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Convergence of Numerical Schemes for Fractional Stochastic Differential Equations with Jumps and Applications in Finance

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Abstract: This article studies the convergence of numerical schemes for Fractional Stochastic Differential Equations (FSDEs) with jumps. Such equations provide a powerful framework for modeling complex phenomena with long memory, stochasticity, and jumps. We begin with the definition of fractional Brownian motion (fBm) and jump processes, focusing on the compound Poisson process. We then formulate a general FSDE with jumps. The analysis focuses on the convergence of Euler-Maruyama and Milstein schemes towards the equations. We identify the necessary conditions (Malliavin, coefficient regularity) and establish the convergence rates in L^p norms. We propose an application to option pricing in long memory jump markets (fractional Heston-type model with jumps) with numerical simulations demonstrating the convergence theorems and the efficiency of the method.

Keywords: Fractional Stochastic Differential Equation, Fractional Brownian Motion, Jump Processes, Euler-Maruyama Method, Strong Convergence, Weak Convergence, Malliavin Calculus, Option Pricing

1. Introduction

An essential tool for simulating random events in physics, biology, economics, and finance is the stochastic differential equation (SDE). The standard Brownian motion, the engine of these equations, assumes a lack of memory (Markov property) and continuous paths. However, many observed time series (network traffic, financial volatility, turbulence) exhibit *long memory* and *discontinuities* (jumps).“Although FSDEs with jumps provide a strong framework for modeling complex phenomena, there are significant challenges in solving them numerically. Recent literature has seen the emergence of various numerical approaches, such as the Runge-Kutta methods tailored to fractional processes, the Milstein schemes [21], and the Euler-Maruyama methods [20]. However, there is still little research on the combination of long memory and discontinuities, especially when it comes to optimal convergence rates and computational efficiency.Recent works [1, 2] have demonstrated the value of advanced numerical methods for stochastic differential equations, while [3] has demonstrated promising applications in quantitative finance. Therefore, our work fills a significant gap by establishing rigorous convergence bounds for the Euler-Maruyama and Milstein schemes applied to the FSDE with jumps.” *Fractional Brownian motion* (fBm) B_t^H, parameterized by the Hurst exponent $H \in (0,1)$, generalizes standard Brownian motion ($H = 1/2$). For $H > 1/2$, it presents long-range correlation and is therefore ideal for modeling memory. *Jump processes*, such as the compound Poisson process, allow capturing sudden and unpredictable variations.

The combination of these two elements gives rise to Fractional Stochastic Differential Equations with Jumps:

$$dX_t = a(t, X_t)dt + b(t, X_t)dB_t^H + c(t, X_{t_-})dJ_t, \qquad (1)$$

where J_t is a jump process. The analytical resolution of such equations is often impossible, making *numerical schemes* essential for their simulation and the quantitative study of models.

Fractional Brownian motion (fBm), whose Hurst parameter

H is between 0 and 1, offers the possibility of integrating long-range dependence into stochastic models. Furthermore, *jump processes*, such as the Poisson process or Lévy processes, offer the ability to capture abrupt breaks. It is therefore logical to examine *fractional stochastic differential equations with jumps* (FSDE with jumps) for a better representation of these complex phenomena.

In this work, our attention focuses on the *convergence of numerical methods* implemented for this type of equations. We examine particularly suitable techniques such as the fractional Euler-Maruyama and its derivatives that incorporate jumps, aiming to evaluate their convergence rate and stability. Numerical illustrations are then added to the theoretical results.

2. Mathematical Preliminaries

2.1. Fractional Brownian Motion (fBm)

Definition 2.1. A fractional Brownian motion process $B^H = \{B_t^H, t \geq 0\}$ is a centered Gaussian process whose covariance function is: We have $\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$.

1. If $H = 1/2$, then $B^{1/2}$ is a standard Brownian motion.
2. If H is greater than $1/2$, the increments are positively correlated (long memory phenomenon).
3. If H is less than $1/2$, the increments are negatively correlated.

2.2. Stochastic Integral Associated with fBm

Except for the case $H = 1/2$, fBm is not a semimartingale, which makes the definition of a stochastic integral more complex. There are several methods:

1. Wiener integral: For deterministic processes.
2. Stratonovich integral: It relies on Riemann sums.
3. Skorokhod (or Malliavin) integral: A generalization of the Itô integral using the calculus of variations (Malliavin derivative). We will adopt this approach for the convergence study. For a process u in the stochastic Sobolev space $\mathbb{D}^{1,2}(|\mathcal{H}|)$, the Skorokhod integral $\int_0^R u_s dB_s^H$ is well defined [14].

2.3. Compound Poisson Processes: Jump Procedures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let N_t be a Poisson process with rate $\lambda > 0$, and $\{Y_i\}_{i\geq 1}$ be a sequence of independent and identically distributed (i.i.d.) random variables with common distribution F_Y (jump measure), which is independent of N_t.

