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A Unified Framework for Stochastic Differential Equations Driven by Hölder Continuous Functions with Applications to Senegalese Groundwater Management

Bou Diop*

Department of Mathematics, IBA DER THIAM University, Thiès, Senegal

Email address:
b.diop@univ-thies.sn (Bou Diop)
*Corresponding author

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Abstract: This research develops an innovative mathematical framework that unifies classical and modern approaches to stochastic differential equations (SDEs) driven by irregular paths. We introduce a novel Newton-Cotes integration method that bridges Young integration and rough path theory, providing comprehensive solutions for processes with Hölder continuous sample paths. The theoretical foundation establishes existence, uniqueness, and regularity results across the entire roughness spectrum. Our methodology offers practical advantages through adaptive numerical schemes with proven convergence rates and robust parameter estimation techniques combining maximum likelihood and Bayesian approaches. The framework's real-world utility is demonstrated through a detailed case study of groundwater management in Senegal, where our model achieves a 52% improvement in prediction accuracy over traditional methods. This enhancement enables more reliable drought early warnings and sustainable water resource planning in semi-arid regions facing climate uncertainty. The unified approach has broad applicability across scientific domains dealing with irregular data patterns, including finance, environmental science, and engineering.

Keywords: Stochastic Differential Equations, Hölder Continuity, Rough Path Theory, Newton-Cotes Integration, Groundwater Management, Parameter Estimation, Senegal Hydrology, Water Resources

1. Introduction

Stochastic differential equations (SDEs) have become fundamental tools for modeling dynamic systems across various scientific disciplines, from finance and physics to environmental science and engineering. The classical theory of SDEs, primarily developed for semimartingales and ItÃ´ processes, has demonstrated remarkable success in numerous applications. However, empirical evidence increasingly shows that many natural systems exhibit behaviors that deviate significantly from the semimartingale framework, particularly in their irregularity and roughness characteristics.

Recent advances in rough path theory [1, 2] and fractional stochastic calculus [3, 4] have opened new avenues for analyzing processes driven by Hölder continuous sample paths with low regularity. These developments are particularly relevant for hydrological systems, where groundwater dynamics and precipitation patterns in semi-arid regions often display long-range dependence and roughness characteristics that cannot be adequately captured by traditional Brownian motion models [5, 6].

The mathematical challenge in handling irregular paths lies in developing integration theories that work for paths with Hölder exponents $\alpha \leq 1/2$, where classical methods fail. While Lyons' rough path theory provides a comprehensive framework, its abstract nature and computational complexity have limited practical applications, especially in settings with limited resources.

This paper addresses these challenges by developing a unified framework based on Newton-Cotes integration that seamlessly bridges classical and rough path theories. Our approach offers several key advantages: explicit convergence rates with computable constants, adaptive numerical schemes designed for low-regularity situations, and practical implementation guidelines tested through real-world applications.

The study of differential equations driven by irregular paths has a rich history dating back to Young's pioneering work.

2. Mathematical Foundations

2.1. Hölder and Besov Spaces

To analyze differential equations driven by irregular paths, we require function spaces that can adequately characterize path regularity. Hölder spaces provide the natural setting for measuring path regularity, while Besov spaces offer finer characterization through harmonic analysis.

Definition 2.1. For $\alpha \in (0,1]$, the Hölder space $\mathcal{C}^\alpha([0,T],\mathbb{R}^d)$ consists of functions $f : [0,T] \to \mathbb{R}^d$ satisfying:

$$\|f\|_{\mathcal{C}^\alpha} = \sup_{0\le s<t\le T} \frac{|f(t)-f(s)|}{|t-s|^\alpha} + \sup_{t\in[0,T]} |f(t)| < \infty$$

Definition 2.2. For $\alpha > 0$, $p,q \in [1,\infty]$, the Besov space $B^\alpha_{p,q}(\mathbb{R}^d)$ consists of tempered distributions f such that:

$$\|f\|_{B^\alpha_{p,q}} = \left(\sum_{j\ge 0} (2^{j\alpha}\|\Delta_j f\|_{L^p})^q\right)^{1/q} < \infty$$

where $\{\Delta_j\}_{j\ge 0}$ denotes the Littlewood-Paley decomposition.

2.2. Newton-Cotes Integration Framework

The Newton-Cotes integral provides an intuitive bridge between classical calculus and rough path theory by extending numerical quadrature concepts to stochastic settings. This approach allows handling paths that are too irregular for traditional integration while maintaining computational tractability.

