

Research Article

Derivation of Worth Growth Rate of an Investor's Portfolio Under Multi-fractal Analysis

Joy Ijeoma Adindu-Dick* 

Department of Mathematics, Imo State University, Owerri, Nigeria

Abstract

The fractal dimension is the basic notion for describing structures that have a scaling symmetry. In finance, multi-fractality is one of the well known facts which characterized non-trivial properties of financial time series. The stock price (or index) fluctuations can be described in terms of long-range temporal correlations by a spectrum of the Holder exponents and a set of fractal dimensions. To forecast the market risk, assessing the stock price indices is the foundation. Multi-fractal has lots of advantages when explaining the volatility of the stock prices. The asset price returns are multi-period market depending on market scenarios which are the measure points. In this work, we use some tools of multi-fractal analysis to derive the worth growth rate of an investor's portfolio for particular and general cases. For the particular case, we considered the situation when the mean interest rate of some stocks does not depend on other stocks in the market. That is, an investor has invested his money in a stock with a linear mean return. Under the general case, we considered a market comprising some units of assets in long position and a unit of the option in short position. Using Ito's formula on the present value of the market, we derived the growth rate of investor's portfolio. Our model equations, which are based on multiplicative processes, capture all the features of the returns. They are tested using data from Zenith Bank of Nigeria stock prices. From our graphs, the worth of investment grows as stock price increases and also decreases with stock price.

Keywords

Multi-fractal Spectrum Model, Worth Growth Rate, Investor's Portfolio, Zenith Bank of Nigeria, Stock Prices

1. Introduction

Financial data has been argued to possess long-term correlation in returns that manifests as volatility clustering, as well as dramatic day-to-day swings that depart notably from normality. Typical signals generated by financial systems are non-trivial structures which can be characterized in terms of the theory of multi-fractals. Interestingly, these structures are to some degree universal in real world, since they come not only from finance but also from diverse fields of science like physics, chemistry or biology. The concept of fractal world was proposed by Mandelbrot in 1980's and was based on

scale-invariant statistics with power law correlations [1]. In subsequent years this new theory was developed and finally it brought a more general concept of multi-scaling. It allows one to study the global and local behavior of the multi-fractal properties of a system. In finance, multi-fractality is one of the well known facts which characterized non-trivial properties of financial time series. The stock price (or index) fluctuations can be described in terms of long-range temporal correlations by a spectrum of the Holder exponents and a set of fractal dimensions.

*Corresponding author: ji16adindudick@yahoo.com (Joy Ijeoma Adindu-Dick)

Received: 30 October 2025; **Accepted:** 13 November 2025; **Published:** 17 December 2025



Copyright: © The Author(s), 2025. Published by Science Publishing Group. This is an **Open Access** article, distributed under the terms of the Creative Commons Attribution 4.0 License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution and reproduction in any medium, provided the original work is properly cited.

Assessing the stock price indices is the foundation of forecasting the market risk. Multi-fractal has lots of advantages in explaining the volatility of the stock prices. Multi-fractal analysis has the ability to illuminate the underlying difference between a mono-fractal like Brownian motion and a considerably more diverse construction like the multi-fractal binomial measure. Thus, combining multi-fractal analysis with an agent-based model has the potential to provide insight into the underlying processes and behaviors of a dynamic economic system that is constantly changing and evolving. Multi-fractal processes have been proposed as a new formalism for modeling the time series of returns in finance. The major attraction of these processes is their capability of generating various degrees of long-memory in different powers of returns – a feature that has been found to characterize virtually all financial prices. Furthermore, elementary variants of multi-fractal models are very parsimonious formalizations as they are essentially one-parameter families of stochastic processes. However, the major attraction of multi-fractal processes is their capability of matching elementary properties of the conditional distribution, i.e. long-term dependence in various powers.

Multi-fractality introduces a new source of heterogeneity through time-varying local regularity in the price path. The concept of local Holder exponent describes local regularity. Multi-fractal processes bridge the gap between locally Gaussian (Ito) diffusions and jump-diffusions by allowing a multiplicity of Holder exponents. [2] developed a portfolio model based on mean multifractal detrended cross-correlation analysis (MF-DCCA).

The multi-fractal Model of assets returns (MMAR) is one of the methods of modeling financial prices, [3-5]. The MMAR is thus an alternative to autoregressive conditionally heteroscedastic (ARCH) and its variants. Multi-fractal processes incorporate long-tailed asset returns and long memory in volatility. Additionally, the MMAR predicts a form of scaling in the moments of returns.

