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Existence of Law Density of a Pair (d-dimensional Diffusion X, First Component Running Maximum M) with Coefficients Depending on (X, M), d $>$ 1

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Abstract: The purpose of this paper is to consider a stochastic differential equation guiding X a continuous d-dimensional diffusion process, the coefficients of which depending on the pair (X, M), where M is the running supremum of the first component X^1. As an application of our work, we could think of a firm the activity of which is characterized by a set of processes $(X^1, \cdots, X^d)$. But one of them, for instance X^1, could be linked to an alarm. Such a $(d+1)-$ dimensional process (X, M) could present a crucial interest in this case where M could be an alarm. Indeed, the possibility of an alarm at time t namely the event $\{\exists s \leq t : \ X_s^1 > u\}$ is identical to the event $\{M_t > u\}$ when the specific (and dangerous) threshold u is exceeded. This means that the law of M is closely linked to the law of the hitting time when X^1 reaches such a dangerous level u. Here is proved that, for all positive real number t, the law of the $(d+1)$-dimensional random vector (X_t, M_t) admits a density with respect to the Lebesgue measure. The solution of such a stochastic differential equation is built using a recursion method. The existence of the density of the law of (X, M) is based on Malliavin calculus. This density is solution of a partial differential equation in a weak sense. Moreover, such a recursive construction will allow to build simulated solutions. So finally such a tool could allow us to build an alarm system to detect the hitting time when the alarm could occur.

Keywords: Running Supremum Process, Joint Law Density, Malliavin Calculus

1. Introduction

Here we look for general (but continuous) cases where this density exists. In $d-$dimensional case, the law of the $(d+1)$-random vector (M_t, X_t) is absolutely continuous, where $M_t := \sup_{s \leq t} X_s^1$. In Nakatsu [19], under more general assumptions than ours, the law of any pairs $(X_t^i, M_t^j), 1 \leq i, j \leq d$ is proved to be absolutely continuous. In the one-dimensional case we got the regularity of this density [7]. Such results are used in [6] where this density is proved to be a weak solution of a partial differential equation. Here is an extension of Téo Balauze's second year of Master's degree report [3] to space of dimension $d > 1$.

The model is as following: let a filtered probability space $(\Omega, (\mathcal{F}_t = \sigma(W_u, \ u \leq t))_{t \geq 0}, \mathbb{P})$ where $W := (W_u, \ u \geq 0)$ is a d-dimensional Brownian motion.

Let X be a d-dimensional diffusion process solution to

$$dX_t = B(X_t, M_t)dt + \sum_{i=1}^{d} A_i(X_t, M_t)dW_t^i, \ \ X_0 = x \in \mathbb{R}^d, \ t > 0, \tag{1}$$

where we denote $M_t := \sup_{s\leq t} X^1_s$, B: $\mathbb{R}^d\times\mathbb{R} \to \mathbb{R}^d$ and A_i: $\mathbb{R}^d\times\mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, d$, are bounded so belonging to Ikeda-Watanabe' spaces, cf. infra Theorem 2.1:

Definition 1.1 Let $\mathcal{A}^{d,r}$ be the set of functions $\mathbb{R}^+ \times \Omega^d \to \mathbb{R}^d \otimes \mathbb{R}^r$ measurable Borelian on $\mathcal{B}(\mathbb{R}^+) \times \mathcal{B}(\Omega^d)$ to $\mathcal{B}(\mathbb{R}^d \otimes \mathbb{R}^r)$ where $\Omega = C(\mathbb{R}^+, \mathbb{R}^d)$.
Moreover $\forall t \geq 0$, $B(t, .)$ and $A(t, .)$ with a fixed t are Borelian.

In case of A, B not bounded but with bounded and continuous differential, we produce the existence of the law of the pair $V_t = (X_t, \sup_{s\leq t} X^1_s) \in \mathbb{R}^{d+1}$.

The main result is set in the first section. Then some useful Balauze's tools are extended to dimension $d > 1$ using the definition of the process V recursively in Section 3. The third section put some Malliavin calculus tools. Finally, the main result is established recursively in Section 5. An Appendix is postponed after the conclusion.

2. The Law of V_t is Absolutely Continuous

Here we quote Ikeda-Watanabe' existence Theorem found in [15] page 169:

Theorem 2.1 Suppose that $A \in \mathcal{A}^{d,r}$ and $B \in \mathcal{A}^{d,1}$ are bounded and continuous. Then for an initial law of X_0 μ with compact support, there exists a solution (X, W) of Equation (1) s.t. the law of X_0 is μ.

Here, the definition of solution can be found in [15] page 161.

Actually we apply this Ikeda-Watanabe' theorem to $\alpha(t, \omega) := A(\omega_t, \sup_{s\leq t} \omega^1_s)$ and $\beta(t, \omega) := B(\omega_t, \sup_{s\leq t} \omega^1_s)$ in case of A, B bounded.

Anyway if A, B are not bounded but satisfy other assumptions we go to prove that for any $t > 0$, the joint law of $V_t := (X_t, M_t)$ admits a density with respect to the Lebesgue measure. For this purpose, we use a recursion scheme for the law of X_t then "Malliavin calculus" (specifically Nualart's results [21]) for the law of the current maximum M.

Let us denote $C^i_b(\mathbb{R}^d)$ the set of the functions i times differentiable not necessarily bounded with bounded derivatives. Our main result is:

Theorem 2.2 We assume that A and B satisfy

$$A \in C^1_b(\mathbb{R}^{d+1}, \mathbb{R}^{d^2}) \text{ and } B \in C^1_b(\mathbb{R}^{d+1}, \mathbb{R}^d). \quad (2)$$

Then the joint law of $V_t = (X_t, M_t)$ admits a density with respect to the Lebesgue measure for all $t > 0$.

The next subsection extends Balauze's definitions and results.