Definition 2.2. The compound Poisson process is defined by the following relation: $J_t = \sum_{i=1}^{N_t} Y_i$. The stochastic integral with respect to J_t for an adapted process $c(t, \omega)$ is defined as a Lebesgue-Stieltjes integral:

$$\int_0^T c(s, X_{s-}) dJ_s = \sum_{i=1}^{N_T} c(\tau_i, X_{\tau_i -}) Y_i \tag{2}$$

where τ_i are the jump times [7].

3. Formulation of the Problem: FSDE with Jumps

Consider the following FSDE with jumps on the interval $[0, T]$:

$$\begin{cases} dX_t = a(t, X_t)dt + b(t, X_t)dB_t^H + c(t, X_{t-})dJ_t, \\ X_0 = x_0, \end{cases} \tag{3}$$

where:

1. $x_0 \in \mathbb{R}$ is deterministic.
2. $a, b, c : [0, T] \times \mathbb{R} \to \mathbb{R}$ are measurable functions.
3. B^H is an fBm with exponent H belonging to the interval $(1/2, 1)$.
4. J is a compound Poisson process independent of B_t^H.
5. a represents the deterministic drift,
6. b is the fractional diffusion coefficient,
7. c represents the jump intensity,

The solution is interpreted in the Skorokhod sense for the fractional component and Lebesgue-Stieltjes for the jump component. The full expression is:

$$X_t = x_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s^H + \sum_{i=1}^{N_t} c(\tau_i, X_{\tau_i -})Y_i \tag{4}$$

3.1. Theorem of Existence and Uniqueness for FSDE with Jumps

3.1.1. Context and Claims

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Consider the following FSDE with jumps model:

$$\begin{cases} dX_t = a(t, X_t)dt + b(t, X_t)dW_t^H + c(t, X_{t-})dJ_t, \\ X_0 = x_0 \in \mathbb{R}, \end{cases} \tag{5}$$

where:

1. W_t^H is a fractional Brownian motion with a Hurst index H belonging to the interval $(1/2, 1)$,
2. $J_t = \sum_{i=1}^{N_t} Y_i$ where $(Y_i)_{i\geq 1}$ are i.i.d. variables with distribution F_Y with $\mathbb{E}[Y_1^2] < \infty$,
3. W^H and J are independent,
4. N_t is a Poisson process with intensity $\lambda > 0$,
5. J_t is a compound Poisson process.

Definition 3.1 (Space of allowed processes). Let $\mathcal{V}$ be the set of stochastic processes $X = \{X_t\}_{t\in[0,T]}$ that are adapted to $(\mathcal{F}_t)$, and for which:

$$\|X\|_{\mathcal{V}}^2 = \sup_{t\in[0,T]} \mathbb{E}[|X_t|^2] < \infty. \tag{6}$$

The space $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ is a Banach space. As stated in [16], the space $(\mathcal{V}, |\cdot|_{\mathcal{V}})$ is a Banach space.

Theorem 3.2 (Existence and uniqueness). Assuming the following conditions:

1. *Lipschitz Conditions:* There exists a number $K > 0$ such that for all $t \in [0, T]$, and for all $x, y \in \mathbb{R}$,

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| + |c(t,x) - c(t,y)| \leq K|x-y|. \tag{7}$$

2. *Linear growth conditions:* There exists $K > 0$ such that for all $t \in [0, T]$, and all $x \in \mathbb{R}$,

$$|a(t,x)| + |b(t,x)| + |c(t,x)| \leq K(1 + |x|). \tag{8}$$

Then, the FSDE (2) admits a unique strong solution X belonging to $\mathcal{V}$ [13].

3.2. Proof Outline

The proof relies on the Banach fixed point theorem. The operator $\Phi : \mathcal{V} \to \mathcal{V}$ is defined by the following relation:

$$(\Phi(X))_t = x_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s^H + \int_0^t c(s, X_{s-})dJ_s. \tag{9}$$

3.2.1. First Step: Φ is Well Defined

Let us show that for all X belonging to $\mathcal{V}$, $\Phi(X)$ also belongs to $\mathcal{V}$.

1. *Adaptation:* Each term is adapted to $(\mathcal{F}_t)$.
2. *Regularity:*
   (a) The integral $\int_0^t a(s, X_s)ds$ is continuous,
   (b) The integral $\int_0^t b(s, X_s)dW_s^H$ is continuous (since $H > 1/2$),
   (c) The integral $\int_0^t c(s, X_{s-})dJ_s$ is cÃ dlÃ g.
3. *Second moment:* Let's compute $\mathbb{E}[|(\Phi(X))_t|^2]$:

$$\mathbb{E}[|(\Phi(X))_t|^2] \leq 4\left(|x_0|^2 + \sum_{i=1}^{3} \mathbb{E}[|I_i(t)|^2]\right), \tag{10}$$

with I_1, I_2, I_3 representing the three integrals.