Current finance literature notes a discrepancy between financial theory, which is largely set in continuous time, and empirical research that tends towards discrete formulation [6]; [7, 8]. This discrepancy is sometimes viewed as a gap to be remedied by improvements in empirical models. Multi-fractality offers a different perspective, showing that a large class of interesting stochastic processes are not covered by the Ito diffusion paradigm. The MMAR contains a restriction, scaling, that generates empirical tests. Scaling describes a specific relationship between data samples of different time scales, e.g. daily, weekly, or monthly returns. Multi-fractal Spectrum Model (MSM) is a stochastic volatility model [9, 10], with arbitrarily many frequencies. It builds on the convenience of regime-switching models, which were advanced in economics and finance [11]. MSM is closely related to the multi-fractal model of Asset Returns (MMAR) [5]. It improves on the MMAR's combinatorial construction by randomizing arrival times, guaranteeing a strictly stationary

process. Our model provides a pure regime-switching formulation of multi-fractal measures, which were pioneered by Benoit Mandelbrot [12-14].

In this work, we present a dynamic stochastic multi-fractal spectrum model (MSM) of variation of the capital market price. We first derive the worth growth rate of an investor's portfolio for the particular and general cases, then, use the data from stock prices of Zenith Bank, Nigeria to analyze the derived model equations.

2. Basic Tools and Preliminaries

Let $(P_t)_{t \geq 0}$ denote the price process of a security, in particular of a stock. To allow comparison of investments in different securities, it is necessary to investigate the rates of return defined by

$$Z_t = \ln P_t - \ln P_{t-1}. \quad (1)$$

As a model for stock prices, the natural candidate is now the multi-fractal process $(X_t^{d,n})_{t \geq 0}$ with the stochastic differential equation [15].

$$dY_t = PY_t dt + Y_t dZ_t^{d,n}. \quad (2)$$

The MSM can be specified in both discrete time and continuous time.

2.1. Discrete Time

Let P_t denote the price of a financial asset, and let

$$z_t = \ln(P_t/P_{t-1}) \quad (3)$$

denote the returns over two consecutive periods. In MSM, returns are specified as

$$r_t = u + \bar{\sigma}(M_{1t}M_{2t} \dots M_{kt})^{1/2} \xi_t, \quad (4)$$

where u and σ are constants and $\{\xi_t\}$ are independent standard Gaussians. Volatility is driven by the first-order latent Markov state vector:

$$M_t = (M_{1t}M_{2t} \dots M_{kt}) \in R_+^k. \quad (5)$$

Given the volatility state, M_t , the next-period multiplier $M_{k(t+1)}$ is drawn from a fixed distribution $\bar{M}$, with probability, γ_k and is otherwise left unchanged. M_{kt} drawn from distribution, $\bar{M}$, with probability γ_k , since the multiplicative measures defined at different stages are drawn from the same distribution. With probability, $1 - \gamma_k$,

$$M_{kt} = M_{k(t-1)} \quad (6)$$

since $\gamma_k + (1 - \gamma_k) = 1$ gives the total probability.

The transition probability is specified by [16]

$$\gamma_k = 1 - (1 - \gamma_1)^{(b^{k-1})}. \quad (7)$$

The sequence γ_k is approximately geometric with $\gamma_k \approx \gamma_1^{b^{k-1}}$ at low frequency. The marginal distribution M has a unit mean, positive support, and is independent of K . In empirical applications, the distribution, M , is often a discrete distribution that can take the values m_0 or $2 - m_0$ with equal probability. The return process, r_t is then specified by the parameters $\theta = \theta(m_0, u, \bar{\sigma}, b, \gamma_1)$.

2.2. Continuous Time

MSM is similarly defined in continuous time. The price process follows the diffusion:

$$\frac{dP_t}{P_t} = u dt + \sigma(M_t) dW_t, \quad (8)$$

where $\sigma(M_t) = \bar{\sigma}(M_{1t} \dots M_{kt})^{1/2}$, W_t is a standard Brownian motion, u and $\bar{\sigma}$ are constants. Each component follows the dynamics: M_{kt} drawn from distribution, M with probability, $\gamma_k dt$. $M_{k(t+dt)} = M_{kt}$ with probability $1 - \gamma_k dt$. The intensities vary geometrically with K : $\gamma_k = \gamma_1^{b^{k-1}}$. When the number of components, $\bar{k}$ goes to infinity, continuous-time MSM converges to a multi-fractal diffusion, whose sample paths take a continuum of local Holder exponents on any finite time interval.

Conditional on the volatility state, the return r_t has Gaussian density.

$$f(r_t | M_t = m^i) = \frac{1}{\sqrt{2\pi\sigma^2(m^i)}} \exp\left[-\frac{(r_t - u)^2}{2\sigma^2(m^i)}\right]. \quad (9)$$

Furthermore, $M_t = M_{1t}, M_{2t}, \dots, M_{kt}$ is the multiplicative measure defined at different stages. The product of k multiplier is defined by [17];

$$\mu(\Delta t) = M_{1t} M_{2t} \dots M_{kt}, \quad (10)$$

with the scaling relationship $E[(\mu(\Delta t))^q] = [E(M)^q]$ or

$$bP - W = b(uP\Delta t + \sigma P\Delta z + D_1\Delta t) - \left\{ \left(\frac{\partial W}{\partial t} + \frac{\partial W}{\partial P} uP + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2 P^2 \right) \Delta t + \frac{\partial W}{\partial P} \sigma P \Delta z \right\}.$$