3. Extension of Balauze [3] Section 3 to a d > 1 Dimension Space

We here extend Balauze's results to dimension d, the proofs seem to be generally extendable. Let $T \in \mathbb{R}_+ \setminus \{0\}$, the first step is his recursive definition:

$$X^0_t = M^0_t = x_0 \in \mathbb{R}^d,$$

$$X^{n+1}_t = x_0 + \int_0^t B(X^n_s, M^n_s)ds + \sum_{i=1}^d A_i(X^n_s, M^n_s)dW^i_s,\ 0 < t \leq T, \quad (3)$$

$$M^{n+1}_t = \sup\{X^{1,n+1}_s, s \leq t\}.$$

Under hypotheses on B and A, Balauze's Proposition 3.1 proof is valuable in dimension $d > 1$, more or less the proof is identical, replacing $|.|$ by norms in $\mathbb{R}^d$: $\|.\|$, with some more multiplicative constants. Thus we quote a little bit generalised to d dimension Balauze's result what we apply to the first component $X^{1,n}$ and its running maximum M^n :

Proposition 3.1 Let A and B satisfy

$$A \in C^1_b(\mathbb{R}^{d+1}, \mathbb{R}^{d^2}) \text{ and } B \in C^1_b(\mathbb{R}^{d+1}, \mathbb{R}^d). \quad (4)$$

We consider the sequence defined above in (3):

$$X^{n+1}_t = x_0 + \int_0^t B(X^n_s, M^n_s)ds + \sum_{i=1}^d A_i(X^n_s, M^n_s)dW^i_s,\ 0 < t \leq T, \quad (5)$$

then $\forall p \geq 2$, there exists a constant $c(p, T, x_0, A, B)$ such that

$$\forall t \in [0, T],\ \forall n \in \mathbb{N},\ E[\sup_{s\leq t} \|X^n_s\|^p] \leq c(p, T, x_0, A, B).$$

Proof: (i) Starting the recurrence $X^0_s = x_0, M^0_s = x^1_0$: This process takes its values in $\mathbb{R}^{d+1}$, is adapted, continuous, and satisfies:

$$E[\sup\{||X^0_s||^p], s \leq t\}] = \|x_0\|^p \leq C.$$

(ii) Assume that the required property is satisfied up to order n. By Definition (3) (X^{n+1}, M^{n+1}) is an adapted continuous process in $\mathbb{R}^{d+1}$. We bound the p-norm

$$\|X_s^{n+1}\|^p \leq 3^{p-1}\left[\|x_0\|^p + \|\int_0^s B(X_u^n, M_u^n)du\|^p + \|\int_0^s \sum_{i=1}^d A_i(X_u^n, M_u^n)dW_u^i\|^p\right].$$

We now deal with the supremum in time:

$$\sup_{s\leq t}\|X_s^{n+1}\|^p \leq 3^{p-1}\left[\|x_0\|^p + \sup_{s\leq t}\left(\int_0^s \|B(X_u^n, M_u^n)\|du\right)^p + \sup_{s\leq t}\left\|\int_0^s \sum_{i=1}^d A_i(X_u^n, M_u^n)dW_u^i\right\|^p\right].$$

The second term above is increasing with respect to s, so

$$\sup_{s\leq t}(\int_0^s \|B(X_u^n, M_u^n)\|du)^p = (\int_0^t \|B(X_u^n, M_u^n)\|du)^p.$$

Since we assume $B \in C_b^1$, B is Lipschitz and denoting $K_B := \sup\|\nabla B\|$:

$$\begin{aligned}\|B(X_u^n, M_u^n)\| &\leq \|B(X_u^n, M_u^n) - B(x_0, x_0)\| + \|B(x_0, x_0)\|\\ &\leq K_B(\|X_u^n - x_0\| + |M_u^n - x_0^1|) + \|B(x_0, x_0)\|.\end{aligned}$$

Otherwise $M_u^n - x_0^1 = \sup_{v\leq u}(X^1)_v^n - x_0^1$, so

$$|M_u^n - x_0^1| \leq \sup_{v\leq u}|(X^1)_v^n - x_0^1| \leq \sup_{v\leq u}\|X_v^n - x_0\|.$$

With respect to multiplicative constants

$$(\int_0^t \|B(X_u^n, M_u^n)\|du)^p \leq (C + \int_0^t \sup_{v\leq u}\|X_v^n - x_0\|du)^p \leq C_1 + C_2 T\sup_{s\leq t}\|X_s^n\|^p \in L^1.$$

(iii) We now deal with the third term $\sum_{i=1}^d \int_0^s A_i(X_u^n, M_u^n)dW_u^i$ which is a continuous local martingale so with Burkholder-Davis-Gundy's inequality

$$\mathbb{E}[\sup_{s\leq t}\|\sum_{i=1}^d \int_0^s A_i(X_u^n, M_u^n)dW_u^i\|^p] \leq C_{BDGp}\mathbb{E}\left[(\int_0^T A_i^k.A_i^j(X_u^n, M_u^n)du)^{p/2}\right].$$

As it is done for the term concerning B, since $A_i^k.A_i^j$ are too Lipschitz, we manage with $A_i^k A_j^k$ to bound this term with a similar bound (Einstein convention is used):

$$\mathbb{E}[\sup_{s\leq t}\|\sum_{i=1}^d \int_0^s A_i(X_u^n, M_u^n)dW_u^i\|^p] \leq C_1' + C_2' T\sup_{s\leq t}\|X_s^n\|^p \in L^1.$$

Finally, there exists a constant $C(p, T, x_0, A, B)$ such that

$$\mathbb{E}\left[(\sup_{s\leq t}\|X_s^n\|_d)^p\right] \leq C(p, T, x_0, A, B). \tag{6}$$

Similarly, Balauze's Theorem 3.2, is extended to d dimension, in case $p = 2$:
Theorem 3.1 Assume B and $A \in C_b^1(\mathbb{R}^{d+1})$ and let the stochastic differential equation

$$dX_t = B(X_t, M_t)dt + \sum_{i=1}^d A_i(X_t, M_t)dW_t^i,\ \ X_0 = x\in \mathbb{R}^d,\ 0 < t \leq T, \tag{7}$$

then

I. the Picard's scheme (3) converges almost surely and uniformly in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ to the process X, a d-dimensional diffusion process,

II. this process X is a solution to the stochastic differential equation (7), where M is the supremum of X^1 and the limit of $M^n_t = \sup_{s\le t}(X^n)^1_t$.

