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Research Article

# Expertise-Based Ranking of Experts in Multicriteria Group Decision Making with Different Preference Representation Formats

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## Abstract

The group decision-making (GDM) is a process of aggregating the preference information of each decision maker in a group into a collective opinion under certain decision criteria. When a decision is made by decision makers (DMs), the group decision or group opinion is a result of the integration of the individual opinions by a mathematical aggregation. An important thing in the integration of the individual opinions is which DMs' opinions should be of importance in the aggregation process. The group opinion is heavily influenced by the level of expertise of the DMs. In real-life group decision making (GDM) problems, preference relations given by decision makers (DMs) are often heterogeneous because of their expertise and different decision habits. The expertise level of the DMs can be estimated using the 'the ability to differentiate consistently', a ratio between discrimination and inconsistency. Each expert has different expertise in different criteria and has his/her limited capacity in constructing of pairwise comparison preference relations. In this paper, we focus on a method to assess expertise-based ranking of experts in GDM with multiple criteria and different preference representation formats. The combination of expertise as 'the ability to differentiate consistently' and the consistency of various types of preference relations is discussed, when experts give their judgements for alternatives in various types of preference relations, such as multiplicative preference relations (MPRs), utility values, preference orderings, even incomplete FPRs or MPRs, including FPRs under certain criterion. Then, since the experts have the limited capacity in constructing of pairwise comparison preference relations in GDM with multiple criteria, we propose a expertise-based ranking method using the adjacent pairwise comparison technique in order to reduce the number of judgements. Finally, expertise-based ranking of experts is applied to aggregate the individual opinions into a group opinion by developing into experts' importance weights in GDM. Numerical analyses show that proposed method is simple than the previous methods and can fill up some gaps in GDM.

## Keywords

Group Decision Making, Ranking, Heterogeneous Preference Relations, Expertise

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## 1. Introduction

The group decision-making (GDM) is a process of aggregating the preference information of each decision maker in a group into a collective opinion under certain decision criteria. When a decision is made by decision makers (DMs), the group decision or group opinion is a result of the integration of the individual opinions by a mathematical aggregation. A important thing in the integration of the individual opinions is which DMs' opinions should be of importance in the aggregation process. The group opinion is heavily influenced by the level of expertise of the DMs. In real-life multi-criteria GDM problems, preference relations provided by DMs are often heterogeneous because of ambiguity of their expertise and different decision habits. DMs' opinions reflect their expertise and customary preference information. This means that DMs' expertise and heterogeneous preference information should be considered simultaneously in the aggregation process.

Weiss and Shanteau [1] proposed the concept of 'the ability to differentiate consistently', which is the ratio between discrimination and inconsistency, to define the DMs' level of expertise. They defined experts as those who are capable of distinguishing between cases that are similar but not exactly the same and of repeating their judgments consistently, and proposed the CWS-Index (the Cochran-Weiss-Shanteau Index), which is the ratio between discrimination and inconsistency, to assess expert' s level of expertise [1, 2]. The CWS-Indexes for the experts yields their ranking; the higher the CWS-Index, the higher is the DM's ranking. However, measuring inconsistency requires repetition, and accordingly the experts need to make judgments more than once.

In [4, 5], the researchers pointed out the irrationality of the repeated evaluation to measure the inconsistency of experts judgements, the authers in [6-11] showed that the additive consistency (AC) of fuzzy preference relations (FPR) could be used to measure the expert's consistency level. But these studies don't consider the ability of the expert to differentiate between similar, but not identical, cases, and produce a experts' ranking based on consistency of the FPRs.

There have been studies to determine the ranking of experts based on their level of assessment, but in these previous researches, the experts' ranking are determined only by the consistency of their assessments, without considering their ability to differentiate.

In sum, the previous studies have not used the comprehensive concept of expertise.

In [3], Herowati et al. proposed a method to determine the raking of the DMs using the combination of expertise as 'the ability to differentiate consistently' and the AC of FPR in GDM with one criterion.

In real-life GDM problems one be often faced with following some situations.

1) The experts may have different expertise for different criteria [3].
2) The preference relations provided by experts are often heterogeneous because they always have different decision habits [12].
3) The experts have their limited capacity in constructing of pairwise comparison preference relations.

To assess expertise-based ranking of experts for these situations appears to bear a certain meaning in the study of GDM problems. Furthermore, if expertise-based ranking of experts is developed into expertise-based experts' importance weights that specify the importance weights of the experts in GDM, we can use these importance weights in the process of aggregating the individual opinions into a group opinion. Research to obtain the experts' importance weight from their ranking has been discussed in another paper [13].

In this paper, we focus on a method to assess expertise-based ranking of experts in GDM with multiple criteria and different preference representation formats.

The combination of expertise as 'the ability to differentiate consistently' and the consistency of various types of preference relations is discussed on the basis of the methodology proposed in [3], when experts give their judgements for alternatives in various types of preference relations, such as multiplicative preference relations (MPRs), utility values, preference orderings, even incomplete FPRs or MPRs, including FPRs under certain criterion. Then, since the experts have the limited capacity in constructing of pairwise comparison preference relations in GDM with multiple criteria, we propose a expertise-based ranking method using the adjacent pairwise comparison technique in order to reduce the number of judgements. Finally, expertise-based ranking of experts is applied to aggregate the individual opinions into a group opinion by developing into experts' importance weights in GDM. Numerical analyses show that proposed method is simple than the previous methods and can fill up some gaps in GDM.

The remainder of this paper is organized as follows. In Section 2 we introduce related preliminaries about expertise levels of experts and different preference relations. In Section 3 we refine the method for ranking the expertise levels of experts when experts provide their judgements for alternatives in MPRs, utility values, preference orderings and incomplete preference relations under one criterion. In Section 4, we propose a expertise-based ranking method using the adjacent pairwise comparison technique in multi-criteria decision making (MCDM). In Section 5 we apply expertise-based ranking of experts to aggregate individual opinions into a group opinion in GDM. In Section 6 a numerical example is discussed to illustrate proposed method, and we conclude the paper in Section 7.

## 2. Preliminary

In this section, we firstly discuss the previous methods related to the expertise level of the experts, then introduce different preference representation formats often used in GDM.

### 2.1. Experts' Expertise Level and FPR's AC

Weiss and Shanteau [1] proposed to combine the concepts of 'consistency' and 'discrimination' to determine the expertise level of a person that becomes 'the ability to differentiate consistently', which is expressed by the CWS-Index as shown in equations (1), (2) and (3) as follows:

$$\text{CWS-Index} = \frac{\text{Discrimination}}{\text{Inconsistency}} = \frac{\text{Variance of different alternatives' values}}{\text{Variance of same alternatives' values}} \tag{1}$$

$$\text{Discrimination} = \frac{\sum_{i=1}^{n} r\left(M_i - GM\right)^2}{n-1} \tag{2}$$

$$\text{Inconsistency} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{r}\left(M_{ij} - M_i\right)^2}{n\left(r-1\right)} \tag{3}$$

where

$r$ : The number of replications,

$M_i$ : The average of individual values for case-$i$,

$GM$ : The grand mean of all individual values,

$n$ : The number of different cases,

$M_{ij}$ : The individual value for replication-$j$ of case-$i$

Equation (2) shows that discrimination expresses the compatibility variance in statistics, and equation (3) shows that inconsistency is the reproducibility variance.