Collecting like terms, we have

$$bP - W = \left\{ (buP + bD_1) - \left(\frac{\partial W}{\partial t} + \frac{\partial W}{\partial P} uP + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2 P^2 \right) \right\} \Delta t + (b\sigma P - \frac{\partial W}{\partial P} \sigma P) \Delta z$$

If we take

$$b = \frac{\partial W}{\partial P}, \quad (19)$$

$$[E(M)^q] = (\Delta t)^{\tau(q)+1}, \quad (11)$$

where $\tau(q) = -\log_b E(M)^q - 1$ [17].

If the latest observation P_t of the designated process at time $t > d$ is conditioned on the information;

$$\varphi_{t-1} = \{P_u - u_u : u = 1, 2, \dots, t-1\} \quad (12)$$

available up to time $t-1$, then

$$P_t | \varphi_{t-1} \sim (u_t, \sigma^2(M_t)) \quad (13)$$

where (using equation (7)),

$$\sigma^2(M_t) = (\Delta t)^{\tau(q)+1} h(\xi_{t-1}, \xi_{t-2}, \dots, \xi_{t-d}, \alpha) \quad (14)$$

and

$$\xi_u = P_u - u_u, u = 1, 2, \dots, t-1. \quad (15)$$

The function $b(\cdot)$ with parameters $\alpha = \{\alpha_k : k = 0, 1, \dots, d\}$ is defined as

$$b(\xi_{t-1}, \xi_{t-2}, \dots, \xi_{t-d}, \alpha) = \alpha_0 + \sum_{k=1}^d \alpha_k \xi_{t-k}^2, \quad (16)$$

so that given the history as in equation (14) of the process up to time $t-1$, the conditional distribution of P_t is normal with mean u_t and variance

$$\sigma^2(M_t) = (\Delta t)^{\tau(q)+1} (\alpha_0 + \sum_{k=1}^d \alpha_k (P_{t-k} - u_{t-k})^2). \quad (17)$$

3. The Model

Let a portfolio comprise of one unit of option in short position and b unit of the assets in long position. At time, T the value of the portfolio is

$$bP - W, \quad (18)$$

measured by the fractal index $C^\phi(E) - W^\phi(E) \neq 0$.

After an elapse of time, Δt , the value of the portfolio will change by the rate $b(\Delta P + D_1\Delta t) - \Delta W$ in view of the dividend received on b units held. By Ito's lemma we have

the uncertainty term disappears, thus the portfolio in this case is temporarily riskless. It should therefore grow in value by the riskless rate in force, hence, we have

$$bP - W = \left\{ (buP + bD_1) - \left(\frac{\partial W}{\partial t} + \frac{\partial W}{\partial P} uP + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2 P^2 \right) \right\} \Delta t + \left(\frac{\partial W}{\partial P} \sigma P - \frac{\partial W}{\partial P} \sigma P \right) \Delta z.$$

That is,

$$bP - W = \left\{ (buP + bD_1) - \left(\frac{\partial W}{\partial t} + \frac{\partial W}{\partial P} uP + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2 P^2 \right) \right\} \Delta t.$$

Since the portfolio in this case is temporarily riskless, it should therefore grow in value by the riskless rate in force. Also, the change in an instantaneous risk free portfolio should equal the exponential growth of placing money in the bank, we have

$$\left\{ (buP + bD_1) - \left(\frac{\partial W}{\partial t} + \frac{\partial W}{\partial P} uP + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2 P^2 \right) \right\} \Delta t = (bP - W)r\Delta t.$$

Thus

$$D_1 \frac{\partial W}{\partial P} - \left(\frac{\partial W}{\partial t} + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2 P^2 \right) = \left(\frac{\partial W}{\partial P} P - W \right) r$$

So

$$\frac{\partial W}{\partial t} + (rP - D_1) \frac{\partial W}{\partial P} + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2 P^2 = rW. \quad (20)$$

Let $D_1 = 0$ (where D_1 is the market price of risk), then the solution of equation (20), which coincides with the solution of

$$\frac{\partial W}{\partial t} + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2 P^2 = 0 \quad (21)$$

with

$$W(P, t) = 0, \quad (22)$$

$$\frac{\partial W(P, t)}{\partial P} = 0 \quad \forall t, \quad (23)$$

and P^2 is assumed constant, is given by

$$W(P, t) = W_0 \exp \left\{ \frac{-2\alpha P^{-2}}{\sigma^2} + \lambda P \right\} e^{rt}$$

with [18]

$$\lambda + \frac{4\alpha P^{-3}t}{\sigma^2} = 0. \quad (24)$$

Where W is the investment output, r the discount rate, α is the fractal exponent and σ^2 the variance of the stock market price.