III. this process is unique up to indistinguishability and satisfies

$$\exists C(x_0, T, B, A),\ E\left(\sup_{t\in[0,T]}\|X_t\|^2\right) \le C(x_0, T, B, A). \tag{8}$$

I. According to (3) with n and $n+1$ and using the standard bound $(\sum_{n=1}^k a_n)^p \le k^{p-1}(\sum_{n=1}^k a_n^p)$ in case $p = 2$:

$$\|X^{n+1}_t - X^n_t\|^2 = \left\| \int_0^t (B(X^n_s, M^n_s) - B(X^{n-1}_s, M^{n-1}_s)ds \right.$$
$$\left. + \sum_{i=1}^d (A_i(X^n_s, M^n_s) - A_i(X^{n-1}_s, M^{n-1}_s))dW^i_s \right\|^2$$

$$\le 2\left[\left\| \int_0^t (B(X^n_s, M^n_s) - B(X^{n-1}_s, M^{n-1}_s)ds \right\|^2 + \right. \tag{9}$$
$$\left. \left\| \int_0^t \sum_{i=1}^d (A_i(X^n_s, M^n_s) - A_i(X^{n-1}_s, M^{n-1}_s))dW^i_s \right\|^2 \right].$$

(i) The first term in (9) is bounded using standard bound in L^2 and Lipschitz function B:

$$\| \int_0^t (B(X^n_s, M^n_s) - B(X^{n-1}_s, M^{n-1}_s)ds\|^2 \le T\int_0^t \|B(X^n_s, M^n_s) - B(X^{n-1}_s, M^{n-1}_s)\|^2 ds$$
$$\le 2TK_B^2 \int_0^t (\|X^n_s - X^{n-1}_s\|^2 + \|M^n_s - M^{n-1}_s\|^2)ds.$$

A useful short lemma:

Lemma 3.1 Let a real sequence of functions in time (a_n), then

$$|\sup_t a^n_t - \sup_t a^{n-1}_t| \le \sup_t |a^n_t - a^{n-1}_t|.$$

It allows to bound $|M^n_s - M^{n-1}_s| \le \sup_{r\le s} |(X^1)^n_r - (X^1)^{n-1}_r| \le \sup_{r\le s} \|X^n_r - X^{n-1}_r\|$ and this first term is bounded as follows

$$\| \int_0^t (B(X^n_s, M^n_s) - B(X^{n-1}_s, M^{n-1}_s)ds\|^2 \le 4TK_B^2 \int_0^t \sup_{r\le s} \|X^n_r - X^{n-1}_r\|^2 ds. \tag{10}$$

(ii) Concerning the second term in (9), it is the norm of a continuous martingale, using Burkholder-Davis-Gundy's inequality, yields

$$\mathbb{E}\left[\left\| \int_0^t \sum_{i=1}^d (A_i(X^n_s, M^n_s) - A_i(X^{n-1}_s, M^{n-1}_s))dW^i_s \right\|^2 \right]$$
$$\le C_{BGD,2} \sum_i \int_0^t \mathbb{E}\left(\|A_i(X^n_s, M^n_s) - A_i(X^{n-1}_s, M^{n-1}_s)\|^2\right) ds.$$

As for the first term, using Lipschitz property of A_i,

$$\|A_i(X^n_s, M^n_s) - A_i(X^{n-1}_s, M^{n-1}_s)\|^2 \le 4K_A^2 \sup_{r\le s} \|X^n_r - X^{n-1}_r\|^2.$$

With respect to a multiplicative constant, yields

$$\mathbb{E}\left[\| \int_0^t \sum_{i=1}^d (A_i(X_s^n, M_s^n) - A_i(X_s^{n-1}, M_s^{n-1}))dW_s^i\|^2\right] \leq C\int_0^t \mathbb{E}[\sup_{r\leq s}\|X_r^n - X_r^{n-1}\|^2]ds. \tag{11}$$

(iii) As a conclusion, using expectation of (10) and bound (11) there exists a constant $D > 0$ such that

$$\mathbb{E}\left[\sup_{t\leq T}\|X_t^{n+1} - X_t^n\|^2\right] \leq D\int_0^T \mathbb{E}[\sup_{r\leq s}\|X_r^n - X_r^{n-1}\|^2]ds. \tag{12}$$

(iv) Let us denote $g_n(t) = \mathbb{E}[\sup_{r\leq t}\|X_r^n - X_r^{n-1}\|^2]$. So the above bound can be written as

$$g_{n+1}(T) \leq D\int_0^T g_n(s)ds.$$

By iteration yields

$$g_{n+1}(T) \leq D^n \int_0^T ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 ..... \int_0^{s_{n-2}} g_1(s_{n-1})ds_{n-1}. \tag{13}$$

(v) We now look at the first term in this series: $g_1(t) = \mathbb{E}[\sup_{r\leq t}\|X_r^1 - x\|^2]$ recalling

$$g_1(t) = \mathbb{E}[\sup_{r\leq t}\| \int_0^r B(x_0, x_0)ds + \sum_{i=1}^d A_i(x_0, x_0)dW_s^i\|^2]$$

Remark that $\int_0^t B(x_0, x_0)ds + \int_0^t \sum_{i=1}^d A_i(x_0, x_0)dW_s^i = tB(x_0, x_0) + \sum_i A_i(x_0, x_0)W_t^i$. Using Doob inequality we bound $\mathbb{E}[\sup_{r\leq s}\|W_r\|^2] \leq 4ds$,

$$g_1(t) \leq 2t\|B(x_0, x_0\|^2 + 2\|A\|^2 4dt \leq Kt$$

with a constant K. Solving the iteration (13)

$$g_{n+1}(t) \leq (DK)^n \frac{t^n}{n!}$$

and $\sqrt{g_{n+1}(t)} \leq \frac{(DKt)^{n/2}}{\sqrt{n!}}$. According to d'Alembert's ratio test, look at

$$\lim_{n\to\infty} \frac{\sqrt{\frac{(DKt)^{n+1}}{(n+1)!}}}{\sqrt{\frac{(DK)^n t^n}{n!}}} = \lim \frac{\sqrt{DKt}}{\sqrt{n+1}} = 0 < 1$$

so the series $\sqrt{(DK)^n \frac{t^n}{n!}}$ is convergent. One deduces the convergence of the positive series $\sqrt{g_{n+1}(t)}$.

(vi) Finally, we look at the convergence in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of the sequence (X^n):

$$\|\sum_{i=0}^k \sup_{s\leq T}\|X_s^{n+i} - X_s^{n+i-1}\|_d\|_{L^2} \leq \sum_{i=0}^k \|\sup_{s\leq T}\|X_s^{n+i} - X_s^{n+i-1}\|_d\|_{L^2}, \tag{14}$$

where the generic term $\|\sup_{s\leq T}\|X_s^{n+i} - X_s^{n+i-1}\|_d\|_{L^2} = \sqrt{g_{n+i}(s)}$ is a convergent series. So the sequence (X^n) is uniformly convergent in $L^2(\Omega, \mathbb{R}^d)$.