In equation (1), to get the CWS - Index, experts are asked to give their assessment repeatedly.

Repeating the measurements are difficult and time-consuming. It's another weakness lies in the possibility to obtain a high CWS-Index score by giving an incorrect assessment consistently. Therefore, the method for comparing experts' expertise needs to be refined.

FPR is one of the most widely used evaluation methods for expert assessment in GDM [8, 14-18].

We assume that $X = \{X_1, X_2, \cdots, X_n\}$ is a set of feasible alternatives to be assessed.

A FPR is determined by a pairwise comparison matrix (PCM) whose entries represent preference degree for two alternatives. The values of a fuzzy PCM $B = \left(b_{ij}\right)_{n\times n}$ range from zero to one and satisfy $b_{ij} + b_{ji} = 1$. If $b_{ij}$ is 0.5, then the alternatives are equally important. If the value of $b_{ij}$ (resp. $b_{ji}$,) is one (resp., zero), then the alternative $X_i$ is the most (resp., least) important than $X_j$.

The AC for FPR can be expressed with the following three relational equations, therefore each entry of the fuzzy PCM $B$ is estimated in three different ways.

$$\varepsilon b_{ik}^{j1} = b_{ij} + b_{jk} - \frac{1}{2},\ j \neq i,k \tag{4}$$

$$\varepsilon b_{ik}^{j2} = b_{jk} - b_{ji} + \frac{1}{2},\ j \neq i,k \tag{5}$$

$$\varepsilon b_{ik}^{j3} = b_{ij} - b_{kj} + \frac{1}{2},\ j \neq i,k \tag{6}$$

Since FPR has property $b_{ij} + b_{ji} = 1$, results estimated from equation (4), (5) and (6) are the same, and for every entry of the FPR $b_{ij}$, the equations produce as many as $n-2$ estimators (since $j \neq i,k$).

All entries $\varepsilon b_{ik}$ of the estimated matrix can be transformed using a transformation function such that the range changes from $[-a, 1+a], a > 0$ to $[0,1]$ [8].

$$f\left(x\right) = \frac{x+a}{1+2a} \tag{7}$$

In [3], Herowati et al. refined the method to measure a person's level of consistency and compare the expertise level of the experts on the basis of the deviation between the values of the estimations using the AC of FPR and the real values given by the expert. In terms of combining the concept of expertise as 'the ability to differentiate consistently' with the AC of FPR, CWS-Index is revised as follows:

$$\text{CWS-Index} = \frac{\text{Variance of different pairwise comparisons between two alternatives}}{\text{Variance of the same pairwise comparisons between two alternatives}} \tag{8}$$

$$\text{Variance of different pairwise comparisons between two alternatives} = \frac{\left(n-1\right)\sum_{i=1}^{N}\left(M_i - GM\right)^2}{N-1} \tag{9}$$

$$\text{Variance of the same pairwise comparisons between two alternatives} = \frac{\sum_{i=1}^{N}\sum_{j=1}^{n-1}\left(M_{ij} - M_i\right)^2}{N\left(n-2\right)} \tag{10}$$

where

$n-1$: The number of replications composed of one real value and $(n-2)$ estimators

$M_i$: The average of individual values for case-$i$

$GM$: The grand mean of all individual values

$n$: The number of different cases

$M_{ij}$: The individual value for replication-$j$ of case-$i$

### 2.2. Other Preference Representation Formats

(1) Utility Values

Assume $U=\{u_1,u_2,\cdots,u_n\}$ is the set of utility values provided by one of the experts. $u_i, i=1,\cdots,n$ represents the utility value corresponding to alternative $X_i$. Generally, $u_i$ range from 0 to 1, and the higher the utility value, the more important the alternative possesses [14, 19-21].

(2) Preference Orderings

Let $O=\{o_1,o_2,\cdots,o_n\}$ be a preference ordering set. This set is the permutation function over the set $\{1,\cdots,n\}$. For example, if an expert provides the ordering $\{3,1,4,2\}$ for four alternatives $\{X_1,X_2,X_3,X_4\}$, then the preference ordering is priority sequence of alternatives, that means the alternative $X_2$ is the best selection among the candidate alternatives, and $X_3$ is the least preferable of the alternatives [14, 19, 20, 22].

(3) Multiplicative Preference Relation [22-24]

In contrast to the utility values and preference orderings, the multiplicative preference relation is always represented by a PCM. Assume the matrix $A=(a_{ij})_{n\times n}$ is the PCM provided by experts. The entry $a_{ij}$ of the PCM is the degree of preference for alternative $X_i$ over $X_j$. The multiplicative PCM of Saaty [24] satisfies $a_{ij}a_{ji}=1$ and $a_{ij}\in[1/9,9]$.

The consistency for the MPR of Saaty [24] can be expressed as follows:

$$a_{ik}=a_{ij}a_{jk}, \forall i,j,k=1,\cdots,n \tag{11}$$

## 3. Ranking of Experts' Expertise According to Different Preference Relations

In this Section we refine the method for ranking the expertise levels of experts when experts provide their judgements for alternatives in MPR, utility values, preference orderings and incomplete preference relations under one criterion.

### 3.1. The Case of MPR

Suppose MPR $A=(a_{ij})_{n\times n}$ is given. MPR $A$ can be converted into FPR $B=(b_{ij})_{n\times n}$ under the transformation $b_{ij}=\frac{a_{ij}}{1+a_{ij}}, b_{ji}=1-b_{ij}, i,j=1,\cdots,n; i<j$. Then we can rank the experts' expertise level using the method in [3]. However, we present the method to directly rank the experts' expertise level from MPR $A$, referring to the method in [3].

According to equation (11), the consistency for MPR can be expressed with the following three relational equations, therefore each entry of the MPR $A$ is estimated in three different ways.

$$\varepsilon a_{ik}^{j1}=a_{ij}a_{jk}, j\neq i,k \tag{12}$$

$$\varepsilon a_{ik}^{j2}=a_{jk}/a_{ji}, j\neq i,k \tag{13}$$

$$\varepsilon a_{ik}^{j3}=a_{ij}/a_{kj}, j\neq i,k \tag{14}$$

Since MPR has property $a_{ij}=1/a_{ji}$, the same results are estimated from equation (12), (13) and (14), and for every entry of the MPR $a_{ij}$, the equations produce as many as $(n-2)$ estimators (since $j\neq i,k$). A expert's level of consistency can be used to measure based on the deviation between the values of the estimations and the real values given by the expert. The consistency level is then used to generate consistency-based ranking of experts.

We perform the following transformation procedure to keep the range of each element $a_{ik}, i\neq k$ of the MPR matrix $A$ within the interval $[1/9,9]$.

1) Perform the logarithm transformation to the base 9 for $a_{ik}, i\neq k$ and the corresponding $(n-2)$ values of estimations $a_{ik}^j, j\neq i,k$.

$$p_{ik}=\log_9 a_{ik}, i\neq k, \quad p_{ik}^j=\log_9 a_{ik}^j, j\neq i,k$$

2) Perform the following transformation so that we have $p_{ik}, p_{ik}^j\in[-1,1]$.

$$t:=\left\{\max p_{ik}^j \,\middle|\, p_{ik}^j>1, i<k; j\neq i,k\right\}$$

$$\varepsilon p_{ik}=p_{ik}/t, \quad \varepsilon p_{ik}^j=p_{ik}^j/t, \quad j\neq i,k$$

3) Perform the antilogarithm transformation for $\varepsilon p_{ik}, \varepsilon p_{ik}^j$ to yield $\varepsilon a_{ik}, \varepsilon a_{ik}^j$.