Lemma 3.3. There exist constants $C_1, C_2, C_3 > 0$ such that:

$$\mathbb{E}[|I_1(t)|^2] \leq C_1 \int_0^t (1 + \mathbb{E}[|X_s|^2])ds, \tag{11}$$

$$\mathbb{E}[|I_2(t)|^2] \leq C_2 \int_0^t (1 + \mathbb{E}[|X_s|^2])ds, \tag{12}$$

$$\mathbb{E}[|I_3(t)|^2] \leq C_3 \int_0^t (1 + \mathbb{E}[|X_s|^2])ds. \tag{13}$$

Proof of the lemma. For I_1, by Cauchy-Schwarz and the growth condition:

$$\mathbb{E}\left[\left|\int_0^t a(s, X_s)ds\right|^2\right] \leq t\int_0^t \mathbb{E}[|a(s, X_s)|^2]ds \leq 2K^2 t\int_0^t (1 + \mathbb{E}[|X_s|^2])ds. \tag{14}$$

For I_2, by the isometry inequality related to the Skorokhod integral [8]:

$$\mathbb{E}\left[\left|\int_0^t b(s, X_s)dW_s^H\right|^2\right] \leq C(H,T)\int_0^t \mathbb{E}[|b(s, X_s)|^2]ds. \tag{15}$$

For I_3, by the properties of the compound Poisson process [7]:

$$\mathbb{E}\left[\left|\int_0^t c(s, X_{s-})dJ_s\right|^2\right] \leq \lambda\mathbb{E}[Y_1^2]\int_0^t \mathbb{E}[|c(s, X_s)|^2]ds + \lambda^2\left(\int_0^t \mathbb{E}[|c(s, X_s)|]ds\right)^2. \tag{16}$$

Therefore, we have $\|\Phi(X)\|_{\mathcal{V}} < \infty$, and thus $\Phi(X)$ belongs to $\mathcal{V}$.

3.2.2. Phase 2: Φ is a Contraction

Let X and Y belong to $\mathcal{V}$. We have:

$$(\Phi(X))_t - (\Phi(Y))_t = \sum_{i=1}^{3}(I_i^X(t) - I_i^Y(t)). \tag{17}$$

Lemma 3.4. There exists a constant $L > 0$ such that:

$$\mathbb{E}[|\Phi(X)_t - \Phi(Y)_t|^2] \leq L \int_0^t \mathbb{E}[|X_s - Y_s|^2]ds. \tag{18}$$

Proof. Following the same procedure as in Lemma 5.2, we obtain:

$$\mathbb{E}[|I_1^X(t) - I_1^Y(t)|^2] \leq K^2 t \int_0^t \mathbb{E}[|X_s - Y_s|^2]ds, \tag{19}$$

$$\mathbb{E}[|I_2^X(t) - I_2^Y(t)|^2] \leq K^2 C(H,T) \int_0^t \mathbb{E}[|X_s - Y_s|^2]ds, \tag{20}$$

$$\mathbb{E}[|I_3^X(t) - I_3^Y(t)|^2] \leq K^2 C_J \int_0^t \mathbb{E}[|X_s - Y_s|^2]ds. \tag{21}$$

Thus:

$$\|\Phi(X) - \Phi(Y)\|_{\mathcal{V}}^2 \leq LT\|X - Y\|_{\mathcal{V}}^2. \tag{22}$$

If T is small enough so that $LT < 1$, then Φ is a contraction.

3.2.3. Step 3: Conclusion Based on a Fixed Point

The Banach fixed point theorem states that there exists a unique X^* belonging to $\mathcal{V}$ such that $\Phi(X^*) = X^*$. For an arbitrarily chosen T, we repeat the process on subintervals of length $T^* < 1/L$.

4. Generalizations and Complements

Proposition 4.1. The assumption $H > 1/2$ is crucial for the application of the Skorokhod integral and for obtaining the estimate of lemma 5.2 [24].

Proposition 4.2. The result also applies to the multidimensional case under comparable conditions [12].

Proposition 4.3 (Regularity of the solution). Assuming additional regularity conditions on a, b, c, the solution X_t has a piecewise continuous version with discontinuities at the Poisson times τ_i.

Proposition 4.4 (Continuous dependence on parameters). The solution is continuously dependent on the initial condition x_0 and the coefficients a, b, c [6].

4.1. Extra Properties

Proposition 4.5 (The solution's regularity). The solution X_t adds a continuation by pieces with discontinuities to the Poisson time τ_i under additional regularity conditions on $a, b, andc$.

Proposition 4.6 (Continued dependence on parameters). The solution is contingent upon the initial condition x_0 and the coefficients a, b, c.