(vii) We now turn to almost sure convergence. From (14) yields the integrable random variable

$$\mathcal{X} := \sum_{i=0}^\infty \sup_{s\leq T}\|X_s^{n+i} - X_s^{n+i-1}\|_d$$

is almost surely finite. Let N such that $\forall\omega \notin N$, $\forall\varepsilon, \exists l(\omega)$ such that

$$\sum_{i=l(\omega)}^\infty \sup_{s\leq T}\|X_s^{n+1}(\omega) - X_s^n(\omega)\|_d < \varepsilon.$$

Let $p, q \geq l(\omega)$,

$$\sup_{s\leq T}\|X_s^p(\omega) - X_s^q(\omega)\|_d \leq \sum_{n\geq l(\omega)} \sup_{s\leq T}\|X_s^{n+1}(\omega) - X_s^n(\omega)\|_d \leq \varepsilon,$$

so the sequence $(X_n(\omega)$ is uniformly convergent in $C([0, T], \mathbb{R}^d)$ and admits a limit $X \in C([0, T], \mathbb{R}^d)$ which states the almost sure convergence of (X^n).

Lemma 3.2 The sequence (M_t^n) goes to $M_t = \sup_{s\leq t} X_s^1$ almost surely and in L^2.

Proof: We use the following bound:

$$|M_s^n - M_s| \leq \sup_{r\leq s}|(X^1)_r^n - (X^1)_r| \leq \sup_{r\leq s}\|X_r^n - X_r\|_d \tag{15}$$

which proves the almost sure convergence. Then, going to the

L^2-norm:

$$|M_s^n - M_s|^2 \leq \sup_{r\leq s} |(X^1)_r^n - (X^1)_r|^2 \leq \sup_{r\leq s} \|X_r^n - X_r\|_d^2. \tag{16}$$

Using now (14), we deduce the uniform convergence in L^2 of the sequence (M_s^n).

II. The Lipschitz so uniform continuity of A, B proves that this limit X (together with $M_t = \sup_{s\leq t} X_s^1$) satisfy the stochastic differential equation (7):

$$\|X_t - x_0 - \int_0^t B(X_s, M_s)ds - \int_0^t \sum_{i=1}^d A_i(X_s, M_s)dW_s^i\|_d \leq$$

$$\|X_t - X_t^{n+1}\|_d + \| \int_0^t (B(X_s, M_s) - B(X_s^n, M_s^n))ds\|_d + \| \int_0^t (A(X_s, M_s) - A(X_s^n, M_s^n))dW_s\|_d$$

Using Lipschitz property of B, and (15)

$$\| \int_0^t (B(X_s, M_s) - B(X_s^n, M_s^n))ds\|_d \leq 2K_B \int_0^t \sup_{r\leq s} \|X_r^n - X_r\|_d ds.$$

Using then the L^2-norm of second term, Lipschitz property of A and (15):

$$\mathbb{E}\left(\left\|\int_0^t (A(X_s, M_s) - A(X_s^n, M_s^n))dW_s\right\|_d^2\right) \leq 2K_A^2 \int_0^t \sup_{r\leq s} \|X_r^n - X_r\|_d^2 ds.$$

We gather these bounds :

$$\mathbb{E}\left(\left\|X_t - x_0 - \int_0^t B(X_s, M_s)ds - \int_0^t \sum_{i=1}^d A_i(X_s, M_s)dW_s^i\right\|_d^2\right) \leq$$

$$3\mathbb{E}\left(\|X_t - X_t^{n+1}\|_d^2\right) + 12K_B^2 \mathbb{E}\left(\int_0^t \sup_{r\leq s} \|X_r^n - X_r\|_d ds\right)^2 + 6K_A^2 \mathbb{E} \int_0^t \sup_{r\leq s} \|X_r^n - X_r\|_d^2 ds.$$

By the previous computations, each of these terms converge to 0 as n goes to $+\infty$ so the stochastic differential equation (7) is satisfied.

III. Uniqueness of the solution of (7).

The bound (8) is a direct consequence of the same bound satisfied for any n (6) and $p = 2$.

We now suppose there exists two solutions (X, M) and $(\tilde{X}, \tilde{M}))$ and we copy paste the proof part I:

$$\|X_t - \tilde{X}_t\|^2 = \| \int_0^t (B(X_s, M_s) - B(\tilde{X}_s, \tilde{M}_s)ds + \sum_{i=1}^d (A_i(X_s, M_s) - A_i((\tilde{X}_s, \tilde{M}_s))dW_s^i\|^2$$

$$\leq 2\left[\left\|\int_0^t (B(X_s, M_s) - B(\tilde{X}_s, \tilde{M}_s)ds\right\|^2 + \left\|\int_0^t \sum_{i=1}^d (A_i(X_s, M_s) - A_i((\tilde{X}_s, \tilde{M}_s))dW_s^i\right\|^2\right].$$

Concerning the first term, we reproduce the proof of (10) replacing the pair X^n, X^{n-1} by the pair $X, \tilde{X}$:

$$\| \int_0^t (B(X_s, M_s) - B(\tilde{X}_s, \tilde{M}_s)ds\|^2 \leq 4TK_B^2 \int_0^t \sup_{r\leq s} \|X_r - \tilde{X}_r\| ds. \tag{17}$$

Similarly, we copy paste (11):

$$\mathbb{E}\left[\|\int_0^t \sum_{i=1}^d (A_i(X_s,M_s)-A_i(\tilde{X}_s,\tilde{M}_s))dW_s^i\|^2\right] \le C\int_0^t \mathbb{E}[\sup_{r\le s}\|X_r-\tilde{X}_r\|^2]ds. \tag{18}$$

Globally, we sum both bounds:

$$\mathbb{E}[\sup_{t\le T}\|X_t-\tilde{X}_t\|^2] \le 8TK_B^2\int_0^T \sup_{r\le s}\mathbb{E}[\|X_r-\tilde{X}_r\|]+2C\int_0^T \mathbb{E}[\sup_{r\le s}\|X_r-\tilde{X}_r\|^2]ds.$$

Gronwall lemma gives the conclusion: $X_r-\tilde{X}_r=0$ almost surely on $[0,T]\times\Omega$.

4. Short Malliavin Calculus Summary

The material of this subsection can be found in Section 1.2 of Nualart [21]. Let $\mathbb{H}=L^2([0,T],\mathbb{R}^d)$ endowed with the usual scalar product $\langle .,.\rangle_{\mathbb{H}}$ and the associated norm $\|.\|_{\mathbb{H}}$. For all $(h,\tilde{h})\in\mathbb{H}^2$:

$$W(h):=\int_0^T h(t)dW_t$$

is a centered Gaussian variable with variance equal to $\|h\|^2_{\mathbb{H}}$. If $\langle h,\tilde{h}\rangle_{\mathbb{H}}=0$ then the random variables $W(h)$ and $W(\tilde{h})$ are independent.