If we assume P^2 to be non-constant, the solution becomes

$$W(P, t) = W_0 e^{-\frac{1}{2}\sigma^2\alpha^2 t + rt} \{A P^{\lambda_1} + B P^{\lambda_2}\} \quad (25)$$

with

$$A\lambda_1 P^{\lambda_1-1} + B\lambda_2 P^{\lambda_2-1} = 0 \quad [18]. \quad (26)$$

Where A and B are arbitrary constants.

For $D_1 \neq 0$, the solution of equation (20) is given as:

$$W(P) = \left(\frac{aq_n^2}{2P} \right)^\beta \left\{ A e^{\lambda_1 \frac{aq_n^2}{2P}} + B e^{\lambda_2 \frac{aq_n^2}{2P}} \right\}, \quad (27)$$

where [17]

$$\lambda_1 = -\frac{2}{z} \pm \sqrt{\frac{4}{z^2} + \frac{8r}{z^2\sigma^2}} \text{ and } \lambda_2 = \pm \frac{1}{z} \sqrt{4 + \frac{8r}{\sigma^2}} \quad (28)$$

3.1. The Particular Case

When the mean interest rate of some stocks does not depend on other stocks in the market, we consider the stochastic differential equation (SDE);

$$dP_t = P_t(udt + \sigma(M_t)dV_t), P_0 = \varepsilon_0. \quad (29)$$

The most probable path $\varphi(t)$ associated with this equation satisfies

$$\varphi'(t) = \left(u - \frac{1}{2} \sigma^2(M_t) \right) \varphi, \varphi(0) = \varepsilon. \quad (30)$$

Equation (30) means that an investor has invested his money in a stock with a linear mean return u_t and volatility $\sigma(M_t)$ and his real return rate is most likely to be given by

$$c(t) = u - \frac{1}{2} \sigma^2(M_t), \quad (31)$$

instead of the usual u .

For $t > 0$ and $b(t) = 2t \log \log \frac{1}{t}$, the singularity spectrum $D(\alpha)$ defined as the Hausdorff dimension of the set where the fractal exponent is equal to α is given by

$$\lim_{t \rightarrow 0} \sup \frac{x_t^d}{b(t)} = \frac{(1+\gamma)^2}{2\theta}, \theta = 1, \gamma \in [0, 1]. \quad (32)$$

Using Ito's Formula on equation (32) leads to the model

equation.

$$-\frac{\partial V}{\partial t} = uP \frac{\partial V}{\partial P} + \frac{1}{2} \sigma^2(M_t) P^2 \frac{\partial^2 V}{\partial P^2} - rV, t \geq t_0, P_0 = \varepsilon_0, \quad (33)$$

where $V = V(P_t, K_t)$ is the investment worth and K_t is the investment over period t . Let the rate of change of worth, V , with respect to time be given as

$$\frac{\partial V(P_t, K_t)}{\partial t} = S_2(P_t, K_t) - C_1(P_t, K_t), \quad (34)$$

where S_2 is the production rate, C_1 , the consumption rate. For a relatively short time, the rate of consumption is very small so that $C_1(P_t, K_t) \rightarrow 0$ as $t \rightarrow 0$. We can then write equation (34) as

$$\frac{\partial V}{\partial t} = S_2(P_t, K_t). \quad (35)$$

Since the rate of change of the worth depends on the investment output at present time, t , we write equation (35) as

$$\frac{dV}{dt} = S_2(K_t) = P_t.$$

Since the differentiation of V on the right hand side of equation (33) is with respect to P (and not K), we can denote $V(P, K)$ by $V(P)$. Hence, we have equation (33) reduced to an ordinary differential equation of the form.

$$uP \frac{dV}{dP} + \frac{1}{2} \sigma^2(M_t) P^2 \frac{d^2 V}{dP^2} - rV = -P.$$

That is,

$$\frac{1}{2} \sigma^2(M_t) P^2 \frac{d^2 V}{dP^2} + uP \frac{dV}{dP} - rV = -P. \quad (36)$$

We then solve equation (36) using Euler's substitution method.

Let $P = e^t$, then

$$\ln P = t, \text{ and } \frac{dt}{dP} = \frac{1}{P}.$$

Also

$$\frac{dV}{dP} = \frac{dV}{dt} \frac{dt}{dP} = \frac{1}{P} \frac{dV}{dt},$$

$$\frac{d^2 V}{dP^2} = \frac{d}{dP} \left(\frac{1}{P} \frac{dV}{dt} \right) = \frac{1}{P} \frac{d}{dP} \left(\frac{dV}{dt} \right) + \frac{dV}{dt} \frac{d}{dP} \left(\frac{1}{P} \right),$$

$$\text{And } \frac{d^2 V}{dP^2} = \frac{1}{P^2} \left(\frac{d^2 V}{dt^2} - \frac{dV}{dt} \right).$$