Now we can calculate the CWS-Index from equations (8), (9), (10) to rank the expertise levels of experts.

### 3.2. The Cases of Utility Values and Preference Orderings

Suppose $U=\{u_1,\cdots,u_n\}$, the set of utility values is given. We can form MPR $U=(u_{ij})_{n\times n}$ in terms of the pairwise comparisons between the utility values, $u_{ij}=u_i/u_j$. Furthermore, MPR $U$ can be converted into FPR $V=(v_{ij})_{n\times n}$ under the following transformation.

$$v_{ij}=\frac{u_{ij}}{1+u_{ij}},v_{ji}=1-v_{ij},i,j=1,\cdots,n;i<j$$

Suppose $O=\{o_1,\cdots,o_n\}$ is the preference ordering set. For preference ordering, in [25] Herrera et al. proposed a nondecreasing function $f$ that assigns ordering values to utilities as follows:

$$u_i=f(n-o_i)=\frac{n-o_i}{n-1},i=1,\cdots,n$$

Then MPR $O=(o_{ij})_{n\times n}$ can be formed in terms of the pairwise comparisons between the utility values, $o_{ij}=u_i/u_j$ as follows:

$$o_{ij}=\frac{n-o_i}{n-o_j},o_i\in\{i=1,\cdots,n\} \tag{15}$$

For four alternatives $\{X_1,X_2,X_3,X_4\}$, if an expert provides the preference ordering $\{o_1,o_2,o_3,o_4\}=\{1,2,3,4\}$, then $o_{14},o_{24},o_{34}$ have the indeterminate forms from equation (15). Even if $n-o_j$ is replaced by 0.0001, when $o_j=n$, we can not distinguish the case of $o_1\neq o_2\neq o_3$. Hence, we revise the equation (15) as follows:

$$o_{ij}=\frac{(n+1)-o_i}{(n+1)-o_j},o_i\in\{i=1,\cdots,n\} \tag{16}$$

Therefore, in the cases of the utility values and preference orderings, the expertise level of experts can be assessed using the equations (8), (9), (10).

### 3.3. The Case of Incomplete MPR

If at least one element in MPR $A^{(h)}=\left(a_{ij}^{(h)}\right)_{n\times n}$ provided by individual expert is unknown, then $A^{(h)}$ is called the incomplete MPR [26]. It is prerequisite for the incomplete MPR to estimate missing pairwise preference values. The values of the estimations obtained from equations (12), (13), (14) can be used to complete the incomplete MPR matrix. In the paper we propose a method to estimate the missing values in the incomplete MPR by modifying a little the equations (12), (13), (14), and study necessary properties.

Each element of MPR $A$ from equations (12), (13), (14) can be estimated as follows:

$$\varepsilon a_{ik}^{j1}=\frac{\sum_{j=1,j\neq i.k}^{n}a_{ij}a_{jk}}{n-2} \tag{17}$$

$$\varepsilon a_{ik}^{j2}=\frac{\sum_{j=1,j\neq i.k}^{n}a_{jk}/a_{ji}}{n-2} \tag{18}$$

$$\varepsilon a_{ik}^{j3}=\frac{\sum_{j=1,j\neq i.k}^{n}a_{ij}/a_{kj}}{n-2} \tag{19}$$

Obiously, when the elements of MPR $A$ are the crisp values, the same results are obataind from equations (17), (18), (19). But it is not so in the case of the interval MPR. So, in order to study hard to the case of the interval MPR in the future, we use single estimated equation by integrating equations (17), (18), (19) as follows:

$$\varepsilon a_{ik}=\frac{\sum_{j=1,j\neq i.k}^{n}\left(a_{ij}a_{jk}+a_{jk}/a_{ji}+a_{ij}/a_{kj}\right)}{3(n-2)} \tag{20}$$

*Theorem 1.* Let $A=(a_{ij})_{n\times n}$ be a multiplicative preference relation. The matrix $A$ is complete consistent if and only if we have $\varepsilon a_{ik}=a_{ik}$ for $\varepsilon a_{ik},i,k=1,\cdots,n$ estimated from the equation (20).

*Proof.* It is obvious for the necessity. One can rewrite the equation (20) as follows:

$$\varepsilon a_{ik}=\frac{\sum_{j=1}^{n}\left(a_{ij}a_{jk}+a_{jk}/a_{ji}+a_{ij}/a_{kj}\right)}{3n} \tag{21}$$

We have $\varepsilon a_{ik}^{j1}=\varepsilon a_{ik}^{j2}=\varepsilon a_{ik}^{j3}$ from the equations (17), (18), (19) and $a_{ki}=1/a_{ik}$. Therefore $\varepsilon a_{ik}=\frac{\sum_{j=1}^{n}a_{ij}a_{jk}}{n},\forall i,k$ hold from the equation (21) and we can see $\sum_{j=1}^{n}a_{ij}a_{jk}=na_{ik}$ from the condition $\varepsilon a_{ik}=a_{ik}$. That is,

$$\begin{aligned}
&a_{11}a_{11}+a_{12}a_{21}+\cdots+a_{1n}a_{n1}=na_{11},\\
&a_{11}a_{12}+a_{12}a_{22}+\cdots+a_{1n}a_{n2}=na_{12},\\
&\quad\cdots\quad\cdots\quad\cdots\\
&a_{11}a_{1n}+a_{12}a_{2n}+\cdots+a_{1n}a_{nn}=na_{1n}
\end{aligned}$$

This is written in form of matrix as $A^T a = na$, where the vector $a$ denote $a=(a_{11},a_{12},\cdots,a_{1n})$. In consequence, $(A^T-nI)a=0$, where $I$ denote the $n$-identity matrix.

Since $a_{ik}\in[1/9,9]$, $a$ is not the zero vector, thus $A^T$ has the largest eigenvalue $\lambda_{\max}=n$.

As proved already, that matrix $A$ is complete consistent if, and only if, we have $\lambda_{\max}=n$. As a result, $A$ is complete consistent matrix.

Now we build an algorithm to estimate the missing elements of the incomplete MPR $A^{(h)}$. First, we will introduce the following notation.

$P=\{(i,k)|i,k\in\{1,\cdots,n\},i\neq k\}$: set of all pairs between alternatives $X_i$ and $X_k$, $U^{(h)}=\{(i,k)\in P|a_{ik}^{(h)}=x\}$: set of the missing elements of $A^{(h)}$,

$K^{(h)}=P\setminus U^{(h)}$: set of the known elements of $A^{(h)}$,

$I_{ik}^{(h)}=\{j\neq i.k|(i,j),(j,k)\in K^{(h)};(i,k)\in U^{(h)}\}$,

The values of $a_{ik}^{(h)}$ can be estimate using the following equation, where $|\cdot|$ denote the cardinality of the set.

$$\varepsilon a_{ik}=\frac{\sum_{j\in I_{ik}^{(h)}}\left(a_{ij}a_{jk}+a_{jk}/a_{ji}+a_{ij}/a_{kj}\right)}{3\left|I_{ik}^{(h)}\right|} \tag{22}$$

Next, the following algorithm is developed to estimate the missing elements of the incomplete MPR $A^{(h)}$.

