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Strong Hub Sets and Strong Hub Number of Hypergraphs

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Abstract: This paper introduces and investigates the concepts of *strong hub sets* and the *strong hub number* in hypergraphs, extending the notion of hub sets defined for graphs. For a hypergraph $H = (V, E)$, a subset $S \subseteq V(H)$ is said to be a *strong hub set* if, for every pair of distinct vertices $u, v \in V(H) - S$, either u and v are adjacent or there exists a strong S-hyperpath joining them. The minimum cardinality of such a set is called the *strong hub number* of H, denoted by $h^*(H)$. Fundamental properties of strong hub sets are established, and relationships between the strong hub number and various hypergraph parameters such as the domination number and connectivity are explored. The effects of vertex deletion and weak deletion on $h^*(H)$ are studied, leading to several sharp bounds and recursive characterizations. In particular, it is shown that under weak deletion of a non cut vertex, the strong hub number remains invariant, while deletion of a cut vertex reduces it according to the structure of the resulting components. Special attention is devoted to *hypertrees*, where structural simplicity allows a precise characterization. This characterization provides an exact formula for $h^*(H)$ in acyclic hypergraphs and establishes the foundational link between connectivity and hub-based path structure in hypertrees.

Keywords: Strong Hub Set, Strong Hub Number, Hyperpath

1. Introduction

In the study of hypergraph parameters, *hub sets* play a significant role in capturing structural connectivity through selected vertex subsets. Building upon this idea, we introduce the notion of a *strong hub set* in hypergraphs. A subset $S \subseteq V(H)$ is said to be a *strong hub set* if, for every pair of non-adjacent vertices $u, v \in V(H) - S$, there exists a *strong hyperpath* P between u and v such that all the internal vertices of P, including the *anchor* and *floater* vertices, belong entirely to S.

This stricter requirement distinguishes strong hub sets from ordinary hub sets: while hub sets ensure connectivity through internal vertices, strong hub sets impose an enhanced dominance over hyperpaths by demanding that even the auxiliary vertices (anchors and floaters) lie completely within S. Hence, a strong hub set may be viewed as a natural extension of hub sets from graphs to hypergraphs, but with an added level of robustness appropriate to the richer structure of hypergraphs. This stronger formulation captures a deeper notion of control and redundancy in connectivity, making strong hub sets a compelling subject of investigation.

Matthew Walsh [7] introduced the concept of hub sets and the *hub number* in graphs, which subsequently inspired extensive research leading to several variants such as open hub sets, total hub sets, and doubly connected hub sets, among others. The study of hub sets continues to be an active and evolving area of research. The extension of the hub set concept to hypergraphs was initiated in [9]. Building on this framework, further investigations into the structural properties and parameters of hub sets in hypergraphs were carried out in our previous work, A Study on Hub Sets in Hypergraphs[15]. These studies provide the foundation for the present work, which focuses on the analysis of strong hub sets in hypergraphs.

In the present work, we introduce and investigate the concept of *strong hub sets* in hypergraphs. We study *minimum strong hub sets* and determine the corresponding *strong hub* *numbers* for various classes of hypergraphs. In addition, we define *pairwise complete hypergraphs* and analyze the behavior of the strong hub number under *vertex weak deletion*. Furthermore, we establish *bounds* on the strong hub number in terms of other hypergraph parameters such as the *domination number* and related invariants.

2. Preliminaries

Definition 2.1. [5, 10, 11] A *hypergraph* $H = (V, E)$ is an ordered pair, where V is a non-empty finite set of vertices and E is a family of non-empty subsets of V, called *edges*, such that $\bigcup_{e \in E} e = V$. Two vertices $u, v \in V$ are said to be *adjacent* if there exists an edge $e \in E$ containing both u and v.

The *degree* of a vertex v, denoted $d(v)$, is the number of edges incident with v, and the *maximum degree* of H is denoted $\Delta(H)$ [4]. The *open neighborhood* of v is $N(v) = \{u \in V : u \text{ is adjacent to } v\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$ [6]. A hypergraph is *r-uniform* if $|e| = r$ for every $e \in E(H)$ [5]. It is *connected* if for any pair of vertices, there exists a path connecting them [4].

Definition 2.2. [1, 14] A subset $S \subseteq V$ is a *dominating set* if for every $v \in V \setminus S$, there exists a vertex $u \in S$ such that u and v are adjacent. The minimum cardinality of a dominating set is the *domination number*, denoted $\gamma(H)$.

A *path* P of length $k \geq 2$ in H is a sequence

$$(x_0, e_1, x_1, e_2, x_2, \ldots, e_k, x_k),$$

where $x_{i-1}, x_i \in e_i$ for $1 \leq i \leq k$ [5]. The vertices $x_1, x_2, \ldots, x_{k-1}$ are called *anchors*, and any vertex $u \in e_i$ that is not an anchor is a *floater* of P [5]. A path is *linear* if any two consecutive edges intersect in exactly one vertex [5]. The *r-uniform linear path* P_n^r and *r-uniform linear cycle* C_n^r are defined with standard intersection properties [5, 11].

A *complete r-partite, r-uniform hypergraph* $H = (V, E)$ has a vertex partition $V = V_1 \cup V_2 \cup \cdots \cup V_r$, with

$$E = \{e \subseteq V : |e| = r \text{ and } |e \cap V_i| = 1 \text{ for all } i\},$$

denoted $K_{n_1,n_2,\ldots,n_r}$ if $|V_i| = n_i$ [4].

A *hypertree* is a connected, acyclic hypergraph [5, 13]. The *weak deletion* of a vertex $v \in V$, denoted $H - v$, is the hypergraph obtained by removing v from V and deleting v from all edges of H [3].