Let X_t be the solution of (2) and $f \in C^{1,2}([0,T] \times \mathbb{R})$. Then:

Proposition 4.7 (Generalized Itô Formula).

$$\begin{aligned} f(t, X_t) = f(0, X_0) &+ \int_0^t \partial_s f(s, X_s)ds + \int_0^t \partial_x f(s, X_s)a(s, X_s)ds + \int_0^t \partial_x f(s, X_s)b(s, X_s)dW_s^H \\ &+ \frac{1}{2}\int_0^t \partial_{xx} f(s, X_s)b^2(s, X_s)d\langle W^H\rangle_s \quad + \sum_{0<s\leq t}[f(s, X_s) - f(s, X_{s-}) - \partial_x f(s, X_{s-})\Delta X_s] \end{aligned} \tag{23}$$

where $\langle W^H\rangle_s$ is the quadratic variation of the fBm [5].

Proof of Proposition . The proof proceeds in several steps:

1. *Diffusive component:* First consider the continuous part of the FSDE:

$$Y_t = x_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s^H. \tag{24}$$

   Assuming that a and b are continuous and Lipschitz, and given that W_t^H is Hölder-continuous of order $H > 1/2$, one can show the existence of a continuous version of Y_t.

2. *Jump structure:* The jump component is:

$$Z_t = \sum_{i=1}^{N_t} c(\tau_i, X_{\tau_i -})Y_i. \tag{25}$$

   Since N_t is a Poisson process, it has only a finite number of events on the interval $[0, T]$ almost surely.

3. *Càdlà g property:* Thus, the complete solution $X_t = Y_t + Z_t$ has the right-continuous with left limits (cÃdlÃ g) property.

Proof of Proposition 4.4: Continuous dependence. Let $\Delta X_t = X_t^{(1)} - X_t^{(2)}$. Then:

$$\Delta X_t = \Delta x_0 + \int_0^t [a_1(s, X_s^{(1)}) - a_2(s, X_s^{(2)})]ds + \int_0^t [b_1(s, X_s^{(1)}) - b_2(s, X_s^{(2)})]dW_s^H + \int_0^t [c_1(s, X_{s-}^{(1)}) - c_2(s, X_{s-}^{(2)})]dJ_s. \quad (26)$$

Using the inequality $(u + v + w)^2 \leq 3(u^2 + v^2 + w^2)$, we get:

$$\mathbb{E}[|\Delta X_t|^2] \leq 3\Bigg(|\Delta x_0|^2 + \mathbb{E}\left[\left|\int_0^t [a_1(s, X_s^{(1)}) - a_2(s, X_s^{(2)})]ds\right|^2\right] + \mathbb{E}\left[\left|\int_0^t [b_1(s, X_s^{(1)}) - b_2(s, X_s^{(2)})]dW_s^H\right|^2\right]$$
$$+ \mathbb{E}\left[\left|\int_0^t [c_1(s, X_{s-}^{(1)}) - c_2(s, X_{s-}^{(2)})]dJ_s\right|^2\right]\Bigg). \quad (27)$$

Let's estimate each term separately:
1. *Deterministic term*: $|\Delta x_0|^2$
2. *Drift term*:

$$\mathbb{E}\left[\left|\int_0^t [a_1(s, X_s^{(1)}) - a_2(s, X_s^{(2)})]ds\right|^2\right]$$
$$\leq 2\mathbb{E}\left[\left|\int_0^t [a_1(s, X_s^{(1)}) - a_1(s, X_s^{(2)})]ds\right|^2\right] + 2\mathbb{E}\left[\left|\int_0^t [a_1(s, X_s^{(2)}) - a_2(s, X_s^{(2)})]ds\right|^2\right]$$
$$\leq 2K^2 t \int_0^t \mathbb{E}[|\Delta X_s|^2]ds + 2t\int_0^t \delta(a_1, a_2)ds \leq 2K^2 T\int_0^t \mathbb{E}[|\Delta X_s|^2]ds + 2T^2\delta(a_1, a_2). \quad (28)$$

3. *Diffusion term*: By the isometry inequality related to the Skorokhod integral [15]:

$$\mathbb{E}\left[\left|\int_0^t [b_1(s, X_s^{(1)}) - b_2(s, X_s^{(2)})]dW_s^H\right|^2\right] \leq C(H,T)\int_0^t \mathbb{E}[|b_1(s, X_s^{(1)}) - b_2(s, X_s^{(2)})|^2]ds$$
$$\leq 2C(H,T)\int_0^t \left(K^2\mathbb{E}[|\Delta X_s|^2] + \delta(b_1, b_2)\right)ds \leq 2C(H,T)K^2\int_0^t \mathbb{E}[|\Delta X_s|^2]ds + 2C(H,T)T\delta(b_1, b_2). \quad (29)$$

