$ through Model 2:

$$\hat{R} = \begin{pmatrix} (0.500,0.500) & (0.500,0.300) & (0.833,0.100) & (0.600,0.200) \\ (0.300,0.500) & (0.500,0.500) & (0.500,0.100) & (0.300,0.167) \\ (0.100,0.833) & (0.100,0.500) & (0.500,0.500) & (0.219,0.700) \\ (0.200,0.600) & (0.167,0.300) & (0.700,0.219) & (0.500,0.500) \end{pmatrix}$$

By Eq. (8), consistency index of IFPR $\hat{R}$ is $CI_2(\hat{R}) = 0.099 < CI_0$ and IFPR $\hat{R}$ is an acceptable. By using Eqs. (18) and (19), calculate consistency matrix $D(\hat{R})$ and consistency index $CI_3(\hat{R})$

$$D(\hat{R})=\begin{pmatrix} 0.0000 & 0.2745 & 0.0007 & 0.0002 \\ 0.1285 & 0.0000 & 0.0003 & 0.0011 \\ 0.0111 & 0.0111 & 0.000 & 0.0000 \\ 0.0023 & 0.0018 & 0.0008 & 0.0000 \end{pmatrix},$$

$$CI_3(\hat{R},\tilde{R})=0.03597<CI_0.$$

In addition, Let us improve the multiplicative consistency of the same IFPR R using the method of Liao and Xu [8]. Calculate multiplicatively consistent IFPR $\bar{R}$ through Model 1:

$$\bar{R}=\begin{pmatrix} (0.500,0.500) & (0.509,0.400) & (0.519,0.333) & (0.560,0.240) \\ (0.400,0.509) & (0.500,0.500) & (0.449,0.367) & (0.489,0.200) \\ (0.333,0.519) & (0.367,0.449) & (0.500,0.500) & (0.409,0.273) \\ (0.240,0.560) & (0.267,0.489) & (0.273,0.409) & (0.500,0.500) \end{pmatrix},$$

$$CI_1(R,\bar{R})=0.233.$$

Denoting the controlling parameter by $\sigma=0.5$ and using Eq. (7), improve the multiplicative inconsistency of IFPR R:

$$\breve{R}=\begin{pmatrix} (0.500,0.500) & (0.504,0.346) & (0.644,0.182) & (0.579,0.219) \\ (0.346,0.504) & (0.500,0.500) & (0.474,0.191) & (0.383,0.200) \\ (0.182,0.644) & (0.191,0.474) & (0.500,0.500) & (0.286,0.437) \\ (0.219,0.579) & (0.200,0.383) & (0.437,0.286) & (0.500,0.500) \end{pmatrix}$$

$$,\ CI_1(\breve{R},\bar{R})=0.083.$$

The results by different methods are calculated at various indices as shown in Table 1.

***Table 1**. The results by different methods.*

Methods	**Consistency Indices** (CI_1,CI_2,CI_3)	**Number of Changed Preference Informatio**
Liao and Xu [8]	$CI_1(\breve{R})=0.083$, $CI_1(\hat{R})=0.967>CI_0$	6 (all but diagonal)
Xu et al. [22]	$CI_2(\hat{R})=0.099$, $CI_2(R)=0.277>CI_0$	3
Our method	$CI_3(R,\tilde{R})=0.0369$, $CI_3(\hat{R},\tilde{R})=0.0360$	0

As shown in Table 1, preference information of acceptable IFPRs constructed by previous methods are changed in process of improving inconsistency but preference information of an acceptable IFPR by our method is constant. In addition, since our method does not calculate any model to construct acceptably consistent IFPR, it is efficient in calculation.

5. Conclusion

In application problems based on IFPRs, improvement of multiplicative inconsistency is primitive to obtain rational results. In order to solve this task, we formulate a formula to straightforwardly calculate multiplicatively consistent IFPR preserving the preference information of every element of an initial IFPR. Based on it, the necessary and sufficient results for the IFPR to be multiplicatively consistent are derived. By using the results, we construct the consistency testing matrix and consistency index that can select the most inconsistent elements and propose a method that revises them by proper intuitionistic fuzzy numbers improving inconsistency as well as preserving the initial preference information. As a result, this method preserves the preference information of every element of the initial IFPR and consistency index converges into zero.

Improving the multiplicative inconsistency of an IFPR is to obtain the rational results not that some preference information between comparative alternatives are newly known. Therefore, the preservation of the preference information of every element of the IFPR that multiplicative inconsistency is improved is natural and justifiable.

Our method differs greatly from previous methods that merely improve the multiplicative inconsistency irrelevantly to the preference information of elements of the initial IFPR through optimal models constructed based on sufficient conditions for IFPRs to be mltiplicatively consistent. In addition, it needs a few calculations in comparison with previous methods because of being not solving any model.

Based on this method, we are going to derive a formula that strightforwardly calculates an underlying priority weight vector in connection with alternatives in group decision making problems.

Abbreviations

IFPR Intuitionistic Fuzzy Preference Relation

Conflicts of Interest

The authors declare no conflicts of interest.

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