Definition 2.3. [4, 6] A *sunflower hypergraph* $\mathrm{SH}(n, p, r)$ is an r-uniform hypergraph with $n = r + (k-1)p$ vertices and k edges (petals), where each edge contains a common subset of $r - p$ vertices (the core) and p distinct vertices.

Definition 2.4. [8] A vertex $v \in V$ is a *separating vertex* if H can be decomposed into two non-empty connected subhypergraphs H_1 and H_2 such that $H = H_1 \oplus H_2$ and $V(H_1) \cap V(H_2) = \{v\}$.

Definition 2.5. [3] The *join* of two hypergraphs H_1 and H_2, denoted $H_1 + H_2$, has vertex set $V(H_1) \cup V(H_2)$ and edge set

$$E(H_1) \cup E(H_2) \cup E_{H_1}^{H_2},$$

where $E_{H_1}^{H_2} = \{e \subseteq V(H_1) \cup V(H_2) : |e| = m,\, e \cap V(H_i) \neq \emptyset \text{ for } i = 1, 2\}, m > 1$.

Definition 2.6. [15] Let $H = (V, E)$ be a hypergraph and $v \in V$. The *contraction* of v in H, denoted H/v, is obtained as follows :

1. Delete v weakly from H.
2. Join each pair of edges in $E_v = \{e \setminus \{v\} : v \in e\}$ to form a hyperedge in H/v, making the neighborhood $N(v)$ pairwise complete.
3. Avoid adding duplicates if $e \cup f$ already exists.

Definition 2.7. [7] A subset $H \subseteq V$ of a graph G is a *hub set* if for any two distinct vertices $u, v \in V \setminus H$, there exists a path P from u to v whose internal vertices lie entirely in H. The minimum cardinality of a hub set is the *hub number* $h(G)$.

3. The Strong Hub Number of Hypergraphs

Definition 3.1. Let $H = (V, E)$ be a hypergraph. Let $S \subseteq V(H)$ and $x_0, x_k \in V(H)$. A strong S-hyperpath (simply strong S-path) between x_0 and x_k is a path $P = (x_0, e_1, x_1, e_2, x_2......, e_k, x_k)$ such that,

1. $x_0, x_k \notin S$
2. $x_i \in S$ for $1 \leq i \leq k-1$ and $e_i \subseteq S$ for $2 \leq i \leq k-1$.
3. $e_1 - \{x_0\} \subseteq S$ and $e_k - \{x_k\} \subseteq S$.

Definition 3.2. Let $H = (V, E)$ be a hypergraph ,then $S \subseteq V$ is a strong hub set of H if for any $u, v \in V - S$ either u and v are adjacent in H or there exists a strong S-path from u to v in H. The minimum cardinality of S is the strong hub number of H, denoted by $h^*(H)$.

Observation: If a hypergraph H contains an isolated vertex v then for any strong hub set S of H, $v \in S$.
Throughout this paper we consider hypergraphs without isolated vertices only.

Theorem 3.1. If $k = max\{|e| : e \in E\}$ then $h^*(H) \leq n-k$. Hence for a r-uniform hypergraph H, $h^*(H) \leq n - r$.

Theorem 3.2. For an r-uniform linear path $H = P_n^r$ with n vertices, $h^*(P_n^r) = n - r$, if $r \geq 2$.

Proof: Let $P_n^r = v_1 e_1 v_r e_2 v_{2r-1}, ...v_n$ be the r-uniform path with $e_1 = \{v_1, v_2, ...v_r\}$, $e_2 = \{v_r, v_{r+1}, ..v_{2r-1}\}$,...and with vertex set, $V(P_n^r) = \{v_1, v_2, ...v_n\}$. If $r = 2$, then H is a path graph P_n and $h^* = n - 2$. If $r > 2$, by above lemma, $h^*(H) \leq n - r$. Thus any subset S of $n - r$ vertices is a strong hub set by taking $V - S$ as a hyper edge. If S' is any subset of V with less than $n - r$ vertices,then $V - S$ contains at least $r + 1$ vertices out of which at least two are non adjacent and there does not exist a strong S'-path connecting these two vertices. So S' is not a strong hub set. This implies S is minimum. Hence $h^*(P_n^r) = n - r$.

Theorem 3.3. For the r-uniform cycle $H = C_n^r$,

$$h^*(C_n^r) = \begin{cases} n-3 & \text{if } r = 2 \\ n-r & \text{if } r > 2 \end{cases}$$

Proof: If $r = 2$, then C_n^r becomes the cycle graph C_n which has hub number $n-3$. Let $r > 2$ and $C_n^r = v_1e_1v_re_2v_{2r-1}, ...v_ne_mv_1$. Then as in the case of P_n^r we can see that for any edge e , $S = V - \{e\}$ is a minimum strong hub set. Hence $h^*(C_n^r) = n - r$, if $r > 2$.

Theorem 3.4. For the complete r-uniform hypergraph K_n^r, $h^*(K_n^r) = 0$.

Proof: The proof is obvious since any pair of vertices in K_n^r are adjacent.

Definition 3.3. A hypergraph $H = (V, E)$ is called *pairwise complete hypergraph* if for every pair of distinct vertices $u, v \in V$, there exists a hyperedge $e \in E$ such that $\{u, v\} \subseteq e$.

Theorem 3.5. Let $H = (V, E)$ be a hypergraph, then $h^*(H) = 0$ if and only if H is pairwise complete.

Theorem 3.6. Let $H = (V, E)$ be an r-uniform star hypergraph with center x and $m \geq 2$ edges. Then

$$h^*(H) = 1 + m(r - 2).$$

Proof: Let the edges be $E_1, E_2, \ldots, E_m$, where each $E_i = \{x\} \cup L_i$ and the leaf sets L_i are pairwise disjoint with $|L_i| = r - 1$.