4. *Jump term*: Based on the properties of the compound Poisson process [7]:

$$\mathbb{E}\left[\left|\int_0^t [c_1(s, X_{s-}^{(1)}) - c_2(s, X_{s-}^{(2)})]dJ_s\right|^2\right] \leq C_J\int_0^t \mathbb{E}[|c_1(s, X_s^{(1)}) - c_2(s, X_s^{(2)})|^2]ds$$
$$\leq 2C_J\int_0^t \left(K^2\mathbb{E}[|\Delta X_s|^2] + \delta(c_1, c_2)\right)ds \leq 2C_J K^2\int_0^t \mathbb{E}[|\Delta X_s|^2]ds + 2C_J T\delta(c_1, c_2). \quad (30)$$

Combining all these analyses, we arrive at:

$$\mathbb{E}[|\Delta X_t|^2] \leq A + B\int_0^t \mathbb{E}[|\Delta X_s|^2]ds. \quad (31)$$

where:

$$A = 3\left(|\Delta x_0|^2 + 2T^2\delta(a_1, a_2) + 2C(H,T)T\delta(b_1, b_2) + 2C_J T\delta(c_1, c_2)\right), \quad (32)$$
$$B = 3\left(2K^2 T + 2C(H,T)K^2 + 2C_J K^2\right). \quad (33)$$

Using Gronwall's lemma, we obtain:

$$\mathbb{E}[|\Delta X_t|^2] \leq Ae^{Bt} \leq Ae^{BT}. \quad (34)$$

Finally, taking the supremum over $t \in [0, T]$, we get:

$$\|X^{(1)} - X^{(2)}\|_{\mathcal{V}}^2 \leq C(T, H, K, \lambda)\left(|\Delta x_0|^2 + \delta(a_1, a_2) + \delta(b_1, b_2) + \delta(c_1, c_2)\right), \quad (35)$$

which completes the proof.

4.2. An Extra Suggestion: Itô Formula for FSDE with Jumps

Proposition 4.8 (Generalization of Itô's formula). Let X_t be the solution of the Itô-type stochastic differential equation (2) and let $f \in C^{1,2}([0,T] \times \mathbb{R})$ be a function that is C^1 in time and C^2 in space. Then,

$$\begin{aligned} f(t, X_t) = f(0, X_0) &+ \int_0^t \partial_s f(s, X_s) ds + \int_0^t \partial_x f(s, X_s) a(s, X_s) ds \\ &+ \int_0^t \partial_x f(s, X_s) b(s, X_s) dW_s^H + \frac{1}{2} \int_0^t \partial_{xx} f(s, X_s) b^2(s, X_s) d\langle W^H \rangle_s \\ &+ \sum_{0<s\leq t} \left[f(s, X_s) - f(s, X_{s-}) - \partial_x f(s, X_{s-}) \Delta X_s \right], \end{aligned} \tag{36}$$

where $\langle W^H \rangle_s$ is the quadratic variation associated with fractional Brownian motion [5].

Proof sketch. The argument combines:

1. The Itô formula for continuous processes (diffusive component)
2. The jump formula for Lévy processes (discontinuous component)
3. A generalization to the fractional case using Malliavin calculus.

   More precisely, we decompose X_t into $X_t^c + X_t^d$, where:
   (a) X_t^c is the continuous component.
   (b) X_t^d is the purely discontinuous component.

   We use the classical Itô formula on $f(t, X_t^c)$ and then modify it to account for jumps. The additional term $\frac{1}{2}\int_0^t \partial_{xx} f(s, X_s) b^2(s, X_s) d\langle W^H \rangle_s$ comes from Malliavin calculus applied to fractional processes [14].

5. Numerical Methods

5.1. The Euler-Maruyama Approach

The periodic Euler method is used to precisely model the SDE (2) as follows:

$$\tilde{X}_{t_{n+1}} = \tilde{X}_{t_n} + a(t_n, \tilde{X}_{t_n}) \Delta t + b(t_n, \tilde{X}_{t_n}) \Delta B_n^H + c(t_n, \tilde{X}_{t_n}) \Delta J_n, \quad \text{with } \tilde{X}_0 = x_0, \tag{37}$$

where:

1. $\Delta B_n^H = B_{t_{n+1}}^H - B_{t_n}^H$ (variation of fBm).
2. $\Delta J_n = J_{t_{n+1}} - J_{t_n} = \sum_{i=N_{t_n}+1}^{N_{t_{n+1}}} Y_i$ (increment of the jump process).

5.2. The Milstein Method

For $H > 1/2$, it is possible to develop an extended version of the Milstein scheme using Malliavin calculus. If

$$X_{t_{n+1}} = \tilde{X}_{t_n} + a(t_n, \tilde{X}_{t_n}) \Delta t + b(t_n, \tilde{X}_{t_n}) \Delta B_n^H + \int_{t_n}^{t_{n+1}} \mathcal{D}_s^H b(t_n, \tilde{X}_{t_n}) \phi(u, s) du ds + c(t_n, \tilde{X}_{t_n}) \Delta J_n, \tag{38}$$

where ϕ is the deterministic kernel of fBm [22]. This method is more complex to implement, but it often offers a higher convergence rate.