*Algorithm 1.*

*Step 1:* Let $U_0^{(h)}=U^{(h)}$ and $t=1$.

*Step 2:* Fist, find $I_{ik}^{(h)}$. Then, obtain the set of pairs of indices $N_t\in U_{t-1}^{(h)}$ that can be get newly in iteration $t$ using the equation (22), where $U_{t-1}^{(h)}$ denote the set of the missing elements of the matrix $A^{(h)}$ after iteration $t-1$.

*Step 3:* If $N_t=\varnothing$, then terminate algorithm, otherwise use the equation (22) to $N_t$ and put the estimated value $\varepsilon a_{ik}$ to the matrix $A^{(h)}$. The value of $a_{ki}$ is given by $\varepsilon a_{ki}=1/\varepsilon a_{ik}$, and then set $t=t+1$; go to step 2.

*Remark 1.* In step 3 of algorithm 1, the case of $N_t=\varnothing$ has two states. One is the state in which we obtained all the missing elements of the matrix $A^{(h)}$, and another is the state in which one or more unknown alternatives come into existence. Except the unknown alternatives, the values of pairwise comparisons between others can be estimated through algorithm 1.

First, let's consider the state that there exist one unknown alternative. Assume that first alternative is unknown without loss of generality. If we suppose that $a_{12}=y,(y>0)$, then from the equation (22) the elements in first row can be expressed as follows:

$$a_{12}=y=c_1y,(c_1=1),$$

$$a_{13}=\frac{a_{12}a_{23}}{1}=a_{23}y=c_2y,$$

$$a_{14}=\frac{a_{12}a_{24}+a_{13}a_{34}}{2}=\frac{(a_{24}+c_2a_{34})}{2}\cdot y=c_3y,$$

$$a_{1i}=\frac{\left(a_{2i}+\sum_{k=2}^{i-2}c_ka_{k+1,i}\right)}{i-2}\cdot y=c_{i-1}y,$$

As seen above, the coefficient $c_{i-1}$ in recurrence equation $a_{1i}=c_{i-1}y$ can be find, since it is composed of the values of pairwise comparisons that are already known. If we assume that $\alpha=\min\{c_1,c_2,\cdots,c_{n-1}\}$, $\beta=\max\{c_1,c_2,\cdots,c_{n-1}\}$, then $a_{1i}\in[1/9,9]$ must be hold for any $i$. That is, $\alpha\cdot y\geq 1/9$, $\beta\cdot y\leq 9$ must be hold. Thus one specifies arbitrary $y$ such that $y\in[1/(9\alpha),9/\beta]$. The elements in first column are calculated from $a_{i1}=1/a_{1i}$.

*Theorem 2.* Let all elements in one row of the incomplete MPR $A^{(h)}$ be unknown. Then the largest eigenvalues of the estimation matrices corresponding to arbitrary $y,y'\in[1/(9\alpha),9/\beta],y\neq y'$ are the same.

*Proof.* Suppose all elements in first row of the incomplete MPR $A^{(h)}$ are unknown, then we estimate the missing elements among the rests except first row and first column by algorithm 1. Let $A,A'$ be the estimated matrices corresponding to arbitrary $y,y'\in[1/(9\alpha),9/\beta],y\neq y'$, and let $\lambda_{\max}(A),\lambda_{\max}(A')$ be those largest eigenvalues, repectively.

If we assume that $y'=cy,c>0$, then we have the following relations between the matrices $A$ and $A'$.

$$a'_{1j}=ca_{1j},a'_{j1}=1/a'_{1j},j=2,\cdots,n,$$

$$a'_{ij}=a_{ij},i,j=2,\cdots,n,a_{ii}=1,i=1,\cdots,n$$

Let $w=(w_1,w_2,\cdots,w_n)^T$, $w'=(w'_1,w'_2,\cdots,w'_n)^T$ be the vectors of the weights obtained from the matrices $A$ and $A'$

by applying the row geometric mean prioritization method [24]. Since $\lambda(A)\frac{1}{n}\sum_{i=1}^{n}\frac{(Aw)_i}{w_i}{}_{max}$, if we show that $\frac{(Aw)_i}{w_i}=\frac{(Aw')_i}{w_i'}$ is hold for all $i\in\{1,\cdots,n\}$, then the proof of theorem is completed.

Let $w_i^*, w_i'^*, i=1,\cdots,n$ be the row geometric mean of the matrices $A, A'$, and let $S, S'$ be its' sums, respectively. Then its' weights are determined by $w_i=w_i^*/S, w_i'^*=w_i'^*/S'$. From the relations between the matrices $A, A'$, we have the following equations.

$$w_1'^*=\sqrt[n]{c^{n-1}}\cdot w_1^*, w_i'^*=\frac{1}{\sqrt[n]{c}}\cdot w_i^*, i=2,\cdots,n$$

Then, for arbitrary $i\in\{2,\cdots,n\}$ we have

$$\frac{(Aw')_i}{w_i'}=1+\sum_{j=1,j\neq i}^{n}a_{ij}\cdot\frac{w_j'}{w_i'}=1+\sum_{j=1,j\neq i}^{n}a_{ij}\cdot\frac{\frac{w_j'^*}{S'}}{\frac{w_i'^*}{S'}}=1+\sum_{j=1,j\neq i}^{n}a_{ij}\cdot\frac{w_j'^*}{w_i'^*}=1+\sum_{j=1,j\neq i}^{n}a_{ij}\cdot\frac{w_j^*}{w_i^*}=1+\sum_{j=1,j\neq i}^{n}a_{ij}\cdot\frac{\frac{w_j^*}{S}}{\frac{w_i^*}{S}}=\frac{(Aw)_i}{w_i}$$

Therefore, we have $\lambda_{\max}(A)=\lambda_{\max}(A')$. This completes the proof of Theorem 2.

From Theorem 2, we can know that the consistency levels of the estimated matrices corresponding to arbitrary $y, y'\in[1/(9\alpha), 9/\beta], y\neq y'$.

Similarly, one can consider even the case that there exist the unknown alternatives more than two.

In sum, the incomplete MPR $A^{(h)}$ has become the complete MPR $A$. As a result, one can assess the expertise levels of experts by using the equations (8), (9), (10).

### 3.4. The Case of Incomplete FPR

If at least one element in FPR $B^{(h)}=\left(b_{ij}^{(h)}\right)_{n\times n}$ provided by individual expert is unknown, then $B^{(h)}$ is called the incomplete FPR [27]. In general, FPR can be converted into MPR under the transformation $a_{ij}=\frac{b_{ij}}{1-b_{ij}}, a_{ji}=1-a_{ij}, i,j=1,\cdots,n; i<j$. Therefore, given the incomplete FPR $B^{(h)}$, the corresponding incomplete MPR $A^{(h)}$ can be obtain. Then, by the method in subsection 2.3 one can obtain the complete MPR $A$ from the incomplete MPR $A^{(h)}$. As a result, from the the equations (8), (9), (10) the expertise levels of experts are assessed from $A$.