From each L_i choose $r - 2$ vertices and let

$$S_0 = \{x\} \cup \bigcup_{i=1}^{m} (\text{chosen } r - 2 \text{ vertices from } L_i).$$

Then $|S_0| = 1 + m(r - 2)$.

For any two vertices $u, v \in V - S_0$, If $u \in L_i$ and $v \in L_j$ with $i \neq j$, the path $P = (u, E_i, x, E_j, v)$ is a strong S_0-path, since $x \in S_0$ and $E_i - \{u\}, E_j - \{v\} \subseteq S_0$. Hence S_0 is a strong hub set and $h^*(H) \leq 1 + m(r - 2)$.

To prove the reverse inequality, let S be any strong hub set.

Case 1: If $x \in S$, then in each edge E_i at most one leaf can lie outside S; otherwise two leaves of E_i outside S cannot both connect to a leaf in another edge. Thus at least $r - 2$ leaves from each E_i belong to S, giving

$$|S| \geq 1 + m(r - 2).$$

Case 2: If $x \notin S$, then all vertices outside S must lie in a single edge, otherwise two leaves from different edges would be neither adjacent nor connected by a strong S-path. Hence

$$|S| \geq |V| - r = (1 + m(r - 1)) - r = (m - 1)(r - 1).$$

In both cases $|S| \geq 1 + m(r - 2)$, and the constructed S_0 attains this bound. Therefore

$$h^*(H) = 1 + m(r - 2).$$

Theorem 3.7. For a sunflower hypergraph $SH(n, p, r)$

$$h^*(SH(n,p,r)) = \begin{cases} 0 & \text{if } k = 1 \\ p & \text{if } k = 2 \\ r-1 & \text{if } p = 1 \text{ and } k \geq r \geq 3 \\ p(k-1) & \text{if } p \geq 2 \text{ and } 3 \leq k \leq r-1 \end{cases}$$

Proof: Consider the sunflower hypergraph $SH(n, p, r)$ with k petals.

Case 1: If $k = 1$ then the sunflower hypergraph becomes an edge, hence $h^* = 0$

Case 2: If $k = 2$ then the hypergraph becomes a path and $h^*(SH(n, p, r)) = n - r = r + (k - 1)p - r = p$.

Case 3: Consider the case $p = 1$ and $k \geq r$ then the common edge subset S of $r - 1$ vertices is clearly a strong hubset of SH, since any pair of vertices outside S is connected by a strong S-path of length two. Clearly it is minimal since $k \geq r$. Hence $h^*(SH(n, p, r)) = r - 1$.

Case 4: Let S be the disjoint union of the vertices in $k - 1$ petals. Then $|S| = p(k - 1)$. Since $V - S$ is an edge, any two vertices outside S lie in the same edge and therefore are adjacent; hence S is a strong hub set. Thus $h^*(H) \leq p(k-1)$.

To prove minimality, let S' be any strong hub set. Because $p \geq 2$, if $u \in P_i$ and $v \in P_j$ come from two different petals $P_i \neq P_j$, then u and v are not adjacent and there is no strong S'-path between them unless at least one of the petals P_i or P_j is entirely contained in S'. Hence S' must contain the vertices of at least $k - 1$ petals, so $|S'| \geq p(k - 1)$. Therefore S is minimal and

$$h^*(\mathrm{SH}(n, p, k)) = p(k - 1).$$

Theorem 3.8. Let $H = K^{(r)}_{n_1,\ldots,n_r}$ be the complete r-partite, r-uniform hypergraph with partite sets $V_1, \ldots, V_r$ where $|V_i| = n_i$ for $i = 1, \ldots, r$. Then

$$h^*\left(K^{(r)}_{n_1,\ldots,n_r}\right) = \begin{cases} 0, & \text{if } n_i = 1 \text{ for every } i, \\ r-1, & \text{if } n_i \geq 2 \text{ for exactly one } i, \\ r, & \text{if } n_i \geq 2 \text{ for at least two indices } i. \end{cases}$$

Proof: Recall $H = K^{(r)}_{n_1,\ldots,n_r}$ has vertex partition $V_1, \ldots, V_r$ and every edge of H contains exactly one vertex from each V_i.

Case 1: $n_i = 1$ for every i.

Then $|V| = r$ and there is exactly one edge $e = V_1 \cup \cdots \cup V_r$ (containing all vertices). Every pair of vertices lies together in the unique edge, so every pair is adjacent; hence the empty set is a strong hub set and $h^*(H) = 0$.

Case 2: Exactly one part, say V_k, satisfies $n_k \geq 2$, while $n_i = 1$ for all $i \neq k$.

Let $S = \bigcup_{i \neq k} V_i$ (so $|S| = r - 1$). Any two vertices lying in different parts are adjacent (they belong to some edge), and the only nonadjacent pairs are pairs of distinct vertices inside V_k. Take $u, w \in V_k, u \neq w$. Since every edge contains exactly one vertex from each part, choose arbitrary vertices $v_i \in V_i$ for $i \neq k$; then $P = u, v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_r, w$ is a strong

S-path from u to w whose internal vertices lie in S. Thus S is a strong hub set, so $h^*(H) \leq r-1$.

To see minimality, any hub set must intersect every edge or otherwise fail to connect some pair inside V_k by an S-path of the required type; hence $|S| \geq r-1$. Therefore $h^*(H) = r-1$.

Case 3: At least two parts have size ≥ 2.
Choose one vertex from each part to form S with $|S| = r$ (for example, pick $s_i \in V_i$ and let $S = \{s_1, \ldots, s_r\}$). For any two distinct vertices u, w in the same part V_j there is a strong S-path $u, s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_r, w$ whose internal vertices lie in S, so S is a strong hub set and $h^*(H) \leq r$.