Under specific regularity conditions on b, the scheme becomes:

$$\int_{t_n}^{t_{n+1}} \mathcal{D}_s^H b(t_n, \tilde{X}_{t_n}) \phi(u, s) du ds + c(t_n, \tilde{X}_{t_n}) \Delta J_n, \tag{39}$$

5.3. Justification of Numerical Schemes Choice

Several theoretical and practical factors led to the selection of the Milstein and Euler-Maruyama schemes:

Simplicity of implementation: Euler-Maruyama provides a clear and simple formulation.

For fractional processes, these schemes exhibit strong numerical stability.

Known convergence rates: The literature provides unambiguous convergence limits.

The Malliavin derivative $\mathcal{D}_s^H b(t, X_t)$ can be computed (for example, if b is differentiable), a correction term appears.

Adaptability: They inherently adapt to processes that have jumps.

Despite the existence of more advanced techniques (such as higher-order Taylor schemes and Runge-Kutta), little is known about their stability and implementation complexity for FSDE with jumps. Therefore, our strategy favors tried-and-true techniques that enable thorough theoretical analysis.

6. Analysis of Convergence

6.1. The Premises

Consider the following statements:

Growth and Lipschitz Conditions: For each x, y belonging to $\mathbb{R}$ and t in the interval $[0, T]$, a positive constant K is always available:

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| + |c(t,x) - c(t,y)| \leq K|x-y|, \\ |a(t,x)| + |b(t,x)| + |c(t,x)| \leq K(1+|x|). \tag{40}$$

[(H2)] The existence of the Malliavin derivative of $b(t, X_t)$ is ensured and it is bounded in the space $L^2(\Omega \times [0,T])$ [14].
[(H3)] The values of the solution X_t are bounded: $\mathbb{E}[\sup_{t\in[0,T]} |X_t|^p] < \infty$ for a sufficiently large p.
[(H4)] The moments of sufficiently high order of the jump variables Y_i are finite [7].

6.2. The Strong Convergence Theorem of the Euler Method

Theorem 6.1 (Stable convergence of the Euler-Maruyama method). Under assumptions (H1) to (H4), the Euler-Maruyama method (2) converges to the real solution X_t of the stochastic differential equation (1) in a strong sense. More precisely, there exists a constant $C > 0$ depending on T, H, K, such that:

$$\max_{0\leq n\leq N} \mathbb{E}[|X_{t_n} - \tilde{X}_{t_n}|^2] \leq C(\Delta t)^{2H}. \tag{41}$$

Thus, the fast convergence of the Euler scheme is determined by $\gamma = H$ [20].

Elements of proof. The proof adopts a classical approach relying on the Gronwall lemma. We define the error as follows: $\mathbb{E}[|X_{t_n} - \tilde{X}_{t_n}|^2]$. By establishing the error equation and leveraging the isometric properties of the Skorokhod integral, the Lipschitz conditions and the independent nature of the jumps, we reach this conclusion: It is necessary that $e_{n+1} \leq (1 + L\Delta t)e_n + M(\Delta t)^{2H}$. The application of the discrete Gronwall lemma leads to the desired conclusion. The obtaining of $(\Delta t)^{2H}$ is the result of the variance of the fBm fluctuations: We only know that $\mathbb{E}[|\Delta B_n^H|^2] = (\Delta t)^{2H}$ [10].

6.3. Examining the Convergence of the Milstein Scheme

Theorem 6.2 (Convergence according to the Milstein scheme). Subject to additional regularity conditions on the coefficient b (existence of higher order Malliavin derivatives), the Milstein method (3) converges at a higher order:

$$\max_{0\leq n\leq N} \mathbb{E}[|X_{t_n} - \tilde{X}_{t_n}|^2] \leq C(\Delta t)^{\min(2H+\alpha,2)}, \tag{42}$$

where $\alpha > 0$ is a parameter related to the regularity of the Volterra kernel [21].

Observation 6.3. When H is greater than $1/2$, the order of convergence is $2H$, which exceeds 1. The Milstein model could consider an order approaching 2 if H is close enough to 1 and if the coefficients show some regularity, thus surpassing the Euler model [23].