## 4. The Adjacent Pairwise Comparison Technique and Expertise-Based Ranking of Experts in MCDM

Let $X=\{X_1,\cdots,X_n\}$, $C=\{C_1,\cdots,C_m\}$ be the sets of feasible alternatives and predefined criteria, respectively. When the experts provide their judgements for alternatives in FPRs and MPRs under $m$ criteria, the above proposed method to rank their expertise levels requires $m\times C_n^2=\frac{mn(n-1)}{2}$ pairwise comparisons for each expert. As the number of alternatives increases, this method calls for increasingly more judgments. It is very difficult for the experts to do so many judgments. From such reason it need to propose the method to assess the expertise levels of experts on the basis of a small number of the judgements as possible. In this section, we propose the method to assess expertise-based ranking of experts using the adjacent pairwise comparison technique in MCDM.

Instead of the conventional technique which reqires pairwise comparisons for the given alternatives under each criterion, the adjacent pairwise comparison technique reqires only $(n-1)$ pairwise comparisons between the mutual adjacent alternatives. When the experts provide their pairwise comparisons for the mutual adjacent alternatives in fuzzy and multiplicative preference format under $m$ criteria, it aims to develop the method to assess expertise-based ranking of experts.

### 4.1. The Adjacent Pairwise Comparison Technique with Fuzzy Preference Format and Expertise-Based Ranking of Experts

The adjacent pairwise comparison technique requires for the experts to judge the relative importance between the alternatives corresponding to arbitrary $i=1,\cdots,n-1$ and $j=i+1$.

Let $E=\{e_1,\cdots,e_K\}$ be given set of experts. The adjacent pairwise comparison matrix which the expert $e_k, k=1,\cdots,K$ provides using fuzzy preference format is shown formally below.

$$B^{(k)}=\left(b_{ij}^{(k)}\right)_{n\times n}=\begin{pmatrix} 0.5 & t_1^{(k)} & b_{13}^{(k)} & \cdots & b_{1n}^{(k)} \\ 1-t_1^{(k)} & 0.5 & t_2^{(k)} & \cdots & b_{2n}^{(k)} \\ 1-b_{13}^{(k)} & 1-t_2^{(k)} & 0.5 & \cdots & b_{3n}^{(k)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1-b_{1n}^{(k)} & 1-b_{2n}^{(k)} & 1-b_{3n}^{(k)} & \cdots & 0.5 \end{pmatrix},$$

where $b_{i,i+1}^{(k)}=t_i^{(k)}\in[0,1], i=1,\cdots,n-1$ are the judgements provided by the expert $e_k$. We estimate other elements in matrix $B^{(k)}$ as follows:

$$b_{ij}^{(k)} = \sum_{r=i}^{j-1}\left(t_r^{(k)} - 0.5\right) + 0.5, i = 1,\cdots,n-1;\ j = i+2,\cdots,n \quad (23)$$

Then, it can be showed easily that the matrix estimated from equation (23) satisfies the AC of FPR [8].

Given the adjacent pairwise comparison matrices of all experts in fuzzy preference format, the adjacent pairwise comparison matrix of group can be build as follows:

$$B = \left(b_{ij}\right)_{n\times n} = \begin{pmatrix} 0.5 & t_1 & b_{13} & \cdots & b_{1n} \\ 1-t_1 & 0.5 & t_2 & \cdots & b_{2n} \\ 1-b_{13} & 1-t_2 & 0.5 & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1-b_{1n} & 1-b_{2n} & 1-b_{3n} & \cdots & 0.5 \end{pmatrix}$$

where $t_i = \dfrac{1}{K}\sum_{k=1}^{K} t_i^{(k)}, i = 1,\cdots,n-1$, and other elements in matrix $B$ are estimated as follows:

$$b_{ij} = \sum_{r=i}^{j-1}\left(t_r - 0.5\right) + 0.5, i = 1,\cdots,n-1;\ j = i+2,\cdots,n$$

It is obvious that the adjacent pairwise comparison matrix of the group satisfies the AC of FPR.

Hence, when the experts provide their judgements in fuzzy preference format with the adjacent pairwise comparison technique, we can reform the CWS-Index for evaluating the expertise level of the experts as follows:

$$\mathrm{CWS}_k = \frac{\text{Variance of different pairwise comparisons between group and expert } e_k}{\text{Variance of the same pairwise comparisons between group and expert } e_k} \quad (24)$$

$$\text{Variance of different pairwise comparisons between group and expert} = \frac{\sum_{i=1}^{n-1}\sum_{j=i}^{n}\left(M_{ij}^{(k)} - GM^{(k)}\right)^2}{C_n^2 - 1} \quad (25)$$

$$\text{Variance of the same pairwise comparisons between group and expert} = \frac{\sum_{i=1}^{n-1}\sum_{j=i}^{n}\left[\left(b_{ij}^{(k)} - M_{ij}^{(k)}\right)^2 + \left(b_{ij} - M_{ij}^{(k)}\right)^2\right]}{C_n^2} \quad (26)$$

where

$M_{ij}^{(k)} = \left(b_{ij}^{(k)} + b_{ij}\right)\Big/2, i = 1,\cdots,n-1; i < j$,

$GM^{(k)} = \dfrac{1}{C_n^2}\sum_{i=1}^{n-1}\sum_{j=i}^{n} M_{ij}^{(k)}$.

## 4.2. The Adjacent Pairwise Comparison Technique with Multiplicative Preference Format

When the expert $e_k, k = 1,\cdots,K$ provides the adjacent pairwise comparison matrix in fuzzy preference format, it can be expressed formally as follows:

$$A^{(k)} = \left(a_{ij}^{(k)}\right)_{n\times n} = \begin{pmatrix} 1 & s_1^{(k)} & a_{13}^{(k)} & \cdots & a_{1n}^{(k)} \\ 1/s_1^{(k)} & 1 & s_2^{(k)} & \cdots & a_{2n}^{(k)} \\ 1/a_{13}^{(k)} & 1/s_2^{(k)} & 1 & \cdots & a_{3n}^{(k)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/a_{1n}^{(k)} & 1/a_{2n}^{(k)} & 1/a_{3n}^{(k)} & \cdots & 1 \end{pmatrix}$$

where $a_{i,i+1}^{(k)} = s_i^{(k)}, i = 1,\cdots,n-1$ are the judgements that the expert $e_k$ provides by vertue of a $9^{(x-1)/8}$-scale, and takes the integer from one to nine. Instead of a nine-point scale developed by Saaty [24], the reason introducing a $9^{(x-1)/8}$-scale is that it does not satisfy an equal ratio relation between the mutual adjacent judgements. We estimate other elements in matrix $A^{(k)}$ as follows:

$$a_{ij}^{(k)} = \prod_{r=i}^{j-1} s_r^{(k)}, i = 1,\cdots,n-1;\ j = i+2,\cdots,n \quad (27)$$

It is trivial that the matrix estimated from equation (27) satisfies complete consistency of MPR.