To see $h^*(H) \geq r$, observe that every edge of H has size r, so any hub set T that is contained in a single edge can have size at most r; but if $|T| < r$ then there is at least one part V_j not represented in T, and because at least two parts have size ≥ 2 one can find distinct vertices in that part which cannot be connected by an T-path whose internal vertices lie entirely in T. Thus no hub set of size $< r$ suffices. Hence $h^*(H) = r$.

Theorem 3.9. Let $H = (V, E)$, $|E| > 1$, be a linear hypertree in which every edge contains at most two cut-vertices. Let

$$C = \{v \in V : d(v) \geq 2\}$$

be the set of cut-vertices, and let $|e|$ be the size of an edge $e \in E$. Then

$$h^*(H) = |C| + \sum_{e \in E} (|e| - 2).$$

Proof: Let $H = (V, E)$, $|E| > 1$, be a linear hypertree in which every edge contains at most two cut-vertices.

Construct a set S_0 as follows:

1. Include all cut-vertices: $C \subseteq S_0$.
2. For each edge $e \in E$, include in S_0 exactly $|e| - 2$ vertices of e *not including the cut-vertex of* e (if any).

By construction, in each edge there are at most two vertices outside S_0, and all cut-vertices are in S_0. Hence

$$|S_0| = |C| + \sum_{e \in E} (|e| - 2).$$

To see that S_0 is a strong hub set, consider any two vertices $u, v \in V - S_0$:

1. If u and v lie in the same edge e, then e has at most two vertices outside S_0, so e itself forms a strong S_0-path connecting u and v.
2. If u and v lie in distinct edges, consider the unique hyperpath connecting them. Each intermediate edge has all vertices in S_0 Therefore, this hyperpath is a strong S_0-path connecting u and v.

Thus, S_0 is a strong hub set, giving

$$h^*(H) \leq |S_0| = |C| + \sum_{e \in E} (|e| - 2).$$

To prove that S_0 is minimum, let S be any strong hub set.

1. *Cut-vertices must belong to* S. Suppose a cut-vertex $x \notin S$. Then x belongs to at least two edges, say e_1 and e_2. Pick a vertex $u \in e_1 \setminus \{x\}$ and a vertex $v \in e_2 \setminus \{x\}$. Any hyperpath connecting u and v must pass through x. Since $x \notin S$, the strong path condition fails (an internal vertex on a strong path must be in S). Hence all cut-vertices are in S: $C \subseteq S$.

2. *At least* $|e| - 2$ *vertices of each edge must belong to* S. Consider an edge e and suppose two vertices $u, v \in e$ lie outside S. Take any vertex w from a different edge. Any path from u to w must start along e. But the strong path condition requires that all but at most one vertex of e be in S. Since u and v are both outside S, this is violated, so no strong path from u to w exists. Therefore, at most two vertices of e can lie outside S, which is equivalent to saying that at least $|e| - 2$ vertices of e must be in S. Summing over all edges and counting each cut-vertex only once gives

$$|S| \geq |C| + \sum_{e \in E} (|e| - 2).$$

Combining the upper and lower bounds, we obtain

$$h^*(H) = |C| + \sum_{e \in E} (|e| - 2),$$

as required.

Theorem 3.10. Let $H = (V, E)$ be a hypergraph with n vertices and let k be the order of maximum pairwise complete sub hypergraph in H, then $h^*(H) \leq n - k$.

Proof: Let Y be the set of all vertices in the maximum pairwise complete sub hypergraph in H. Then $S = V - Y$ is obviously a strong hub set, since any pair of vertices in Y are adjacent. Hence $h^*(H) \leq n - k$.

Theorem 3.11. Let $S \subseteq V$ then H/S is pairwise complete if S is a hubset of H.

Proof: Let S be a hub set of H and $x, y \in V - S$. By definition, either x and y are adjacent in H or there is an S-path from x to y in H with internal vertices in S. If e is a hyperedge containing x and y and there is no path from x to y, then still there exists an edge containing x and y in H/S . Thus x and y are adjacent in H/S. Now if P is a path from x to y with internal vertices in S, then P will be contracted to a hyperedge $f \subseteq N(S)$ containing x and y. Hence x and y are adjacent in H/S.Thus H/S is pairwise complete.

Remark 3.1. The converse of the above theorem holds for 2-uniform hypergraphs (that is, for ordinary graphs) but does not hold in general for hypergraphs.

For example, consider the star hypergraph H with

$$V(H) = \{x, 1, 2, 3, 4, 5, 6\} \quad \text{and}$$

$$E(H) = \{\{x, 1, 2, 3\}, \{x, 4, 5\}, \{x, 6\}\}.$$

Clearly, the weak deletion H/x is pairwise complete since

$$E(H/x) = \{\{1, 2, 3\}, \{4, 5\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 6\}, \{4, 5, 6\}\}.$$

However, the singleton set $\{x\}$ is not a hub set of H.

Theorem 3.12. Let H/S be complete then S is a hub set of H if S is a hubset of the sub hypergraph induced by N[S].

Remark 3.2. If S is a minimal strong hub set of $H = H_1 \oplus H_2$ and v is a separating vertex of H then,

1. $v \in S$.

2. If $S_1 = S \cap V(H_1)$ and $S_2 = S \cap V(H_2)$, then S_1 and S_2 will be strong hub sets for H_1 and H_2 respectively. Hence $h^*(H_1) \leq h^*(H_1 \oplus H_2)$ and $h^*(H_2) \leq h^*(H_1 \oplus H_2)$.

Theorem 3.13. Let H be a connected hypergraph with $H = H_1 \oplus H_2$ and separating vertex v, then $h^*(H_1 \oplus H_2) \geq h^*(H_1) + h^*(H_2)$.