Substituting accordingly in equation (36) gives

$$\frac{1}{2} \sigma^2(M_t) \left(\frac{d^2 V}{dt^2} - \frac{dV}{dt} \right) + u \frac{dV}{dt} - rV = -e^t. \quad (37)$$

For the homogeneous part, let $V = e^{\lambda t}$ be the solution of equation (37). Therefore

$$\frac{1}{2} \sigma^2(M_t) \lambda^2 e^{\lambda t} - \frac{1}{2} \sigma^2(M_t) \lambda e^{\lambda t} + u \lambda e^{\lambda t} - r e^{\lambda t} = 0$$

Simplifying and collecting like terms, we have

$$\frac{1}{2} \sigma^2(M_t) \lambda^2 + \left(u - \frac{1}{2} \sigma^2(M_t) \right) \lambda - r = 0 \quad (38)$$

Solving equation (38) which is quadratic in λ gives

$$\lambda_1 = \frac{-\left(u - \frac{1}{2} \sigma^2(M_t)\right) + \sqrt{\left(u - \frac{1}{2} \sigma^2(M_t)\right)^2 + 2 \sigma^2(M_t) r}}{\sigma^2(M_t)} \quad (39)$$

which is the positive characteristics root of the equation. Our complementary function becomes.

$V_c = A e^{\lambda_1 t}$. But $t = \ln P$, hence

$$V_c = A e^{\lambda_1 \ln P} = A e^{\ln P^{\lambda_1}} = A P^{\lambda_1} \quad (40)$$

We now obtain the particular integral using the method of undetermined coefficients using equation (37).

Let $V_p = A e^t$, then $V_p' = A e^t$ and $V_p'' = A e^t$.

Substituting in equation (37) gives

$$\frac{1}{2} \sigma^2(M_t) A e^t - \frac{1}{2} \sigma^2(M_t) A e^t + u A e^t - r A e^t = -e^t.$$

Simplifying we have

$$(uA - rA) e^t = -e^t.$$

Equating coefficient of like terms, we have

$$uA - rA = -1.$$

Solving for A in the above equation gives

$$A = \frac{1}{r-u}.$$

But $V_p = A e^t$, substituting the value of A , we have

$$V_p = \frac{e^t}{r-u}.$$

But $t = \ln P$, hence

$$V_p = \frac{e^{\ln P}}{r-u} = \frac{P}{r-u}. \quad (41)$$

But the solution of equation (36) is $V = V_c + V_p$. Hence, from equations (40) and (41), the solution of equation (36) is:

$$V(P) = A P^{\lambda_1} + \frac{P}{r-u}, \quad (42)$$

with

$$A\lambda_1 \hat{P}^{\lambda_1} + \frac{\hat{P}}{r-u} = 0, \quad (43)$$

where

$$\lambda_1 = \frac{-\left(u - \frac{1}{2}\sigma^2(M_t)\right) + \sqrt{\left(u - \frac{1}{2}\sigma^2(M_t)\right)^2 + 2\sigma^2(M_t)r}}{\sigma^2(M_t)}, \quad (44)$$

is the positive characteristics root of equation (36).

We have assumed here that $V(P)$ is twice differentiable such that

$$V(0) = 0 \text{ and } \frac{dV(P)}{dP} = 0. \quad (45)$$

Using equations (4), (11), (15) and (17), we have equation (42) as

$$V(P) = AP^{\lambda_1} + \frac{(\Delta t)^{-(\tau(q)+1)}P}{(\alpha_0 + \sum_{k=1}^d \alpha_{k+1}(P_{t-k} - u_{t-k})^2)}, \quad (46)$$

with using equation (45)

$$A\lambda_1 \hat{P}^{\lambda_1} - \frac{u(\Delta t)^{-(\tau(q)+1)}}{\sigma(\hat{P}_u - u_u)^2} = 0. \quad (47)$$

Equation (46) is the worth growth rate of an investor under MSM.

3.2. The General Case

Generally, markets are neither ideal nor complete in the real world. Therefore, equation (33) can hardly be seen as real world behaviour of a stock market price.

Under the following dynamics

$$dP_t = \alpha(t)P_t dt + \sigma(M_t)P_t dV(t),$$

Furthermore,

$$\frac{\partial \bar{V}(\bar{P}, t)}{\partial \bar{P}} = 0, \forall t, \quad (54)$$

with

$$\left(u - \frac{1}{2}\sigma^2(M_t)\right) + \sqrt{\left(u - \frac{1}{2}\sigma^2(M_t)\right)^2 + 2\sigma^2(M_t)r} + \frac{4 \ln\left(\frac{P_{t+\Delta t} - P_t}{\Delta t}\right)}{\bar{P}^3(\Delta t)^{\tau(q)+1}h(\xi_{t-1}, \xi_{t-2}, \dots, \xi_{t-d}, \alpha)\bar{P}_u - u_u} = 0. \quad (55)$$

we have the MSM version of the parabolic partial differential equation with $D_1 = 0$ as;

$$-\frac{\partial V(P, t)}{\partial t} = rP \frac{\partial V(P, t)}{\partial P} + \frac{1}{2}\sigma^2(M_t)P^2 \frac{\partial^2 V(P, t)}{\partial P^2} - rV(P, t),$$

$$\forall (P, t) \in (0, \infty) \times (0, T), \quad (48)$$

where $\alpha(t) = \ln\left(\frac{P_{t+\Delta t} - P_t}{\Delta t}\right)$ is the rate of stock price changes at time t .