Let $\mathcal{S}$ denote the class of smooth random variables F defined as following:

$$F=f(W(h_1),...,W(h_n)) \tag{19}$$

where $n\in\mathbb{N}$, $h_1,...,h_n\in\mathbb{H}$ and f belongs to $C_b(\mathbb{R}^n)$.

Definition 4.1 The Malliavin derivative of a smooth variable F defined in (19) is the $\mathbb{H}$ valued random variable given by

$$DF=\sum_{i=1}^n \partial_i f(W(h_1),...,W(h_n))h_i.$$

The following is Nualart Proposition 1.2.1 page 26 in [21]:

Proposition 4.1 The operator D is closable from $L^p(\Omega)$ into $L^p(\Omega,\mathbb{H})$ for any $p\ge 1$.

For any $p\ge 1$, let us denote $\mathbb{D}^{1,p}$ the domain of the operator D in $L^p(\Omega)$, meaning that $\mathbb{D}^{1,p}$ is the closure of the class of smooth random variables $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p}=\left(\mathbb{E}[|F|^p]+[\mathbb{E}[\|DF\|^p_{\mathbb{H}}]\right)^{1/p}.$$

Malliavin calculus is a powerful tool to prove the absolute continuity of random variables law. For instance, Nualart's Propositions 2.1.10 and 2.1.11 [21] are the tools to deduce the absolute continuity of M^n law from this of $(X^n)^1$ law:

Proposition 4.2 Let X_t^1, $t\in[0,T]$, continuous, satisfying
(i) $E(\sup_t(X_t^1)^2)<\infty$.
(ii) $\forall t,\ X_t^1\in\mathbb{D}^{1,2}$,
(iii) $D_.X_t^1$ admits a continuous version,
(iv) $\mathbb{E}[\sup_t\|DX_t^1\|^2_H]<\infty$,
then $M_t=\sup_{s\le t}X_s^1\in\mathbb{D}^{1,2}$.

Moreover suppose that
(v) $\|DX_t^1\|_H\neq 0$ on the set $\{t: X_t^1=M\}$.

Then the law of M is absolutely continuous.

Moreover, looking at the proof of this proposition, yields

$$\|DM_t\|^2_H\le\sup_{s\le t}\|DX_s^1\|^2_H. \tag{20}$$

Lemma 4.1 (Prop 4.19 Balauze or Prop 1.3.8 page 43 Nualart [21]) Let u a continuous and adapted process taking its value in $\mathbb{R}^d$ s.t.
(i) $\mathbb{E}(\sup_{s\in[0,T]}\|u_s\|^2)<+\infty$,
(ii) for any $s\in[0,T]$, $u_s\in\mathbb{D}^{1,2}$,
(iii) for any $t\in[0,T]$, the process $D_t(u_.)$ is continuous and adapted,
(iv) $\sup_{t\in[0,T]}\left[\mathbb{E}\left(\sup_{s\in[t,T]}\|D_t(u_s)\|^2\right)\right]<+\infty$.

Then $\int_0^T u_sds\in\mathbb{D}^{1,2}$ and

$$D(\int_0^T u_sds)=\int_0^T D(u_s)ds\ ;\ \forall t\in[0,T],\ D_t(\delta(u)=u_t+\delta(D_t(u)).$$

Proof in Appendix.

5. Law Density of (X_t^n, M_t^n) Obtained Recursively

I. The initial pair $(X_t^0,M_t^0)=(x_0,x_0)$ goes to

$$X_t^1=x_0+B(x_0,x_0)t+A(x_0,x_0)W_t\in\mathbb{R}^d$$

which is a vectorial Gaussian random variable $\mathcal{N}(x_0+B(x_0,x_0)t,A\tilde{A}t)$, the law of it is absolutely continuous. Moreover the density of the law of the pair (W_t,M_t^1) in $\mathbb{R}^{d+1}$ where M_t^1 is the running maximum of W_t^1 is well known:

$$p(x,m;t) = \frac{2(2m-x^1)}{\sqrt{2\pi t^3}}\exp(-\frac{2m-x^1}{2t})(\frac{1}{\sqrt{2\pi t}})^{n-1}\exp(-\frac{\sum_{i=2}^d (x^i)^2}{2t}).$$

We now suppose that the law of (X_t^n, M_t^n) is absolutely continuous and denote its density $g_n(x,m,t)$:
Consider a function $f \in C_K^2(\mathbb{R}^d)$ and Ito formula using Definition (3), $\quad t > 0$:

$$\begin{aligned} f(X_t^{n+1}) &= f(x_0) + \int_0^t \frac{\partial f}{\partial x^i}(X_t^{n+1})B^i(X_s^n, M_s^n)ds + \frac{\partial f}{\partial x^i}(X_t^{n+1})A_i(X_s^n, M_s^n)dW_s^i + \\ \frac{1}{2} & D^2 f(X_t^{n+1})A\tilde{A}(X_s^n, M_s^n)(X_s^n, M_s^n)ds \end{aligned} \tag{21}$$

where, for simplicity and now on, we omit the sums using Einstein convention. We integrate under probability $\mathbb{P}$ the above equality noting that the second term is a martingale (recall that here the support of function f is compact so the integrand is bounded):

$$E[f(X_t^{n+1})] = f(x_0) + E\left[\int_0^t \frac{\partial f}{\partial x^i}(X_t^{n+1})B^i(X_s^n, M_s^n)ds\right] + \frac{1}{2}E\left[\int_0^t D^2 f(X_t^{n+1})A\tilde{A}(X_s^n, M_s^n)ds\right].$$

Then, according to the recursive definition (5) of the process X^{n+1}, its law conditionally to (X_s^n, M_s^n) is a Gaussian law, denoted as $p_{n+1}(.|X_s^n, M_s^n)$

$$\begin{aligned} E[f(X_t^{n+1})] &= f(x_0) + \int_0^t ds \int_{\mathbb{R}^d}\int_{x^1\le m}\left[\int_{\mathbb{R}^d}\frac{\partial f}{\partial y^i}(y)p_{n+1}(y|x,m)dy\right]B^i(x,m)g_n(s,x,m)dxdm \\ &+ \frac{1}{2}\int_0^t ds \int_{\mathbb{R}^d}\int_{x^1\le m}\left[\int_{\mathbb{R}^d} D_y^2 f(y)p_{n+1}(y|x,m)dy\right]A\tilde{A}(x,m)g_n(s,x,m)dxdm. \end{aligned} \tag{22}$$