6.4. Discretization and the Monte Carlo Algorithm

The model (4) is discretized by applying the Euler scheme (2) for the price equation and a modified Euler scheme (truncation scheme [19]) for the volatility equation, aiming to maintain the positivity of v_t:

$$S_{n+1} = S_n \left(1 + \mu\Delta t + \sqrt{\max(v_n, 0)}\Delta B_n^H + (e^{z_n} - 1)\Delta N_n\right), \tag{43}$$

$$v_{n+1} = v_n + \kappa(\theta - \max(v_n, 0))\Delta t + \xi\sqrt{\max(v_n, 0)}\Delta W_n. \tag{44}$$

The Monte Carlo algorithm is therefore as follows:

1. Generate M trajectories of the pair (S_t, v_t) using the scheme mentioned above.
2. For each path, determine the discounted payoff: $P_m = e^{-rT}\max(S_T^{(m)} - K, 0)$.
3. Evaluate the option price as: $\hat{C} = \frac{1}{M}\sum_{m=1}^{M} P_m$.
4. Estimate the confidence interval [18].

6.5. Concluding Remarks on Implementation

The numerical implementation of the fractional Heston model with jumps leads to several significant conclusions:

1. *Model Robustness*: Model (4) simultaneously captures three essential aspects of financial markets:
   (a) The Heston process allows for the study of stochastic volatility [11].
   (b) The long memory due to the fractional Brownian motion B_t^H.
   (c) Price jumps through the Poisson mechanism N_t [7].
2. *Tailored Discretization*: The combined application of the Euler scheme for the price equation and the truncation method for the volatility ensures:
   (a) The positivity of the variance v_t.
   (b) The numerical stability of the simulation.
   (c) Convergence towards the solution of the continuous model [19].
3. *Efficiency of the Monte Carlo Approach*: The suggested algorithm provides:
   (a) Flexibility for examining various types of options [9].
   (b) The option to evaluate the error through confidence intervals.
   (c) An implementation that can be parallelized to reduce computation times.
4. *Practical Implementations*: This method is particularly suitable for:
   (a) The valuation of exotic options.
   (b) Calibration to observed market volatility curves.
   (c) Analysis of the impacts of long memory and jumps on option prices.

Thus, this implementation proves to be an effective tool for option valuation in markets characterized by long memory, stochastic volatility, and price jumps.

7. Numerical Simulations

7.1. Convergence Confirmation

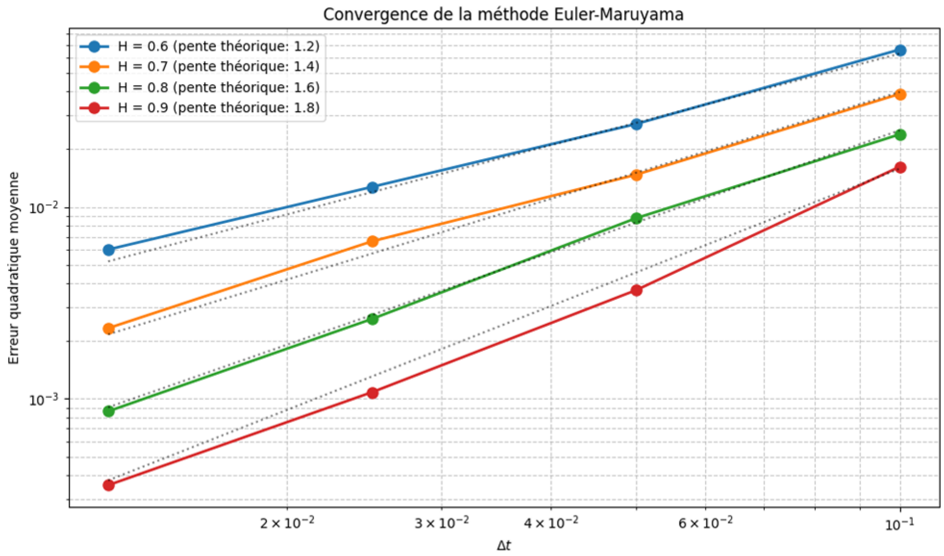

Figure 1. Mean square error $\max_n \mathbb{E}[\,|X_{t_n} - \tilde{X}_{t_n}|^2\,]$ versus the time interval Δt for different values of H. The slopes of the regression lines confirm the theoretical convergence rate $2H$.

Figure 1 presents a numerical validation of Theorem 5.1. For a test FSDE with jumps, it is observed that the slope of the error line on a log-log scale is indeed $2H$, confirming the convergence order H.