Given the adjacent pairwise comparison matrices of all experts in multiplicative preference format, the adjacent pairwise comparison matrix of experts' group can be build as follows:

$$A = \left(a_{ij}\right)_{n\times n} = \begin{pmatrix} 1 & s_1 & a_{13} & \cdots & a_{1n} \\ 1/s_1 & 1 & s_2 & \cdots & a_{2n} \\ 1/a_{13} & 1/s_2 & 1 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/a_{1n} & 1/a_{2n} & 1/a_{3n} & \cdots & 1 \end{pmatrix}$$

where $a_{i,i+1} = s_i = \sqrt[K]{\prod_{k=1}^{K} s_i^{(k)}}, i = 1,\cdots,n-1$, and other elements in matrix $A$ are estimated as follows:

$$a_{ij} = \prod_{r=i}^{j-1} s_r, i = 1,\cdots,n-1; j = i+2,\cdots,n$$

The adjacent pairwise comparison matrix of the group, $A$ is also the complete consistent MPR. Therefore, when the experts provide their judgements in multiplicative preference format with the adjacent pairwise comparison technique, we can reform the CWS-Index for evaluating the expertise level of the experts as follows:

$$\text{CWS}_k = \frac{\text{Variance of different pairwise comparisons between group and expert } e_k}{\text{Variance of the same pairwise comparisons between group and expert } e_k} \tag{28}$$

$$\text{Variance of different pairwise comparisons between group and expert} = \frac{\sum_{i=1}^{n-1}\sum_{j=i}^{n}\left(M_{ij}^{(k)} - GM^{(k)}\right)^2}{C_n^2 - 1} \tag{29}$$

$$\text{Variance of the same pairwise comparisons between group and expert} = \frac{\sum_{i=1}^{n-1}\sum_{j=i}^{n}\left[\left(a_{ij}^{(k)} - M_{ij}^{(k)}\right)^2 + \left(a_{ij} - M_{ij}^{(k)}\right)^2\right]}{C_n^2} \tag{30}$$

where

$M_{ij}^{(k)} = \sqrt{a_{ij}^{(k)} a_{ij}}, i = 1,\cdots,n-1; i < j$,

$GM^{(k)} = \sqrt[C_n^2]{\prod_{i=1}^{n-1}\prod_{j=i}^{n} M_{ij}^{(k)}}$.

## 4.3. Comprehensive Ranking of Experts' Expertise Level Under Multiple Criteria

Next, when the experts provide their pairwise comparison judgements for the mutual adjacent alternatives in fuzzy and multiplicative preference format under $m$ criteria, we consider the comprehensive ranking method of their expertise level.

Let $\text{CWS}_{kj}$ express the expertise level of the expert $e_k, k = 1,\cdots,K$ obtained under each criterion $C_j, j = 1,\cdots,m$ from the adjacent pairwise comparison technique. Let $r_{kj}$ be the ranking by $\text{CWS}_{kj}$.

Then, one can make ranking matrix for the expertise levels of experts, $R = \left(r_{kj}\right)_{K\times m}$.

In order to complete the comprehensive ranking for the expertise levels of experts, we can apply the Borda Count to the matrix $R$ [29]. This method ranks the expertise levels of the experts according to the total score of each expert with respect to the various criteria. For the entries of the matrix, $r_{kj}, k = 1,\cdots,K; j = 1,\cdots,m$, the score of each expert, $q_k$ is given as below:

$$q_k = Km - \sum_{j=1}^{m} r_{kj}, k = 1,\cdots,K \tag{31}$$

The higher the score $q_k$, the higher is the expert's ranking.

If the criteria have different relative importance weights, we get the synthetic expertise level of the expert $e_k, k = 1,\cdots,K$ as below and rank according to their maginitude.

$$q_k = \sum_{j=1}^{m} \text{CWS}_{kj} \times \lambda_j, k = 1,\cdots,K \tag{32}$$

where $\lambda_j$ denote the importance weight of the criterion $C_j$ and $\lambda_j > 0, \sum_{j=1}^{m} \lambda_j = 1$ hold.

One can apply the method of entropy to the matrix $R$ to determine the objective weights of the criteria. Entropy method is well known as a kind of objective weighting methods to measure the usefull information of obtained data, the smaller the information entropy is, the bigger the weight of the criterion is [28].

Let $\Lambda = (\lambda_1,\cdots,\lambda_m)$ be the vector of weights of the criteria obtained by the method of entropy. $\Lambda = (\lambda_1,\cdots,\lambda_m)$ is calculated as follows:

$$\lambda_j = \frac{1 - P_j}{m - \sum_{j=1}^{m} P_j}, j = 1,\cdots,m,$$

where $P_j = -\frac{1}{\ln K}\sum_{k=1}^{K} p_{ij} \ln p_{ij}$, shows the value of information entropy of the criterion $C_j$; and

$p_{kj} = \frac{\text{CWS}_{kj}}{\sum_{i=1}^{K} \text{CWS}_{ij}}$, shows the standard expertise level of the expert $e_k$ under the criterion $C_j$. Next, one can get the score $\text{CWS}_k$ from the equation (32) and ranks the experts.

## 5. Weighting for Expertise Levels of Experts

Expertise-based ranking of experts can be applied to aggregate the individual opinions into a group opinion by developing into experts' importance weights in GDM.

The vector the total scores for the expertise levels of the experts, $Q = (q_1, \cdots, q_K)$ is regarded as the original data sequence. As CWS-indices to evaluate the expertise levels of the experts are formed of the ratios of the variances, they may have large differences. For which reason, to improve the smoothness of the original data sequence under the surcumstance of remaining its basic characteristic, we apply the following logarithmic function transformation to deal with the original data sequence.

$$r_k = \log_t (q_k + 1), k = 1, \cdots, K ,$$

where $t = \min_{k \in \{1, \cdots, K\}} \{q_k + 1\}$. It is clear that $r_k \geq 1, k = 1, \cdots, K$.

Let $W = (w_1, \cdots, w_K)$ be a vector of the experts' importance weights. $W$ is determined as follows:

$$w_k = \frac{r_k}{\sum_{i=1}^{K} r_i}, k = 1, \cdots, K$$

Suppose that the decision matrices provided by the experts, $D^{(k)} = \left(d_{ij}^{(k)}\right)_{m \times n}, k = 1, \cdots, K$, are given, where $d_{ij}^{(k)}$ indicates the evaluation value of the attribute of the alternative $X_j, j = 1, \cdots, n$ for the index $C_i, i = 1, \cdots, m$ provided by the expert $e_k, k = 1, \cdots, K$. Then, we can obtain the aggregated decision matrix of the group, $D = (d_{ij})_{m \times n}$, by considering the experts' importance weights as follows:

$$d_{ij} = \sum_{k=1}^{K} w_k d_{ij}^{(k)}$$

If the vector of the importance weights of the criteria is given, we compute the synthetic score of the alternative $X_j$ as follows:

$$s_j = \sum_{i=1}^{m} \lambda_i d_{ij}, j = 1, \cdots, n$$

The higher the value $s_j$, the greater is the importance of the alternative $X_j$.

## 6. Illustrative Example

In order to show the workability of the proposed method, we start with a numerical example to illustrate it. Suppose there are four people who are expert at judging the five senses on the wet goods. These experts are expressed as $E = \{e_1, e_2, e_3, e_4\}$. In order to judge the five senses on the wet goods, they ought to be assess four wet goods: alcoholic liquor, distilled liquor, brewage and fruit wine, with various sensible characteristics such as bouquet, taste and clearness. The wet goods form a set of four alternatives $X = \{X_1, X_2, X_3, X_4\}$. And the sensible characteristics form a set of three criteria $C = \{C_1, C_2, C_3\}$.