Proof: Let S_1 and S_2 be smallest strong hub sets of H_1 and H_2 respectively. Then $S' = S_1 \cup S_2$ will be a strong hubset for H if and only if $v \in S_1 \cup S_2$. If $v \notin S'$ then let $S = S' \cup \{v\}$. If $u, w \notin S$, then we have three cases.

Case 1: If $u, w \in V(H_1) - S_1$, then either u and v are adjacent in H_1 or there exist a strong S_1-path in H_1.

Case 2: $u, w \in V(H_2) - S_2$, then either u and v are adjacent in H_2 or there exist a strong S_2-path in H_2.

Case 3: $u \in V(H_1) - S_1$ and $w \in V(H_2) - S_2$. Since $V(H_1) \cap V(H_2) = \{v\}$, using properties of strong hub set and connectedness there is a strong S_1-path from u to v in H_1, and a strong S_2-path from v to w in H_2. Hence there is a strong S-path from u to v in H. If there are no such paths then $S = S_1 \cup S_2 \cup \{u, w\}$ will be a strong hub set of $H_1 \oplus H_2$. Hence $h^*(H_1 \oplus H_2) \geq h^*(H_1) + h^*(H_2)$. If H_1 or H_2 or both are pairwise complete then since v is a separating vertex, $H_1 \oplus H_2$ is not pairwise complete. In these cases we have $h^*(H_1 \oplus H_2) > h^*(H_1) + h^*(H_2)$.

Theorem 3.14. For any hypergraph $H = (V, E)$, $\gamma(H) \leq h^*(H) + 1$.

Proof: Let S be a strong hub set in H. Then any $a, b \in V - N[S]$ are adjacent in H. Hence $V - N[S]$ induces a pairwise complete sub hypergraph. Then a vertex $x \in V - N[S]$ will dominate all other vertices in $V - N[S]$. Now let $a \notin S$ then either $a \in V - N[S]$ or $a \in N(S)$. If $a \in V - N[S]$, then x will dominate a . If $a \in N(S)$ then since S is a hub set, a is adjacent to some vertex in S. Hence $S \cup \{x\}$ will be a dominating set of H. Thus $\gamma(H) \leq h^*(H) + 1$. If $V - N[S]$ is empty then S will be a dominating set and the inequality holds.

Theorem 3.15. Let H_1 and H_2 be two hypergraphs and $H = H_1 + H_2$, then

$$h^* = \begin{cases} 0 & \text{if } m > 2 \text{ or both } H_i\text{'s are pairwise complete.} \\ 1 & \text{if } m = 1 \text{ and exactly one } H_i \text{ is pairwise complete.} \\ 2 & \text{if } m = 2 \text{ and } H_1, H_2 \text{ are not pairwise complete.} \end{cases}$$

Proof: Let H_1 and H_2 be two hypergraphs. If both $H_i's$ are pairwise complete then $H_1 + H_2$ becomes pairwise complete and hence $h^*(H_1 + H_2) = 0$.

Case 1: $m > 2$.

Since $m > 2$ any two vertices in $V(H_1 + H_2)$ is adjacent as there are all possible combinations of edges of m vectors with at least one vertex from $V(H_i)$ .Thus $H_1 + H_2$ is pairwise complete hence $h^*(H_1 + H_2) = 0$

Case 2: $m = 2$.

Take $S = \{x, y\}$ where $x \in V(H_1), y \in V(H_2)$, then for any $u, v \notin S$ we have three sub cases

1. If $u \in V(H_1), v \in V(H_1)$, then either u and v are adjacent in H_1 or there exists a path $P = u - y - v$ from u to v with internal vertex in S in $H_1 + H_2$.
2. If $u \in V(H_1)$,$v \in V(H_2)$, then u and v are adjacent with edge $e = uv$ in $H_1 + H_2$.
3. If $u \in V(H_2), v \in V(H_2)$, then either u and v are adjacent in H_2 or there exists a path $P = u - x - v$ from u to v with internal vertex in S in $H_1 + H_2$.

Thus S is a hubset of $H_1 + H_2$ and $h^*(H_1 + H_2) \leq 2$. If exactly one of H_1 or H_2 is pairwise complete, then $H_1 + H_2$ is not complete. Let H_1 be pairwise complete and take $S = \{v\}$, where $v \in V(H_1)$. Then for $x, y \in V(H_2)$ either x and y are adjacent or there is an S-path $x - v - y$. Hence S is a minimal hub set of $H_1 + H_2$ and $h^*(H_1 + H_2) = 1$. If H_1 and H_2 are not pairwise complete then $H_1 + H_2$ is not pairwise complete and then S is minimal since $\gamma(H_1 + H_2) = 2$. Hence $h^*(H_1 + H_2) = 2$.

By the above theorem if $m > 2$, $H_1 + H_2$ becomes pairwise complete so we take the case $m = 2$ only in the following corollaries.

Corollary 3.1. Let H be a hypergraph that is not pairwise complete and v be a vertex not in H, then $h^*(H + \{v\}) = 1$.

Corollary 3.2. $h^*(K_n^r + H) = \gamma(K_n^r + H)$ for any positive integer n and hypergraph H if and only if H is not a pairwise complete hypergraph.

Proof: Let H be a hypergraph such that $h^*(K_n^r + H) = \gamma(K_n^r + H)$ for any positive integer n. If H is pairwise complete, then $K_n^r + H$ becomes pairwise complete.Thus $h^*(K_n^r + H) = 0$ and $\gamma(K_n^r + H) = 1$. Hence H is not pairwise complete.

Conversely, suppose H is not pairwise complete.Let $S = \{u\}$ where $u \in K_n^r$. Since any $v \notin S$ is adjacent to u, S is a minimal dominating set. By Theorem 3.14, S is a minimal hub set of $K_n^r + H$. Hence $h^*(K_n^r + H) = \gamma(K_n^r + H)$.