To remove the effect of r from equation (48), we let $r = 0$, and set

$$\bar{V} = e^{-rt}V \quad (49)$$

and

$$\bar{P} = e^{-rt}P, \quad (50)$$

so that equation (48) becomes;

$$-\frac{\partial \bar{V}(\bar{P}, t)}{\partial t} = (0)\bar{P} \frac{\partial \bar{V}(\bar{P}, t)}{\partial \bar{P}} + \frac{1}{2}\sigma^2(M_t)\bar{P}^2 \frac{\partial^2 \bar{V}(\bar{P}, t)}{\partial \bar{P}^2} - (0)\bar{V}(\bar{P}, t).$$

$$-\frac{\partial \bar{V}(\bar{P}, t)}{\partial t} = \frac{1}{2}\sigma^2(M_t)\bar{P}^2 \frac{\partial^2 \bar{V}(\bar{P}, t)}{\partial \bar{P}^2},$$

where $V(P, t)$ is the investment output, r the linear discount rate and P the stock prices.

Hence, we have

$$\frac{1}{2}\sigma^2(M_t)\bar{P}^2 \frac{\partial^2 \bar{V}(\bar{P}, t)}{\partial \bar{P}^2} + \frac{\partial \bar{V}(\bar{P}, t)}{\partial t} = 0. \quad (51)$$

From equation (21) with $\sigma^2 = \sigma^2(M_t)$, we have a special solution of the form:

$$\bar{V} = \bar{V}_0 \exp\left\{\frac{-2\alpha t \bar{P}^{-2}}{\sigma^2(M_t)} + \lambda \bar{P}\right\}. \quad (52)$$

Using equations (4), (14) and (15) into equation (52), we have

$$\bar{V}(\bar{P}, t) = \bar{V}(0) \exp\left[-\frac{\left(2 \ln\left(\frac{P_{t+\Delta t} - P_t}{\Delta t}\right)\bar{P}^{-2} + \sqrt{\left(u - \frac{1}{2}\sigma^2(M_t)\right)^2 + 2\sigma^2(M_t)r}\right)}{(\Delta t)^{\tau(q)+1}(\alpha_0 + \sum_{k=1}^d \alpha_k \xi_{t-k}^2)}\right]. \quad (53)$$

Again, solving for $\bar{P}$ in equation (55) gives (using equation (53) and if λ is as in equation (44));

$$\bar{P} = \left[\frac{4 \ln(P_t - P_{t+\Delta t})(\Delta t)^{-\tau(q)}}{\left(\left(u - \frac{1}{2} \sigma^2(M_t) \right) + \sqrt{\left(u - \frac{1}{2} \sigma^2(M_t) \right)^2 + 2 \sigma^2(M_t) r} \right) (\alpha_0 + \sum_{k=1}^d \alpha_k \xi_{t-k}^2)} \right]^{1/3}. \quad (56)$$

Substituting equation (56) in equation (50) using equations (4) and (44) gives

$$P = \left[\frac{4 \ln(P_t - P_{t+\Delta t})(\Delta t)^{-\tau(q)}}{\lambda (\alpha_0 + \sum_{k=1}^d \alpha_k \xi_{t-k}^2)} \right]^{1/3} e^{(u + \sigma(M_t) \xi_t)}$$

$$= \left[\frac{4 \ln(P_t - P_{t+\Delta t})(\Delta t)^{-\tau(q)}}{\lambda (\alpha_0 + \sum_{k=1}^d \alpha_k \xi_{t-k}^2)} \right]^{1/3} \exp \{ u + (\Delta t)^{-(\tau(q)+1)} (\alpha_0 + \sum_{k=1}^d \alpha_k \xi_{t-k}) P_t - u_t \},$$

(using equations (4), (14) and (15)). We therefore have the future price as

$$P_t = \left[\frac{4 \ln(P_t - P_{t+\Delta t})(\Delta t)^{-\tau(q)}}{\lambda (\alpha_0 + \sum_{k=1}^d \alpha_k \xi_{t-k}^2)} \right]^{1/3} \exp \{ u + (\Delta t)^{-(\tau(q)+1)} (\alpha_0 + \sum_{k=0}^d \alpha_{k+1} (P_t - u_t)^2) \}. \quad (57)$$

In general, we have the growth rate of the investor's portfolio as (using equations (49) and (53))

$$V = \bar{V}(0) \exp \left[- \left\{ \frac{2 \ln \left(\frac{P_t + \Delta t - P_t}{\Delta t} \right) \bar{P}^{-2} + \sqrt{\left(u - \frac{1}{2} \sigma^2(M_t) \right)^2 + 2 \sigma^2(M_t) r}}{(\Delta t)^{\tau(q)+1} (\alpha_0 + \sum_{k=1}^d \alpha_k \xi_{t-k}^2)} \right\} \right] \exp \left\{ \left(u + \bar{\sigma}(M_{1t} M_{2t} \dots M_{kt})^{1/2} \xi_t \right) t \right\} \quad (58)$$

growth rate of investor's portfolio for the general case, equation (53) are tested using data from Zenith Bank of Nigeria stock prices. The data below in Table 1 are stock prices of Zenith Bank, Nigeria between January and August.