Since we use a function $f \in C^2(\mathbb{R}^d)$ with compact support we can use Fubini Theorem and commute the order of integrations:

$$\int_{\mathbb{R}^d}\int_{x^1\le m}\left[\int_{\mathbb{R}^d}\frac{\partial f}{\partial y^i}(y)p_{n+1}(y|x,m)dy\right]B^i(x,m)g_n(s,x,m)dxdm =$$

$$\int_{\mathbb{R}^d}\frac{\partial f}{\partial y^i}(y)\left[\int_{\mathbb{R}^d}\int_{x^1\le m} p_{n+1}(y|x,m)B^i(x,m)g_n(s,x,m)dxdm\right]dy$$

We now operate an integration by parts

$$\int_{\mathbb{R}^d}\frac{\partial f}{\partial y^i}(y)\left[\int_{\mathbb{R}^d}\int_{x^1\le m} p_{n+1}(y|x,m)B^i(x,m)g_n(s,x,m)dxdm\right]dy =$$

$$-\int_{\mathbb{R}^d} f(y)\frac{\partial}{\partial y^i}\left[\int_{\mathbb{R}^d}\int_{x^1\le m} p_{n+1}(y|x,m)B^i(x,m)g_n(s,x,m)dxdm\right]dy.$$

Similarly, concerning the second term in (22), we operate two integrations by parts:

$$\int_{\mathbb{R}^d}\int_{x^1\le m}\left[\int_{\mathbb{R}^d} D_y^2 f(y)p_{n+1}(y|x,m)dy\right]A\tilde{A}(x,m)g_n(s,x,m)dxdm =$$

$$\left[\int_{\mathbb{R}^d} f(y)D_y^2\left[\int_{\mathbb{R}^d}\int_{x^1\le m} p_{n+1}(y|x,m)A\tilde{A}(x,m)g_n(s,x,m)\right]dxdm\right]dy.$$

Using the fact that f is with compact support, derivation and integration can commute so yields the law of X_t^{n+1} is absolutely continuous and admits a density h_{n+1} by integration for $m \ge x^1$ as following:

$$h_{n+1}(x,t) = \delta(x_0) - \int_0^t \left[\int_{\mathbb{R}^d}\int_{x^1\le m}\frac{\partial}{\partial y^i}p_{n+1}(y|x,m)B^i(x,m)g_n(s,x,m)dxdm\right] \tag{23}$$

$$+\left[\int_{\mathbb{R}^d}\int_{x^1\le m} D_y^2 p_{n+1}(y|x,m)A\tilde{A}(x,m)g_n(s,x,m)dxdm\right]$$

that actually is a partial differential equation in a weak sense.

II. Absolute continuity of the law of M^{n+1}: To use Proposition 4.2 we have to prove five points recursively:

1. (a) $E(\sup_t (X_t^1)^2) < \infty$.
2. (b) $\forall t,\ X_t^1 \in \mathbb{D}^{1,2}$,
3. (c) $D_{\cdot} X_t^1$ admits a continuous version,
4. (d) $\mathbb{E}[\sup_t \|DX_t^1\|_H^2] < \infty$,
5. (e) $\|DX_t^1\|_H \neq 0$ on the set $\{t : X_t^1 = M\}$

and the conclusion will be the absolute continuity of the law of M^{n+1}

1. A first step is to prove the five points for $n = 0$: since X_t^0 is the constant $x_0 \in \mathbb{R}^d$, it satisfies DX^0 is null so obviously the points (a) to (d) are satisfied. We now look at $X_t^1 = x_0 + \int_0^t B(x_0, x_0)ds + \int_0^t \sum_{i=1}^d A_i(x_0, x_0)dW_s^i$, meaning $X_t^1 = x_0 + B(x_0, x_0)t + \sum_{i=1}^d A_i(x_0, x_0)W_t^i$, so

$$D_s^i(X_t^1) = A_i(x_0, x_0)\mathbb{1}_{[0,t]}(s).$$

so (e) is satisfied for $n = 1$.

2. We now suppose that these five points are satisfied up to order $n \in \mathbb{N}$.

(a) It is a consequence of Balauze's Proposition 3.1 which is above extended to d-dimension in Proposition 3.1.

(b) Remark that X^n continuous and $\mathcal{F}-$adapted implies that M^n is too. So (3) shows that (X^{n+1}, M^{n+1}) is continuous and $\mathcal{F}$-adapted. Moreover, by recursion $(D_t(X_{\cdot}^n), D_t(M_{\cdot}^n))$ is continuous and $\mathcal{F}$-adapted: using Nualart prop 1.2.3. page 28 extended to integrals, the Malliavin derivative of (3) satisfies the following, using Section 4, Lemma 4.1 (and Nualart Prop 1.3.8 page 43 (1.57)), Malliavin derivative of Equation (3):

$$\begin{aligned} D_r X_t^{n+1} &= D_r\Big[\int_0^t B(X_s^n, M_s^n)ds + \int_0^t \sum_{i=1}^d A_i(X_s^n, M_s^n)dW_s^i\Big] \\ &= \int_0^t [\partial_x B(X_s^n, M_s^n).D_r X_s^n + \partial_m B(X_s^n, M_s^n)D_r M_s^n]ds + \sum_{i=1}^d A_i(X_r^n, M_r^n) \\ &\quad + \sum_{i=1}^d \int_0^t (\partial_x A_i(X_s^n, M_s^n).D_r X_s^n + \partial_m A^i(X_s^n, M_s^n)D_r M_s^n)dW_s^i. \end{aligned} \tag{24}$$

Thus $\forall t$, actually $X_t^{n+1} \in \mathbb{D}^{1,2}$ meaning point (b).

(c) By recursion assumption, $\forall t \in [0, T]$, the process $(D_t(X_{\cdot}^n), D_t(M_{\cdot}^n))$ is continuous and $\mathcal{F}$-adapted. According to (24), $D_t(X_{\cdot}^{n+1})$ is continuous and $\mathcal{F}$-adapted.

Using now Nualart Proposition 2.1.10 [21] to prove that $M^{n+1} \in \mathbb{D}^{1,2}$: assumptions of this proposition are (a) and (b), so $M^{n+1} \in \mathbb{D}^{1,2}$.