Theorem 3.16. Let S be a strong hubset of a hypergraph $H = (V, E)$. If $V - S$ is an independent set then $V - N[S] = \phi$.

Proof: Let $V - S$ be independent. Since S is a strong hubset the sub hypergraph induced by $V - N[S]$ is pairwise complete. If $u \in V - N[S]$ there must be an edge e such that $u \in e \subseteq V - S$. Which is a contradiction.Thus $V - N[S] = \phi$.

Theorem 3.17. If S is a strong hub set of H and $v \notin S$ is a pendent vertex then either $V - S$ is a hyperedge or any other vertex in $V - S$ is connected to v by a strong S-path.

Proof: If v is a pendent vertex then $d(v) = 1$, so for any $u \notin S$ either there will be exactly one hyperedge e such that $u, v \in e = V - S$ or for any $u \notin S$ there is a strong S-Path from u to v with an edge e_v containing v common to all such S-paths. Thus any other vertex in $V - S$ is connected to v by a strong S-path.

Theorem 3.18. Let H be a connected hypergraph with n vertices. If $|e| > 2$ for at least one hyperedge e of H, then

$$h^*(H) < n - 2.$$

Moreover, $h^*(H) = n - 2$ if and only if H is a 2-uniform path on n vertices.

Proof: Let H be a connected hypergraph with n vertices, and suppose there exists a hyperedge e in H such that $|e| > 2$. If H is complete, then every pair of vertices is adjacent, and hence $h^*(H) = 0$. Assume that H is not complete, and let

$$k = \max\{|e| : e \in E(H)\}.$$

Since $k > 2$, we have $h^*(H) < n - k < n - 2$.

We also know that for a 2-uniform path P_n^2,

$$h^*(P_n^2) = n - 2.$$

Conversely, suppose $h^*(H) = n - 2$. Then H must be a 2-uniform connected graph (since if any edge had size greater than 2, the inequality above would hold strictly).

Let $S = \{v_1, v_2, \ldots, v_{n-2}\}$ be a smallest strong hub set of H, and let $u, v \notin S$. By the definition of a strong hub set, either u and v are adjacent, or there exists a strong S-path P from u to v whose internal vertices lie in S. Without loss of generality, assume that u and v are not adjacent.

If every vertex of S lies on P, then H is a path. Suppose instead that there exists at least one vertex of S not in P. Let

$$W = \{w \in S : w \notin P\} \neq \varnothing.$$

Let $w \in W$. Since H is connected, there exists a path P_1 joining w to some vertex $x \in P$.

If all vertices of P_1 lie in S, let y be the vertex adjacent to x in P_1. Then $S - \{y\}$ is also a strong hub set of H, contradicting the minimality of S.

If P_1 contains either u or v, then H contains a cycle, which implies $h^*(H) > n - 2$, a contradiction.

Hence, such a vertex w cannot exist, and therefore $W = \varnothing$. Thus, all vertices of S lie on the path P, and H is a 2-uniform path with internal vertices in S.

Theorem 3.19. Given any two positive integers n and k with $2 \leq k < \lceil n/2 \rceil$, there exists a connected hypergraph with n vertices such that

$$h^*(H) = k = \gamma(H).$$

Proof: Let $r = \lceil (k + 3)/2 \rceil$. Consider the complete r-uniform hypergraph K_{n-k}^r on the vertex set

$$V(K_{n-k}^r) = \{v_1, v_2, \ldots, v_{n-k}\}.$$

Construct an r-uniform connected hypergraph H with n vertices by adding k new vertices

$$\{u_1, u_2, \ldots, u_k\}$$

and joining them to K_{n-k}^r by adding the following edges:

$$\begin{aligned}
e_1 &= \{v_1, v_2, v_3, \ldots, v_{r-1}, u_1\},\\
e_2 &= \{v_2, v_3, v_4, \ldots, v_r, u_2\},\\
e_3 &= \{v_3, v_4, v_5, \ldots, v_{r+1}, u_3\},\\
&\vdots\\
e_k &= \{v_k, v_1, v_2, \ldots, v_{r-2}, u_k\}.
\end{aligned}$$

Let $S = \{v_1, v_2, \ldots, v_k\}$. Clearly, S is a dominating set of H, and hence $\gamma(H) \leq k$.

Now, for any $u_i, u_j \in V - S$ with $i \neq j$, there exists an S-path

$$u_i\, e_i\, v_i\, e_j\, u_j.$$

Also, for any $u_i, v_j \notin S$, the path

$$u_i\, e_i\, v_i\, f_{i,j}\, v_j,$$

where $f_{i,j}$ is an edge connecting v_i and v_j, is an S-path. Thus, S is a strong hub set of H.

If S' is any strong hub set of H, then by the construction of H, for each $i = 1, 2, \ldots, k$, either $u_i \in S'$ or $v_i \in S'$. Hence, $|S'| \geq k$, implying $h^*(H) \geq k$. Therefore, S is a minimum strong hub set, and $h^*(H) = k$.

Furthermore, note that if we take $S = \{u_1, u_2, \ldots, u_k\}$, this set is also a hub set but not a dominating set, since the vertex v_{k+1} is not dominated by any member of S. Hence, we conclude that

$$h^*(H) = k = \gamma(H).$$

Theorem 3.20. Given any two positive integers n and k with $2 \leq k < \lceil n/2 \rceil$, there exists a connected hypergraph with n vertices such that

$$h^*(H) = k = \gamma(H).$$

Proof: Let K_{n-k}^k be the complete k-uniform hypergraph on $n - k$ vertices with vertex set

$$V(K_{n-k}^k) = \{u_1, u_2, \ldots, u_{n-k}\}.$$

Construct a hypergraph H by adding k new vertices

$$\{v_1, v_2, \ldots, v_k\}$$

and adding k new edges $u_i v_i$ for $i = 1, 2, \ldots, k$. Thus, H is a connected hypergraph with n vertices.