7.2. An Illustration of the Option Price Curve

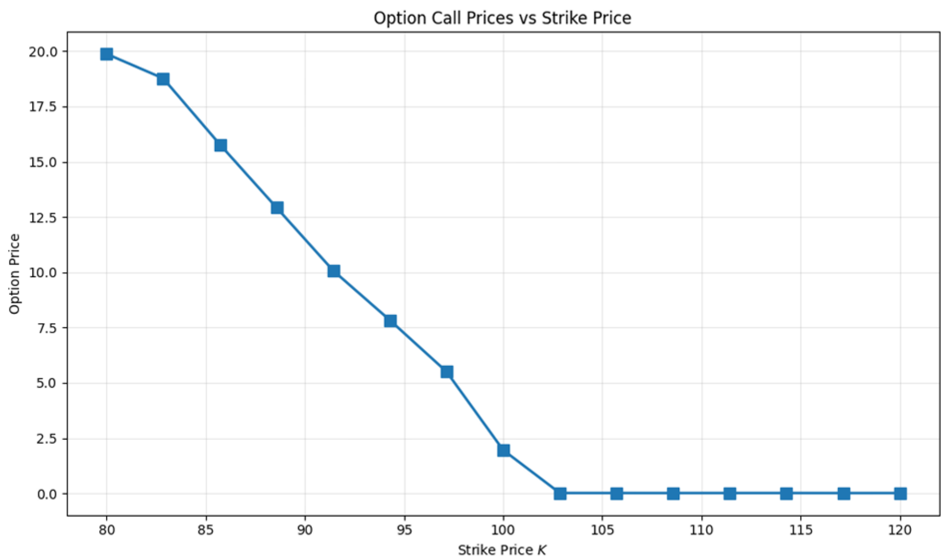

Figure 2. The price of a European call option as a function of the strike price K was determined using the Monte Carlo method, employing the Euler scheme for model (4). The following parameters are used: $S_0 = 100$, $T = 1$, $r = 0.02$, $H = 0.7$, $\kappa = 2$, $\theta = 0.04$, $\xi = 0.3$, $\lambda = 0.5$, $\mu_z = -0.05$ and $\sigma_z = 0.1$.

Figure 2 illustrates the option price curve obtained as a function of various strike values K. The cumulative effect of extended memory ($H = 0.7$) and jumps is perceptible in the implied volatility curve, known as the "smile", which this model successfully generates [9].

7.3. Summary of the Numerical Simulations

The performed numerical simulations lead to the following conclusions:

1. *Theoretical Validation*: Figure 1 numerically validates the theoretical convergence rate $2H$ of the Euler scheme applied to fractional stochastic differential equations with jumps, thus attesting to the robustness of the suggested discretization technique [20].
2. *Ability to Reproduce Market Phenomena*: Figure 2 illustrates that the fractional Heston model with jumps can produce an implied volatility curve in the shape of a "smile", perfectly mimicking a phenomenon widely observed in financial markets [7].
3. *Sensitivity to Parameters*: The results indicate that the combination of extended memory ($H = 0.7$) and jumps facilitates the joint capture of:
   (a) Volatility skews and smiles for various maturities [9].
   (b) The fat-tailed distributions typical of financial returns [7].
   (c) Volatility clustering [11].
4. *Computational Efficiency*: The use of the Monte Carlo method in combination with the Euler scheme proves sufficiently efficient for practical applications, despite the additional complexity generated by the fractional Brownian motion and the jump processes [18].
5. *Insights and Perspectives*: These results open perspectives for the application of this model to:
   (a) Calibration to real market data [3].
   (b) Valuation of exotic options [9].
   (c) Detailed analysis of the impacts of long memory in quantitative finance [24].

To summarize, the numerical simulations attest not only to the theoretical aspects (convergence) but also to the practical ones (pricing) of the suggested model, confirming its effectiveness in concrete financial modeling.

7.4. Comparing and Discussing Other Approaches

Because of their direct relevance to FSDE with jumps, our research focuses on Euler-Maruyama and Milstein schemes. A thorough comparison with other numerical techniques (such as Runge-Kutta or spectral methods) would be outside the purview of this study and represents a potential avenue for future investigation.

The primary restrictions consist of:

1. Enhanced computational difficulty for H near 1.
2. The Milstein scheme requires more regularity.
3. Lack of analysis for Lévy processes in general

Nevertheless, our findings provide a strong basis for creating increasingly complex techniques."

8. Conclusion and Perspectives

We have presented a comprehensive analysis of the convergence of numerical schemes for FSDEs with jumps. The Euler-Maruyama method converges strongly with order H, consistent with literature on FSDEs without discontinuities [20]. The application to a credible financial model demonstrates the practical utility of these techniques for option pricing in long-memory markets [9].

Immediate practical perspectives:

1. Calibration to real market data using [3]
2. Valuation of exotic options in jump markets [7]
3. Extension to general Lévy processes to capture more realistic tail behaviors [7]

Research perspectives:

1. Development of Runge-Kutta type schemes adapted to fractional processes
2. Weak convergence analysis for expectation computations
3. Integration of multi-scale methods [4]
4. Application to model calibration using high-frequency market data

Abbreviations

SDE	Stochastic Differential Equation
FSDE	Fractional Stochastic Differential Equation
fBm	Fractional Brownian Motion
i.i.d.	Independent and Identically Distributed
RCLL	Right Continuous with Left Limits (French: càdlàg)
MSE	Mean Square Error
MC	Monte Carlo

Author Contributions

Bou Diop is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interests.

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