(d) Look at (24): since ∇B and ∇A are bounded and using page 109 Nualart, $\|D_t M\|_H^2 \leq \sup_{s\leq t} \|DX_s^1\|_H^2$ (i.e. (20)) yields:

$$\|D_r(X^{n+1})_t^1\|_H \leq \int_0^t 2\|\nabla B^1\|\|D_r(X^n)_s^1\|_H ds + \sum_{i=1}^d \|A_i^1(X_r^n, M_r^n)\|_H + \tag{25}$$

$$\|\int_0^t (\partial_x A_i^1(X_s^n, M_s^n)D_r(X^n)_s^i + \partial_m A_i^1(X_s^n, M_s^n)D_r M_s^n)dW_s^i\|_H.$$

Firstly remark that the second term $\sum_{i=1}^d \|A_i^1(X_r^n, M_r^n)\|_H$ does not depend on t.

Then using the boundness of the gradients ∇B the first term is bounded as

$$\int_0^t 2\|\nabla B^1\|\|D_r(X^n)_s^1\|_H ds \leq 2K_B \int_0^t \sup_{s\leq t} \|D_r(X^n)_s^1\|_H ds.$$

Finally ∇A bounded, $\|D_t M\|_H^2 \leq \sup_{s\leq t} \|DX_s^1\|_H^2$ (cf.(20)), and Burkholder-Davis-Gundy (for $m = 1/2$) inequality provide the following bound of the third term:

$$E\left[\|\int_0^t (\partial_x A_i(X_s^n, M_s^n)D_r(X^n)_s^i + \partial_m A_i(X_s^n, M_s^n)D_r M_s^n)dW_s^i\|_H\right] \leq$$

$$2K_A C_{BGD1/2}\left(E[\int_0^t \sup_{s\leq t} \|D_r(X^n)_s^i\|_H^2 ds]\right)^{\frac{1}{2}}.$$

(e) Let $s \leq t$ such that $X_s^{1,n+1} = M_t^{n+1}$. Using the same lines as the proof of Proposition 2.1.11 [21], we prove that almost surely $\{s : X_s^1 = M_t\} \subset \{s : DM_t = DX_s^1\}$; so on the set $\{s : X_s^1 = M_t\}$ we have $DM_t = DX_s^1$; if there exists s such that $\|DX_s^1\|_H = 0$ we would have also $\|DM_t\|_H = 0$, so $DM_t = 0$, and using Nualart p.25 Lemma 1.2.1 (1.30) : for any $u \in$ the domain of δ, $E[M_t\delta(u)] = E[\int u_s D_s M_t ds]$, so M_t would be 0 which is a contradiction and finally $\|D(X^n)^1_s\|_H \neq 0$ on the set $\{s : (X^n)^1_s = M^n_t\}$.

III. So, recursively, the process (X^n, M^n) law is absolutely continuous. We denote $g_n(t,.,.)$ the density of its law. Actually $X^n \to X$ uniformly in L^2 and almost surely. We come back to the recursive equation (21) and let n go to infinity with $f \in C^2_K$:

$$f(X_t) = f(x_0) + \int_0^t \frac{\partial f}{\partial x^i}(X_s)B^i(X_s, M_s)ds + \int_0^t \frac{\partial f}{\partial x^i}(X_s)A_i(X_s, M_s)dW^i_s + \frac{1}{2}\int_0^t A\tilde{A}(X_s, M_s)D^2 f(X_s)ds. \tag{26}$$

Let a bounded function f. Since $f(X^n_t) \to f(X_t)$ almost surely and uniformly in L^2 and Lebesgue dominated theorem denoting $h_n(t,x) = \int_{x^1 \le m} g_n(t,x,m)dm$):

$$E[f(X^n_t)] = \int_{\mathbb{R}^{d+1}} f(x)h_n(t,x)dx \to E[f(X_t)].$$

Thus the law of X is absolutely continuous: indeed let a Borel set $A \subset \mathbb{R}^d$ with null Lebesgue measure:

$$\mathbb{P}\{X \in A\} = \lim_n \mathbb{P}\{X_n \in A\} = 0,$$

which means that the law of X is absolutely continuous;

Concerning the law of the running maximum M we now use Proposition (4.2) (v): suppose that on the set $\{t : X^1_t = M\}$, $t \in [0,T]$, $\|DX^1_t\|_H = 0$. So actually $X^1_t = E(X^1_t)$ in case of null Malliavin derivative. But on this set $M = X^1_t = E(X^1_t)$, meaning the first component of process X could be a constant on this set. But this is a contradiction with the definition of process X solution of stochastic differential equation (26) which includes a Brownian part

$$X^1_t = x^1_0 + \int_0^t B^1(X_s, M_s)ds + A^1_i(X_s, M_s)dW^i_s.$$

This could not be a constant and $\|DX^1_t\|_H \neq 0$ which proves that the law of M is absolutely continuous using Proposition (4.2) (v).

IV. Then, the law of (X) is absolutely continuous, so using (26) and some integration by parts, denoting $h(.,t)$ the density of the law of X_t, the density g of (X, M) is solution to the following:

$$\begin{aligned} h(x,t) &= \int_{x^1 \le m} g(s,x,m)dm \\ &= \delta_{x_0} + \int_0^t \int_{m \ge x^1} \left[-\frac{\partial(B^i g)}{\partial x^i}(s,x,m) + \frac{1}{2}D^2_x[A\tilde{A}g] \right](s,x,m)dsdm, \end{aligned} \tag{27}$$

so , in a weak sense since the regularity of g is not proved:

$$g(t,x,m) = \delta_{x_0,x^1_0}(x,m) + \int_0^t \left[-\frac{\partial(B^i g)}{\partial x^i}(s,x,m) + \frac{1}{2}D^2_x[A\tilde{A}g](s,x,m) \right] ds,\ m \ge x^1. \tag{28}$$

6. Conclusion

We emphasize here that such results on the law of the pair (M, X) could be used when we are interested in barrier options, since the law of time activation (or extinction) is strongly related to that of the couple (M, X). Moreover, for concrete applications, the tools built in [13, 14] would be used.

Finally, the recursive construction could be useful for simulations processes, up to detect the events when a barrier could be reached.

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There were no financial or commercial conflict of interests to declare which arose during the preparation or publication process of this article.