Let $S = \{u_1, u_2, \ldots, u_k\}$. Clearly, S is a minimum dominating set of H, and hence $\gamma(H) = k$.

By Theorem 3.14, we have $h^*(H) \geq k-1$. Now, observe the following:

1. For any $u_i, u_j \notin S$, the vertices are adjacent in H.
2. For any $u_i, v_j \notin S$, there exists an S-path with ordered vertex sequence $u_i - u_j - v_j$.
3. For any $v_i, v_j \notin S$, there exists a strong S-path $v_i - u_i - u_j - v_j$.

If S' is any minimum strong hub set of H, then by the construction of H, for each $i = 1, 2, \ldots, k$, either $u_i \in S'$ or $v_i \in S'$. Thus, $|S'| \geq k$, which implies $h^*(H) \geq k$. Therefore, S is a minimum strong hub set of H, and hence

$$h^*(H) = k.$$

4. Weak Deletion of a Vertex and Strong Hub Number

Strong hub sets capture the essential connectivity of a hypergraph by ensuring that all non-hub vertices are either adjacent or connected via paths passing through the hub set. Investigating the effect of weak deletion, the removal of a vertex from the hypergraph, helps reveal the stability and robustness of these hub structures. Analyzing how weak deletion impacts the strong hub number highlights critical and non-critical vertices, providing structural insights and guiding applications in network control, monitoring, and fault-tolerant design.

Theorem 4.1. $h^*(K_n^r - v) = 0$, for any $v \in V(H)$

Theorem 4.2. If $H = P_n^r$, then $h^*(P_n^r - v) = n - (r+1)$ for any floater vertex v of H.

Lemma 4.1. Let $H = (V, E)$ be a hypertree, in which every edge contains at most two cut-vertex. Let $v \in V$ be a vertex with $\deg_H(v) = 1$ (i.e. a non–cut-vertex). Then

$$h^*(H - v) = h^*(H) - 1.$$

Proof: Since $d(v) = 1$, the vertex v lies in exactly one edge, say e. Removing v does not change the set C of cut vertices, so the set of all cut vertices C' for $H - v$ equals the set C for H. The only change occurs in the edge-size, the size of e decreases by one, while all other edges are unchanged. Hence,

$$\sum_{f \in E(H-v)} (|f| - 2) = \sum_{f \in E(H)} (|f| - 2) - 1.$$

Therefore, applying the defining formula for h^* to $H - v$ gives

$$h^*(H - v) = |C| + \Big(\sum_{f \in E(H)} (|f| - 2) - 1 \Big) = h^*(H) - 1,$$

as claimed.

Theorem 4.3. Let $H = (V, E)$ be a hypertree and let $v \in V$ be a cut-vertex. Suppose that deleting v disconnects H into $k \geq 2$ connected components $H_1, H_2, \ldots, H_k$. Then the strong hub number of $H - v$ satisfies

$$h^*(H - v) = \min_{1 \leq j \leq k} \left\{ \sum_{i \neq j} |V(H_i)| + h^*(H_j) \right\},$$

In particular,

$$h^*(H - v) \geq h^*(H),$$

Proof: Since v is a cut-vertex, its removal disconnects H into k components $H_1, H_2, \ldots, H_k$.

In $H - v$, any pair of vertices belonging to different components has no path connecting them. To form a strong hub set in $H - v$, we must ensure that all such pairs are connected via internal hub vertices.

1. Consider choosing one component H_j to contribute only its minimal strong hub set S_j of size $h^*(H_j)$.

2. For the remaining $k - 1$ components, every vertex must be included in the hub set; otherwise, vertices in different components would remain disconnected.

3. Therefore, for each choice of H_j, the total number of hubs required is

$$\sum_{i \neq j} |V(H_i)| + h^*(H_j).$$

4. Since any of the k components could be chosen as H_j, we take the minimum over all j to minimize the total hub set.

Thus, we obtain

$$h^*(H - v) = \min_{1 \leq j \leq k} \left\{ \sum_{i \neq j} |V(H_i)| + h^*(H_j) \right\}.$$

Finally, because weak deletion of a cut-vertex disconnects the hypergraph, we must include at least one vertex from each disconnected component. Hence,

$$h^*(H - v) \geq h^*(H),$$

so the strong hub number does not decrease and typically increases after deleting a cut-vertex.

Theorem 4.4. Let S be a smallest strong hub set of a connected hypergraph $H = (V, E)$. Then, for any vertex $v \in S$,

1. $h^*(H - v) \geq h^*(H)$, if v is a cut vertex in an S-path;
2. $h^*(H - v) = h^*(H) - 1$, if v is an anchor or a floater in any S-path in H but not a cut vertex.

Proof: Let S be a smallest strong hub set in H. If $v \in S$, then $S - \{v\}$ is not a strong hub set in H. Hence, there exist vertices $x, y \notin S - \{v\}$ such that x and y are not adjacent in H, and there is no path between x and y whose internal vertices lie entirely in $S - \{v\}$.

We consider the following cases.

Case 1: Suppose v is a cut vertex in an S-path between x and y in H. Then, in $H - v$, the path P between x and y is split into two disjoint paths. Since S is minimal, we may assume that P is the only S-path joining x and y. Consequently, there is no x-y path in $H - v$, and hence $S - \{v\}$ fails to be a strong hub set in $H - v$.