4. Testing Our Model Equations

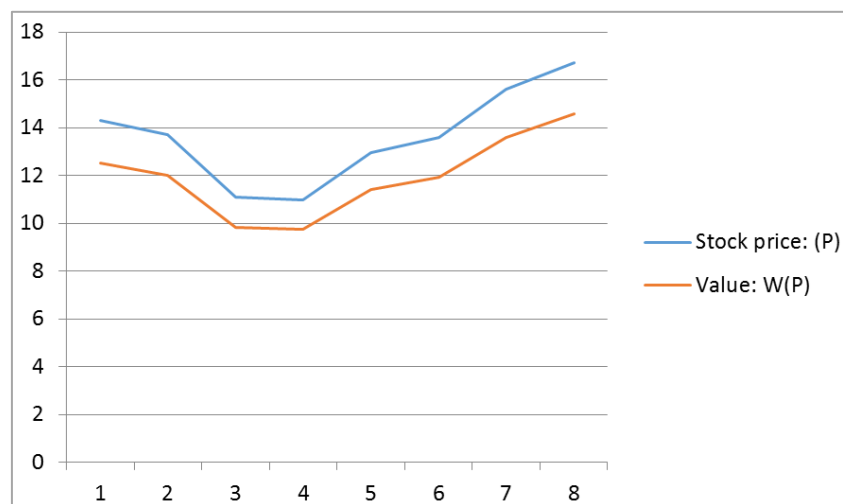
Our model equations, worth growth rate of investor's portfolio for the particular case, equation (46) and worth

Table 1. Stock of Zenith bank Nigeria.

S/No	Stock Price	Volume	Value (₦)
1	14.30	5979214	84469072.19
2	13.69	21971420	297229102.30
3	11.08	21263057	234828711.10
4	10.98	12262680	133454007.40
5	12.97	19737082	255511920.50
6	13.60	56728977	770381004.30
7	15.60	4308491	67613293.42
8	16.71	15512358	259157916.10

Table 2. Worth growth rate for the particular case.

Stock price: (P)	Value: V (P)
14.3	12.52
13.69	12.01
11.08	9.82
10.98	9.74
12.97	11.41
13.6	11.94
15.6	13.6
16.71	14.56

**Figure 1.** Our worth growth rate for the particular case.

From Figure 1, the worth of investment grows as stock price increases and also decreases with stock price. In other words, the worth and the stock price move in the same direction.

Table 3. Worth growth rate for the general case.

Stock price: ($\bar{P}$)	Value: $\bar{V}(\bar{P}, t)$
14.3	0.55
13.69	0.917
11.08	0.939
10.98	0.902
12.97	0.931
13.6	0.976
15.6	0.64
16.71	0.9

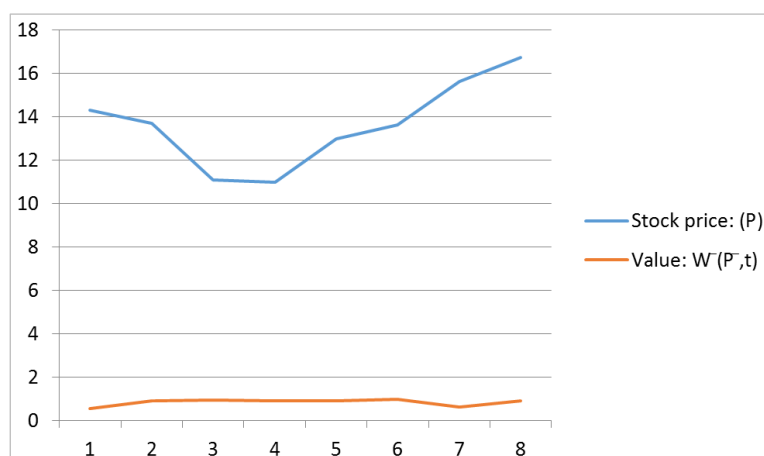


Figure 2. Our worth growth rate for the general case.

Figure 2 shows that the worth of investment increases as stock price increases but not in the same direction.