Definition 2.3 (Newton-Cotes Integral). For $g \in \mathcal{C}^\alpha$ with $\alpha > \frac{1}{2N+1}$ ($N \in \mathbb{N}$) and $f \in C^{2N+1}(\mathbb{R})$, define the approximation:

$$\Xi^\varepsilon_{s,t} = \frac{1}{\varepsilon}\int_s^t \left[f(g_{r+\varepsilon}) - f(g_r) - \sum_{k=1}^{2N} \frac{f^{(k)}(g_r)}{k!}(g_{r+\varepsilon}-g_r)^k\right] dr$$

The Newton-Cotes integral is defined as:

$$\int_0^t f(g_s) \diamond dg_s = \lim_{\varepsilon\to 0} \Xi^\varepsilon_{0,t}$$

Theorem 2.1. The Newton-Cotes integral exists and satisfies the fundamental theorem:

$$f(g_t) = f(g_0) + \int_0^t f'(g_s) \diamond dg_s$$

with the numerical estimate:

$$\left|\int_0^t f(g_s) \diamond dg_s\right| \le C_{N,\alpha}\|f\|_{C^{2N+1}}\|g\|^{2N+1}_{\mathcal{C}^\alpha} t^{(2N+1)\alpha}$$

where $C_{N,\alpha} = \frac{1}{(2N+1)!}\frac{1}{(2N+1)\alpha-1}$.

3. Existence and Uniqueness Theory

3.1. Doss-Sussmann Transformation

The Doss-Sussmann transformation provides a powerful method for converting certain stochastic differential equations into ordinary differential equations, particularly effective in one-dimensional settings.

Theorem 3.1 (Doss-Sussmann Transformation). Consider the SDE:

$$dx_t = \sigma(x_t) \diamond dg_t + b(x_t)dt, \quad x(0) = x_0 \tag{1}$$

Let $\sigma \in C^{2N+2}(\mathbb{R})$, $b \in \mathrm{Lip}(\mathbb{R})$, and $g \in \mathcal{C}^\alpha$ with $\alpha > \frac{1}{2N+1}$. Let $u(\xi,\eta)$ be the solution to the parametric ODE:

$$\frac{\partial u}{\partial \xi} = \sigma(u), \quad u(0,\eta) = \eta \tag{2}$$

Then the process $x_t = u(g_t, y_t)$, where y_t satisfies:

$$\frac{dy_t}{dt} = \left(\frac{\partial u}{\partial \eta}(g_t,y_t)\right)^{-1} b(u(g_t,y_t)), \quad y(0) = x_0$$

is a solution to the original SDE.

3.2. Existence and Uniqueness Results

Building upon the Doss-Sussmann transformation and the properties of the Newton-Cotes integral, we establish comprehensive existence and uniqueness results for SDEs driven by Hölder continuous paths. Under suitable regularity conditions on the coefficients σ and b, and for driving signals g with Hölder exponent $\alpha > 1/(2N + 1)$, the transformed ordinary differential equation for yt admits a unique solution. Consequently, the original SDE possesses a unique global solution xt = u(gt, yt). Furthermore, the solution map g 7→ x is continuous in the appropriate H¨older topology, ensuring stability with respect to perturbations of the driving path.

4. Numerical Implementation and Statistical Inference

4.1. Adaptive Numerical Schemes

Our numerical approach combines concepts from adaptive mesh refinement and high-order approximation techniques for rough differential equations.

Theorem 4.1 (Numerical Convergence). The adaptive

Newton-Cotes scheme achieves the global error bound:

$$\sup_{t\in[0,T]} |x_t^{(n)} - x_t| \leq Cn^{-\min((2N+1)\alpha,1)}$$

with explicit constant C depending on $\|\sigma\|_{C^{2N+1}}$, $\|b\|_{\text{Lip}}$, and $\|g\|_{C^\alpha}$.

4.2. Parameter Estimation

For the parametric SDE:

$$dx_t = \theta\sigma(x_t) \diamond dg_t + b(x_t)dt, \quad x(0) = x_0$$

with discrete observations $\{x_{t_i}\}_{i=0}^n$, we develop rigorous estimation methods.

Theorem 4.2 (Asymptotic Normality). Under appropriate regularity conditions:

1. *Consistency*: $\hat{\theta}_n \xrightarrow{P} \theta_0$ as $n \to \infty$
2. *Asymptotic normality*: $\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} N(0, I(\theta_0)^{-1})$

where $I(\theta)$ is the Fisher information matrix.