The experts were asked to fill their judgements in the blank space of Table 1, with the MPRs $a_{ij}$. $a_{ij}$ represents the relative importance of the alternative $X_i$ to the alternative $X_j$.

***Table 1.*** *The evaluation form.*

| | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|---|---|---|---|---|
| $X_1$ | 1 | $\boxed{\phantom{x}}$ | $\boxed{\phantom{x}}$ | $\boxed{\phantom{x}}$ |
| $X_2$ | | 1 | $\boxed{\phantom{x}}$ | $\boxed{\phantom{x}}$ |
| $X_3$ | | | 1 | $\boxed{\phantom{x}}$ |
| $X_4$ | | | | 1 |

Suppose that the experts' judgements under each criterion in the evaluation form of Table 1 are given as Table 2.

***Table 2.*** *The experts' judgements under each criterion.*

| $a_{ij}$ | $C_1$ | | | | $C_2$ | | | | $C_3$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
| $a_{12}$ | 2 | 2 | 2 | 1 | 4 | 5 | 1 | 5 | 7 | 2 | 3 | 3 |
| $a_{13}$ | 6 | 4 | 3 | 4 | 4 | 7 | 4 | 5 | 7 | 3 | 4 | 4 |
| $a_{14}$ | 8 | 5 | 6 | 3 | 6 | 8 | 4 | 4 | 6 | 8 | 8 | 5 |
| $a_{23}$ | 3 | 2 | 2 | 1 | 3 | 6 | 3 | 6 | 7 | 1 | 3 | 3 |
| $a_{24}$ | 5 | 3 | 5 | 1 | 3 | 6 | 1 | 6 | 8 | 6 | 3 | 3 |
| $a_{34}$ | 8 | 5 | 5 | 1 | 4 | 6 | 3 | 3 | 5 | 6 | 8 | 6 |

For example, from Table 2, the multiplicative judgement matrix of the expert $e_1$ under the criterion $C_1$ is formed as follows:

$$A^{(1)} = \left(a_{ij}^{(1)}\right)_{4\times 4} = \begin{pmatrix} 1 & 2 & 6 & 8 \\ 1/2 & 1 & 3 & 5 \\ 1/6 & 1/3 & 1 & 8 \\ 1/8 & 1/5 & 1/8 & 1 \end{pmatrix}$$

The estimation values of the entries of the matrix $A^{(1)}$ are obtained from the equation (12) as Table 3.

***Table 3.*** *The estimation values of the entries of matrix* $A^{(1)}$.

| $a_{ij}^{(1)}$ | **Before transformation** | | | **After transformation ( $\log_9 a_{ij}^{(1)}$ )** | | |
|---|---|---|---|---|---|---|
| | **Actual judgement** | **Estimation value** | | **judgement** | **Estimation value** | |
| $a_{12}^{(1)}$ | 2 | 2 | 1.6 | 0.315 | 0.315 | 0.214 |
| $a_{13}^{(1)}$ | 6 | 6 | 1 | 0.815 | 0.815 | 0 |
| $a_{14}^{(1)}$ | 8 | 10 | 0.75 | 0.946 | 1.048 | -0.131 |
| $a_{23}^{(1)}$ | 3 | 3 | 0.625 | 0.5 | 0.5 | -0.214 |
| $a_{24}^{(1)}$ | 5 | 4 | 24 | 0.732 | 0.631 | 1.446 |
| $a_{34}^{(1)}$ | 8 | 1.333 | 1.667 | 0.946 | 0.131 | 0.232 |

Applied the transformation procedure in subsection 3.1 to Table 3, one can get Table 4, and calculate CWS-index from the equations (8), (9), (10).

**Table 4.** *The transformation values of the entries of matrix* $A^{(1)}$.

| $a_{ij}^{(1)}$ | Transformation procedure in subsection 3.1 | | | $M_i$ | $(M_i - GM)^2$ | $\sum_{j=1}^{n-1}(M_{ij} - M_i)^2$ |
|---|---|---|---|---|---|---|
| | Judgement | Estimation value | | | | |
| $a_{12}^{(1)}$ | 1.615 | 1.615 | 1.384 | 1.538 | 1.397 | 0.036 |
| $a_{13}^{(1)}$ | 3.451 | 3.451 | 1.000 | 2.634 | 0.007 | 4.006 |
| $a_{14}^{(1)}$ | 4.211 | 4.913 | 0.820 | 3.315 | 0.354 | 9.584 |
| $a_{23}^{(1)}$ | 2.137 | 2.137 | 0.723 | 1.666 | 1.111 | 1.334 |
| $a_{24}^{(1)}$ | 3.043 | 2.608 | 9.000 | 4.883 | 4.680 | 25.514 |
| $a_{34}^{(1)}$ | 4.211 | 1.220 | 1.424 | 2.285 | 0.189 | 5.585 |
| sum | | | | | 7.74 | 46.05 |

$$\text{Variance of different pairwise comparisons between two alternatives} = \frac{(4-1)\times 7.74}{6-1} = 4.644$$

$$\text{Variance of the same pairwise comparisons between two alternatives} = \frac{46.059}{6\times(4-2)} = 3.838$$

$$\text{CWS-Index} = \frac{\text{Variance of different pairwise comparisons between two alternatives}}{\text{Variance of the same pairwise comparisons between two alternatives}} = \frac{4.644}{3.838} = 1.21$$

As above, one can find the CWS-indices for every criterion and every expert as Table 5.

**Table 5.** *CWS-indices and ranking of the experts under the criteria.*

| Criterion | Result | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|---|
| $C_1$ | CWS-index | 1.21 | 0.984 | 1.678 | 0.812 |
| | Ranking | $e_4 \prec e_2 \prec e_1 \prec e_3$ | | | |
| $C_2$ | CWS-index | 0.834 | 0.822 | 0.76 | 0.787 |
| | Ranking | $e_3 \prec e_4 \prec e_2 \prec e_1$ | | | |
| $C_3$ | CWS-index | 0.748 | 2.225 | 0.767 | 0.686 |
| | Ranking | $e_4 \prec e_1 \prec e_3 \prec e_2$ | | | |

Table 5 shows that the experts $e_3, e_1, e_2$ have the highest sensible ability in the criteria 'bouquet', 'taste', 'clearness', respectively. While the expert $e_3$ has the lowest sensible ability in the criterion 'taste', the expert $e_4$ in the criteria 'bouquet' and 'clearness' is so.

Next, we apply the adjacent pairwise comparison technique to the data of Table 2 to evaluate the experts' sensible ability. For example, according to the discussion of subsection 4.2, the multiplicative judgement matrix of the experts' group under the criterion $C_1$ can be formed as follows:

$$A = (a_{ij})_{4\times 4} = \begin{pmatrix} 1 & 1.682 & 3.13 & 11.77 \\ 0.595 & 1 & 1.861 & 6.999 \\ 0.319 & 0.537 & 1 & 3.761 \\ 0.085 & 0.143 & 0.266 & 1 \end{pmatrix}$$

Then, we can obtain the sensible ability of the experts under the criterion $C_1$ as follows:

$$\text{CWS}_{11} = 2.225, \text{CWS}_{12} = 0.874, \text{CWS}_{13} = 2.094, \text{CWS}_{14} = 0.283$$

The CWS-Indices for all criteria are given as Table 6.