Appendix

Here the proof of Lemma 4.1 =Prop 4.19 Balauze extended to d-dimension:
Let u a continuous and adapted process taking its value in $\mathbb{R}^d$ such that:
(i) $\mathbb{E}(\sup_{s\in[0,T]}\|u_s\|^2) < +\infty$,
(ii) for any $s \in [0,T], u_s \in \mathbb{D}^{1,2}$,
(iii) for any $t \in [0,T]$, the process $D_t(u_.)$ is continuous and adapted,
(iv) $\sup_{t\in[0,T]}\left[\mathbb{E}\left(\sup_{s\in[t,T]}\|D_t(u_s)\|^2\right)\right] < +\infty.$
Then $\int_0^T u_s ds \in \mathbb{D}^{1,2}$ and

$$D(\int_0^T u_s ds) = \int_0^T D(u_s)ds\ ;\ \forall t \in [0,T],\ D_t(\delta(u) = u_t + \delta(D_t(u)).$$

Proof: 1. We consider the Riemann sums $S_k = \frac{T}{k}\sum_{j=1}^k u_{jT/k}$. So $\forall k$

$$E[\|S_k\|^2] \leq \frac{T^2}{k} C_k \sum_{j=1}^k \|u_{jT/k}\|^2 \leq T^2 C_k E\left[\sup_{s\in[0,T]}\|u_s\|^2\right]. \tag{29}$$

Since the process u is continuous, S_k goes a.s. to $\int_0^T u_s ds$. We now look at

$$\|S_k - \int_0^T u_s ds\|^2 = \|\frac{T}{k}\sum_{j=1}^k u_{jT/k} - \int_0^T u_s ds\|^2 = \frac{1}{k^2}\|\sum_{j=1}^k \int_0^T (u_{jT/k} - u_s)ds\|^2 \leq$$

$$\frac{1}{k^2}C_k\sum_{j=1}^k \|\int_0^T (u_{jT/k} - u_s)ds\|^2 \leq \frac{T}{k^2}C_k\sum_{j=1}^k \int_0^T \|u_{jT/k} - u_s\|^2 ds \leq$$

$$\frac{T}{k^2}C_k 4\sum_{j=1}^k \int_0^T \sup_{r\in(0,T)}\|u_r\|^2 ds = D_k \sup_{r\in(0,T)}\|u_r\|^2.$$

This bound is integrable using Assumption (i). So dominated convergence Lebesgue Theorem yields that S_k goes to $\int_0^T u_s ds$ in $L^2(\Omega)$:

$$\lim_{k\to\infty} E[\|S_k - \int_0^T u_s ds\|^2] = E[\lim_{k\to\infty}\|S_k - \int_0^T u_s ds\|^2] = 0$$

2. Using Assumption (ii), u_s so $S_k \in \mathbb{D}^{1,2}$ and $D(S_k) = \frac{T}{k}\sum_{j=1}^k D(u_{jT/k})$, look at

$$E\left(\|D(S_k)\|^2_{L^2[0,T]}\right) = E\left(\left[\int_0^T \|\frac{T}{k}\sum_{j=1}^k D_s(u_{jT/k})\|^2 ds\right]\right)$$

$$\leq C(k,T)\sup_{0\leq s\leq T} E[\sup_{s\leq r\leq T}\|D_s(u_r)\|^2] < \infty \tag{30}$$

using Assumption (iv) above.
3. Nualart's Lemma 1.2.3. page 28 proves that $\int_0^T u_s ds \in \mathbb{D}^{1,2}$.
4. By linearity, $D_t(S_k) = \frac{T}{k}\sum_{j=1}^k D_t u_{jT/k}$ and with Assumption (iii), k going to infinity, almost surely:

$$D_t(S_k) \to \int_0^T D_t(u_s)ds,\ \forall t \in [0,T].$$

In L^2,

$$\|D_t(S_k) - \int_0^T D_t(u_s)ds\|^2 = \frac{1}{k^2}\|\sum_{j=1}^k \int_0^T (D_t(u_{jT/k}) - D_t(u_s))ds\|^2 \leq C(k,T)\sup_{t\leq r\leq T}\|D_t(u_r)\|^2.$$

By integration on $[0, T]$ and using Assumption (iv)

$$E\left(\int_0^T \|D_t(S_k) - \int_0^T D_t(u_s)ds\|^2 dt\right) \leq C(k,T)E\left(\int_0^T \sup_{t\leq r\leq T} \|D_t(u_r)\|^2 dt\right) \leq$$

$$TC(k,T)\sup_t E[\sup_{t\leq r\leq T} \|D_t(u_r)\|^2] < \infty.$$

Dominated convergence Theorem yields

$$\lim_{k\to\infty} E\left(\int_0^T \|D_t(S_k) - \int_0^T D_t(u_s)ds\|^2 dt\right) = E\left(\int_0^T \lim_{k\to\infty} \|D_t(S_k) - \int_0^T D_t(u_s)ds\|^2 dt\right) = 0$$

meaning that $D(S_k) \to \int_0^T D(u_s)ds$. The operator D is continuous in L^2 so $S_k \to \int_0^T u_s ds$ implies $D(S_k) \to D(\int_0^T u_s ds)$ and

$$D(\int_0^T u_s ds) = \int_0^T D(u_s)ds.$$

Assumptions (i) and (ii) imply that $u \in \mathbb{D}^{1,2}(L^2([0,T])$.

We now look at

$$E\left(\int_0^T \|D_t(u_s)ds\|^2 ds\right) = \int_0^T E\left(\|D_t(u_s)\|^2\right) ds \leq T \sup_{t\in 0,T]} E\left(\sup_{t\leq s} \|D_t(u_s)\|^2\right) < \infty$$

according to Assumption (iv).

Assumption (iii) gives that the process $D_t(u_.)$ is adapted. So using Nualart (1.30) page 24, $\forall t \in [0,T]$, $D_t(u_.) \in Domain(\delta)$ and $\delta(D_t(u)) = \int_t^T D_t(u_s)dB_s$.

We bound this martingale using Burkholder-Davis-Gundy inequality:

$$\begin{aligned} E\left(\int_0^T \|\int_t^T D_t(u_s)dB_s\|^2 dt\right) &\leq \int_0^T C_{BDG} E(\int_t^T \|D_t(u_s)\|^2 ds)dt \\ &\leq T^2 C_{BDG} \sup_{t\in[0,T]} E(\sup_{t\leq s\leq T} \|D_t(u_s)\|^2) < \infty. \end{aligned}$$

Process u is adapted, we use Nualart Prop 1.3.8. page 43 (1.57) or Lemma 4.1:

$$D_t(\delta(u) = u_t + \delta(D_t(u))$$

so is ended the proof of the lemma.

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