If (x, y) is the only such pair, then

$$S' = (S - \{v\}) \cup \{x, y\}$$

is a strong hub set in $H - v$. Therefore, $h^*(H - v) > h^*(H)$. If there exists a vertex $w \notin S$ such that there is another path from x to y whose internal vertices lie in S except for w, then

$$S' = (S - \{v\}) \cup \{w\}$$

is a strong hub set of $H - v$. In either situation, we have

$$h^*(H - v) \geq h^*(H).$$

Case 2: Suppose $v \in S$ is an anchor or a floater vertex, but not a cut vertex in any S-path of H. Then, for any such S-path P connecting vertices x and y, the path $P - v$ remains a path from x to y in $H - v$ whose internal vertices lie in $S - \{v\}$.

Hence $S - \{v\}$ is a strong hub set of $H - v$. Thus,

$$h^*(H - v) = h^*(H) - 1.$$

Theorem 4.5. Let S be a smallest strong hub set of a connected hypergraph $H = (V, E)$. If $v \notin S$, then

$$h^*(H - v) \leq h^*(H).$$

Proof: Since $v \notin S$, there exists a vertex $u \in V - S$ such that either

1. u and v are adjacent in H, or
2. there exists a strong S-path from u to v whose internal vertices lie entirely in S.

In case (1), deleting v does not affect the existing adjacencies among the vertices of $V - \{v\}$, and hence S remains a strong hub set of $H - v$. Thus,

$$h^*(H - v) = h^*(H).$$

In case (2), suppose there is a strong S-path P from v to some vertex $u \in V - S$ with internal vertices in S. If deleting v does not break all such strong S-paths, then S continues to serve as a strong hub set in $H - v$, and again $h^*(H - v) = h^*(H)$.

However, if P contains a vertex $w \in S$ that appears only in this path, then upon deleting v, the vertex w may become redundant. In that case, $S - \{w\}$ may serve as a smallest strong hub set in $H - v$, implying that

$$h^*(H - v) \leq h^*(H).$$

Hence, in all possible cases, we have

$$h^*(H - v) \leq h^*(H).$$

5. Conclusion

In this paper, we introduced and studied the concept of *strong hub sets* in hypergraphs as a natural extension of hub sets. Unlike ordinary hub sets, which require only the internal vertices of a hyperpath to belong to the hub set, strong hub sets impose a stricter condition by demanding that even the anchor and floater vertices of a strong hyperpath be contained in the set. This refinement yields the new parameter called the *strong hub number* $h^*(H)$, which we investigated for various classes of hypergraphs.

We explored fundamental properties of strong hub sets and established relationships between the strong hub number and other well-known hypergraph invariants, such as the domination number. Connections with vertex weak deletion were also highlighted, showing how vertex removal operations influence the value of $h^*(H)$. Moreover, we provided bounds and extremal characterizations for the strong hub number, thereby situating this parameter within the broader framework of intersection-based measures in hypergraphs.

The study of strong hub sets not only generalizes hub-based notions from graphs to hypergraphs but also strengthens them by introducing robustness in connectivity through strong hyperpaths. These results open up further directions for research, including algorithmic aspects of computing $h^*(H)$, tight bounds for specific classes of hypergraphs, and potential applications in network design, fault tolerance, and information systems.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Acharya B., Domination in Hypergraphs, *AKCE J. Graphs. Combin.*, 4 NO. 2 (2007), 111-126. https://doi.org/10.1080/09728600.2007.12088825

[2] M. Amin Bahmanian and Mateja Sajna, Hypergraphs: connection and separation., *arXiv: 1504. 04274v2 [math.CO]*, 2 (28 May 2015). https://doi.org/10.20429/tag.2015.020205

[3] Amitesh Sarkar, Anirban Bnerjee., Joins of hypergraphs and their spectra., *arXiv: 1912. 12921v1 [math CO]*, 1(30 Dec 2019). https://doi.org/10.1016/j.laa.2020.05.029

[4] Berge C., Graphs and Hypergraphs, *North-Holland, Amsterdam* 6(1973).

[5] Berge C., Hypergraphs, *North-Holland Mathematical Library, New York*, Volume-45 (1989).

[6] Ioan Tomescu., On the chromaticity of sunflower hypergraphs SH(n, p, h)., *Discrete Mathematics.* 307 (2007), 781-786. https://doi.org/10.1016/j.disc.2006.07.026

[7] Matthew Walsh, The hub number of a graph, *International Journal of Mathematics and computer Science.*, 1 (2006), 117, 124.

[8] Regina I. Ttshkevich, Vadim E. Zverovich, Line hypergraphs : A survey, *Acta Applicandae Mathematicae.*, 52 (1998), 209-222. https://doi.org/10.1023/A:1005963110362

[9] K. K. Mythili, C. Nandhini, Hub parameters of Hypergraph, *J. Indones. Math. Soc*, 30 01 (2024), 100-109. https://doi.org/10.22342/jims.30.1.1443.100-109

[10] V. I. Voloshin, Introduction to Graph and Hypergraph Theory, *Nova Science Publishers*, 2009.

[11] A. Bretto, Hypergraph Theory: An Introduction, *Springer*, 2013. https://doi.org/10.1007/978-3-319-00080-0

[12] C. Thomassen, Connectivity in hypergraphs, *J. Combin. Theory Ser. B*, 37(2) (1984), 244–249.

[13] I. E. Zverovich, On hypertrees and their applications in network theory, *Discrete Math. Algorithms Appl.*, 11(3) (2019), 1950045.

[14] X. Zhang, Y. Li, Domination in hypergraphs and its applications, *Discuss. Math. Graph Theory*, 36(2) (2016), 325–340.

[15] K. S. Shama, P. M. Divya, and T. V. Ramakrishnan, A Study on Hub Sets in Hypergraphs, *Int. J. Math. And Appl.*, vol. 13 (2) (2025), pp. 71–81.