***Table 6.*** *CWS-indices and ranking using the adjacent pairwise comparison technique.*

| Criterion | Result | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|---|
| $C_1$ | CWS-index | 2.225 | 0.874 | 2.094 | 0.283 |
| | Ranking | $e_4 \prec e_2 \prec e_3 \prec e_1$ | | | |
| $C_2$ | CWS-index | 0.129 | 0.198 | 0.08 | 0.071 |
| | Ranking | $e_4 \prec e_3 \prec e_1 \prec e_2$ | | | |
| $C_3$ | CWS-index | 0.108 | 0.223 | 0.185 | 0.087 |
| | Ranking | $e_4 \prec e_1 \prec e_3 \prec e_2$ | | | |

Compared Table 6 with Table 5, the experts have the same ranking in the criteria 'clearness', while there exist a little difference in the criteria 'bouquet', 'taste'.

Now, according to Table 5 and Table 6, we form the evaluation matrices for the experts' sensible ability, $R = \left(\mathrm{CWS}_{kj}\right)_{4\times 3}$, respectively; then we complete the comprehesive ranking for their sensible ability by the Borda Count in subsection 4.3.

***Table 7.*** *The comprehesive ranking by the Borda Count.*

| | Result | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|---|
| Table 5 | (equation (31)) | 6 | 7 | 5 | 0 |
| | Ranking | $e_4 \prec e_3 \prec e_1 \prec e_2$ | | | |
| Table 6 | (equation (31)) | 6 | 6 | 5 | 1 |
| | Ranking | $e_4 \prec e_3 \prec e_1 \sim e_2$ | | | |

As seen in Table 7, the comprehensive ranking of the experts obtained by the adjacent pairwise comparison technique includes the result obtained by the traditional pairwise comparisons. This shows the acceptability of the adjacent pairwise comparison technique.

Now, we suppose that the pairwise comparison judgements of the experts for the alternative $X_1$ under each criterion are unknown. For example, the incomplete multiplicative judgement matrix of the expert $e_1$ under the criterion $C_1$ is made from Table 2 as bellow.

$$A^{(1)} = \left(a_{ij}^{(1)}\right)_{4\times 4} = \begin{pmatrix} 1 & x & x & x \\ x & 1 & 3 & 5 \\ x & 1/3 & 1 & 8 \\ x & 1/5 & 1/8 & 1 \end{pmatrix}$$

Applied the algorithm 1 of subsection 3.3, we can estimate the missing elements of the matrix $A^{(1)}$. Refered to the Remark 1 in subsection 3.3, since

$$a_{12} = y = c_1 y, (c_1 = 1),$$

$$a_{13} = \frac{a_{12}a_{23}}{1} = a_{23}y = c_2 y = 3y,$$

$$a_{14} = \frac{a_{12}a_{24} + a_{13}a_{34}}{2} = \frac{(a_{24} + c_2 a_{34})}{2} \cdot y = c_3 y = \frac{29}{2} y,$$

$$\alpha = \min\{c_1, c_2, c_3\} = 1, \quad \beta = \max\{c_1, c_2, c_3\} = \frac{29}{2},$$

we can take $y = 1/3$ to ensure $y \in [1/(9\alpha), 9/\beta]$. Therefore we have $a_{12} = 1/3$, $a_{13} = 1$, $a_{14} = 29/6$. Similarly, one can estimate the incomplete multiplicative judgements of the experts for the alternative $X_1$ under each criterion (Table 8).

***Table 8.*** *The estimation of the incomplete judgements of the experts for the alternative* $X_1$.

| $a_{ij}$ | $C_1$ | | | | $C_2$ | | | | $C_3$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
| $a_{12}$ | 1/3 | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 1/2 | 1 | 1/2 | 1 |
| $a_{13}$ | 1 | 2 | 2 | 1 | 3 | 6 | 6 | 6 | 7/2 | 1 | 3/2 | 3 |
| $a_{14}$ | 29/6 | 13/2 | 15/2 | 3/2 | 11/2 | 9 | 7 | 6 | 21/4 | 9 | 19/4 | 15/2 |

According to the method in subsection 3.1, the CWS-Indices and ranking are obtained as Table 9.

***Table 9.*** *The CWS-Indices and ranking for the incomplete judgements.*

| Criterion | Result | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|---|
| $C_1$ | CWS-Index | 5.51 | 2.581 | 4.737 | 0.2 |
| | Ranking | $e_4 \prec e_2 \prec e_3 \prec e_1$ | | | |
| $C_2$ | CWS-Index | 2.132 | 2.282 | 0.728 | 3.386 |
| | Ranking | $e_3 \prec e_1 \prec e_2 \prec e_4$ | | | |
| $C_3$ | CWS-Index | 4.254 | 5.977 | 2.742 | 1.877 |
| | Ranking | $e_4 \prec e_3 \prec e_1 \prec e_2$ | | | |

And we do the comprehensive ranking their sensible ability by Borda Count (Table 10).

***Table 10.*** *The comprehensive ranking by Borda Count.*

| Result | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|
| $q_k$ | 6 | 6 | 3 | 3 |
| Ranking | $e_4 \sim e_3 \prec e_1 \sim e_2$ | | | |

Obviously, the result of Table 10 includes the result of Table 7. This shows the acceptability of the method proposed in subsection 3.3.

## 7. Conclusion

This paper mainly studies on the method to assess expertise-based ranking of experts in GDM with multiple criteria and different preference representation formats. First of all, we discuss on the combination of expertise as 'the ability to differentiate consistently' and the consistency of various types of preference relations on the basis of the methodology proposed in [3], when experts give their judgements for alternatives in various types of preference relations, such as multiplicative preference relations (MPRs), utility values, preference orderings, even incomplete FPRs or MPRs under certain criterion. In particular, we develop an algorithm to estimate the missing elements of the incomplete MPR and show the properties supporting the algorithm. At the same time, we propose a expertise-based ranking method using the adjacent pairwise comparison technique in order to reduce the number of the pairwise comparison judgements provided by the experts in GDM with multiple criteria. Finally, expertise-based ranking of experts is applied to aggregate the individual opinions into a group opinion by developing into experts' importance weights in GDM. Through the numerical analyses we confirm that proposed method is acceptable and can fill up some gaps in GDM. Further study may extend to incomplete preference relations under linguistic [30], interval [16], intuitionistic fuzzy [33], hesitant fuzzy linguistic [31] and interval-valued Pythagorean fuzzy [34], Fermatean fuzzy [32] environment in GDM.

## Abbreviations

| | |
|---|---|
| GDM | Group Decision Making |
| DM | Decision Maker |
| CWS-Index | Cochran-Weiss-Shanteau Index |
| AC | Additive Consistency |
| FPR | Fuzzy Preference Relation |
| MPR | Multiplicative Preference Relation |
| MCDM | Multi-Criteria Decision Making |
| PCM | Pairwise Comparison Matrix |

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## Author Contributions

**SongHo Sim:** Methodology, Validation

**SinHyok Kim:** Writing – original draft, Writing – review & editing

## Conflicts of Interest

The authors declare no conflicts of interest.

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