5. Conclusion

This work used some tools of multi-fractal analysis to derive the worth growth rate of an investor's portfolio for particular and general cases. Multi-fractal analysis allows one to predict, as well as possible, the future volatility of an asset over a certain time horizon, or even better, to predict that probability distribution of all possible price paths between now and a certain future date. It also provides a well-defined procedure to filter the series of past price changes, and to compute the weights of the different future paths.

Figure 1 shows that the worth of investment grows as stock price increases and also decreases with stock price. This means that the worth and the stock price move in the same direction. On the other hand, Figure 2 shows that the worth of investment increases as stock price increases but not in the same direction. Our model equations, which were based on multiplicative processes, captured all the features of the returns, hence, from our graphs the worth of investment grows as stock price increases and also decreases with stock price. For the particular cases, the worth and stock price move at the same rate but for the general case, the worth picks up gradually as stock price increases. Hence, our models can be used in the measurement of random behaviour of equity returns.

Multi-fractal spectrum model captures a multitude of important features of data. These include long memory in volatility, long tails, scaling, and high variability in returns at high frequencies.

Abbreviations

ARCH	AutoRegressive Conditionally Heteroscedastic
MF-DCCA	Multi-Fractal Detrended Cross-Correlation Analysis

MMAR	Multi-Fractal Model of Assets Returns
MSM	Multi-fractal Spectrum Model
SDE	Stochastic Differential Equation

Author Contributions

Joy Ijeoma Adindu-Dick is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflicts of interest.

References

- [1] Mandelbrot, B. B., (1982). *The fractal geometry of nature*. New York: Freeman.
- [2] Jia, L., Xu, W., Linliu, Z., & Qianying, F. (2021). Research on the Portfolio Model based on Mean-MF-DCCA under Multifractal Feature Constraint. *Journal of Computational and Applied Mathematics*, 386, 113264. <https://doi.org/10.1016/j.cam.2020.113264>
- [3] Calvet, L., Fisher, A., & Mandelbrot, B. B. (1997). A multifractal model of asset returns. *Working Paper*: Yale University.
- [4] Mandelbrot, B. B., (1997). *Fractals and scaling in finance: Discontinuity, Concentration, Risk* New York: Springer-Verlag.
- [5] Mandelbrot, B. B., Fisher, A., & Calvet, L., (1997). A multifractal model of asset returns. *Discussion Paper*, Cowles Foundation Yale University. 1164-1166.
- [6] Melino, A., (1994). Estimation of continuous-time models in finance, in advances in econometrics: *Sixth World Congress*. Cambridge. Cambridge University Press.
- [7] Drost, F. C., & Werker, B. J. (1996). Closing the GARCH gap: Continuous time GARCH modeling. *Journal of Econometrics*, 74, 59-84. [https://doi.org/10.1016/0304-4076\(95\)01755-8](https://doi.org/10.1016/0304-4076(95)01755-8)

- [8] Rossi, P. E., (1997). *Modeling stock market volatility: Bridging the gap to continuous time*. New York: Academic Press.
- [9] Wiggins, J. B., (1987). Option values under stochastic volatility: Theory and empirical estimates. *Journal of Financial Economics*, 19 (2), 351-372.
[https://doi.org/10.1016/0304-405X\(87\)90009-2](https://doi.org/10.1016/0304-405X(87)90009-2)
- [10] Taylor, S. J., & Steven, (2008). *Modeling financial time series* (2nd ed.). New Jersey: World Scientist.
- [11] Hamilton, J. D., (2008). *Regime-Switching models: New Palgrave Dictionary of Economics* (2nd ed.). Palgrave: McMillan Ltd.
- [12] Mandelbrot, B. B., (1983). *The fractal geometry of nature* (Updated and augm. ed.). New York: Freeman.
- [13] Mandelbrot, B. B., (2006). Intermittent turbulence in self-similar cascades: Divergence of high moments and dimension of the carrier. *Journal of Fluid Mechanics*, 62 (2), 331-358. <https://doi.org/10.1017/S0022112074000711>
- [14] Mandelbrot, B. B., & Berger, J. M., (1999). *Multifractals and 1/f noise: wild self-affinity in physics* (1963-1976). (Repr. ed.). New York, NY [u. a.]: Springer.
- [15] Eberlein, E., & Keller, U. (1995). Hyperbolic distributions in finance. *Bernoulli*, 1(3), 281-299.
<https://doi.org/10.3150/bj/1193667819>
- [16] Calvet, L. E., & Fisher, A. J. (2004). How to forecast long-run volatility: Regime switching and the estimation of multifractal processes. *Journal of Financial Econometrics*, 2, 49-83.
<https://doi.org/10.1093/jfec/2.1.49>
- [17] Calvet, L. E., & Fisher, A. J. (1996). Multifractality in assets returns: Theory and evidence. *Working paper*: Yale University.
- [18] Osu, B. O. & Adindu-Dick J. I. (2014). Optimal prediction of the expected value of assets under fractal scaling exponent. *Applied Mathematics and Sciences: An International Journal (MathSJ.)*, Vol. 1, No 3, 41-51.