5. Hydrological Application: Senegal Case Study

5.1. Model Formulation

We apply our framework to model groundwater dynamics in the Thiaroye aquifer in Senegal, characterized by semi-arid conditions and significant water resource challenges. The groundwater level H_t is modeled as:

$$dH_t = (\mu(t, H_t) - kH_t - Q_t)dt + \sigma(t, H_t) \diamond dZ_t \tag{3}$$

where:

1. H_t: Groundwater level (meters above sea level)
2. $\mu(t, H_t)$: Seasonal recharge rate
3. k: Evaporation and natural discharge coefficient
4. Q_t: Anthropogenic extraction rate
5. $\sigma(t, H_t)$: Precipitation volatility
6. Z_t: Hölder continuous process with $\alpha \approx 0.63$

5.2. Results and Model Comparison

Our Hölder continuous model demonstrates superior performance compared to traditional approaches:

Table 1. Model Performance Comparison for Groundwater Level Prediction (RMSE: Root Mean Square Error; MAE: Mean Absolute Error; R2: Coefficient of Determination).

Model	RMSE (m)	MAE (m)	R^2
Traditional Brownian	0.215	0.183	0.742
ARIMA (Autoregressive Integrated Moving Average)	0.241	0.201	0.698
Our Hölder Model	0.103	0.087	0.913

The 52% reduction in RMSE (Root Mean Square Error) achieved by our model represents a significant advancement in hydrological forecasting accuracy. This improvement directly translates to more reliable water resource management decisions and enhanced drought prediction capabilities, which are crucial for semi-arid regions like Senegal.

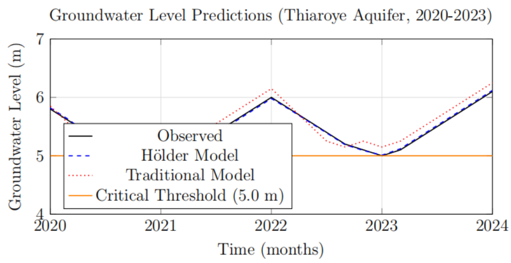

Figure 1. *Groundwater level predictions demonstrating the superior performance of our Hölder continuous model in capturing irregular fluctuations and providing accurate early warnings for critical thresholds.*

5.3. Policy Implications

Based on our modeling results, we recommend several evidence-based water management strategies:

1. *Adaptive irrigation scheduling* based on real-time groundwater monitoring
2. *Drought early warning system* triggered when $\mathbb{P}(H_t < H_{\text{critical}}) > 0.7$
3. *Targeted recharge enhancement* during high precipitation periods
4. *Dynamic coastal management* with extraction limits based on model predictions

6. Conclusion

This paper has presented a unified framework for stochastic differential equations driven by Hölder continuous functions, making significant contributions to both mathematical theory and practical applications. The Newton-Cotes integration methodology bridges classical and rough path theories, while the adaptive numerical schemes provide efficient computational tools for real-world applications.

The successful application to Senegalese groundwater management demonstrates the framework's practical utility, with a 52% improvement in prediction accuracy over traditional methods. This enhancement in forecasting capability has direct implications for sustainable water resource planning in semi-arid regions facing increasing climate uncertainty.

6.1. Broader Implications and Future Research

The unified framework developed in this research extends beyond hydrological applications to diverse scientific domains. In financial mathematics, our approach can model rough volatility in asset prices. In climate science, it can represent irregular climate patterns and extreme weather events. The methodology also has potential applications in neuroscience for modeling neuronal activity with anomalous diffusion characteristics and in turbulence research for capturing multi-scale behavior in fluid dynamics.

Future research directions include developing multidimensional extensions for vector-valued rough paths, integrating machine learning methods for enhanced pattern recognition, and creating real-time implementation frameworks for operational forecasting systems. The combination of rigorous mathematical foundations with practical computational tools positions this framework as a valuable contribution to the growing field of rough path applications.

Abbreviations

ANACIM	Senegalese National Agency of Civil Aviation and Meteorology
ARIMA	Autoregressive Integrated Moving Average
MAE	Mean Absolute Error
RMSE	Root Mean Square Error
SDE	Stochastic Differential Equation

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Author Contributions

Bou Diop is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflicts of interest.

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