American Journal of Applied Mathematics
2025; 13(6): 412-418
http://www.sciencepublishinggroup.com/j/ajam
doi: 10.11648/j.ajam.20251306.13

Homological Functor $\tilde{H}_n : \mathrm{Comp}(\mathscr{A}) \longrightarrow Ab$ Where $\mathscr{A}$ Is an Abelian Category and n is an Integer in $\mathbb{Z}$

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To cite this article:
Ablaye Diallo. (2025). Homological Functor $\tilde{H}_n : \mathrm{Comp}(\mathscr{A}) \longrightarrow Ab$ Where $\mathscr{A}$ Is an Abelian Category and n is an Integer in $\mathbb{Z}$.
American Journal of Applied Mathematics, 13(6), 412-418. https://doi.org/10.11648/j.ajam.20251306.13

Received: 10 October 2025; **Accepted:** 27 October 2025; **Published:** 6 December 2025

Abstract: The study of the homological functor has been carried out in particular cases of abelian categories, notably in the category of left A-modules (resp. right A-modules) and Categories of graded A-modules where A is a ring. We call category of complexes of an abelian category $\mathscr{A}$ the category denoted by $\mathrm{Comp}(\mathscr{A})$ whose objects are the complexe sequence of $\mathscr{A}$ and morphisms are the chain maps of $\mathscr{A}$. In this paper, we consider the functor $\tilde{H}_n$ define by the composition of functors $H_n \circ \mathrm{Hom}_{\mathrm{Comp}}(\mathscr{A})(X,-)$, where $\mathrm{Hom}_{\mathrm{Comp}}(\mathscr{A})(X,-)$ is define by $\mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(\mathrm{Ab})$, the functor H_n is define by $\mathrm{Comp}(\mathrm{Ab}) \to \mathrm{Ab}$ and X is a projective object of $\mathscr{A}$. In this article, we study how the homological functor $\tilde{H}_n$ define by $\mathrm{Comp}(\mathscr{A}) \longrightarrow Ab$ where $\mathscr{A}$ is an abelian category and n is an integer in $\mathbb{Z}$ transforms a short exact sequence into a long exact sequence. The main results of this paper are: the construction of the connecting morphism ϕ_n associated to the covariant functor $\tilde{H}_n$, and show that $\tilde{H}_n$ transforms all short exact sequence of morphisms in $\mathrm{Comp}(\mathscr{A})$ into a long exact sequence of morphisms in Ab, where X is an projectif object in $\mathscr{A}$ and $\mathscr{A}$ is an abelian category.

Keywords: Abelian Category, Homological Functor, Category of Abelian Groups

1. Introduction

The main objective of this paper is to study the exactness of the homological functor in an arbitrary abelian category $\mathscr{A}$ by developing certain concepts in homological algebra. We are inspired by the homological functor $H_n : \mathrm{Comp}(\text{A-Mod}) \longrightarrow \mathrm{Ab}$ in the category of left A-modules A-Mod (resp. right A-modules Mod-A) and complexes of left A-modules Comp(A-Mod) (resp. complexes of right A-modules Comp(Mod-A)). Some authors have already worked on additive categories, abelian categories and the homological functor: Pierre Gabriel [1], Charles A Weibel [2], Peter J Freyd [3], Joseph J Rotman [4], [5], Aurelien Djamen and Antoine Touze [6], Arij Benkhadra [7], Alex Heller [8], Carol Peercy Walker [9], Manfred Hartl and Bruno Loiseau [10], Deren Luo [11], Lidia Angeleri Hugel [12], Sebastian Posur [13], Aurelien Djament and Antoine Touze [14], Xi Tang and Zhao Yong Huang [15]. Thus, the paper is structured as follows:

In *Section 2.1*, entitled Preliminaries, we define the basic concepts in an arbitrary abelian category.

In *Section 3*, we show that:

Let

$$(0) \longrightarrow (Y,\alpha) \xrightarrow{f} (Z,\beta) \xrightarrow{g} (T,\theta) \longrightarrow (0)$$

be a short exact sequence of morphisms in $\mathrm{Comp}(\mathscr{A})$, where $\mathscr{A}$ is an abelian category. Then:

1. The morphism called the connecting morphism is defined by $\forall n \in \mathbb{Z}$

$$\begin{array}{rccc} \phi_n : & \tilde{H}_n((T,\theta)) & \longrightarrow & \tilde{H}_{n+1}((Y,\alpha)) \\ & \overline{k_{n+1}} & \longmapsto & \overline{f^{*-1}_{n+2}(\beta^*_{n+1}(g^{*-1}_{n+1}(k_{n+1})))} \end{array}$$

2. The sequence

$$\cdots \longrightarrow \tilde{H}_n((Y,\alpha)) \xrightarrow{\tilde{H}_n(f)} \tilde{H}_n((Z,\beta)) \xrightarrow{\tilde{H}_n(g)}$$

$$\tilde{H}_n((T,\theta)) \xrightarrow{\phi_n} \tilde{H}_{n+1}((Y,\alpha)) \xrightarrow{\tilde{H}_{n+1}(f)} \tilde{H}_{n+1}((Z,\beta)) \xrightarrow{\tilde{H}_{n+1}(g)}$$

$$\tilde{H}_{n+1}((T,\theta)) \xrightarrow{\phi_{n+1}} \cdots$$

is a long exact sequence of morphisms of abelian groups. That is, for all $n \in \mathbb{Z}$:

$$\begin{cases} \mathrm{Im}(\tilde{H}_n(f)) = \ker(\tilde{H}_n(g)) \\ \mathrm{Im}(\tilde{H}_n(g)) = \ker(\phi_n) \\ \mathrm{Im}(\phi_n) = \ker(\tilde{H}_{n+1}(f)) \end{cases}$$

2. Preliminaries

Definition 2.1. [*Abelian Category*]

An abelian category is a category $\mathscr{A}$ that satisfies the following conditions:

1. the category $\mathscr{A}$ has a zero object;
2. for every pair of objects X and Y in $\mathscr{A}$, the set $\mathrm{Hom}_{\mathscr{A}}(X,Y)$ is endowed with an abelian group structure, with composition law written additively ($(\mathrm{Hom}_{\mathscr{A}}(X,Y),+)$ is an abelian group);
3. composition in $\mathscr{A}$ is bilinear with respect to these additions;
4. every finite family of objects in $\mathscr{A}$ admits a coproduct;
5. every morphism in $\mathscr{A}$ has a kernel and a cokernel;
6. every monomorphism in $\mathscr{A}$ is the kernel of its cokernel;
7. every epimorphism in $\mathscr{A}$ is the cokernel of its kernel.

Definition 2.2 (Left Exact Sequence in $\mathscr{A}$).

Let (S) be:

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$$

a short sequence of morphisms in $\mathscr{A}$. Then (S) is said to be *left exact* if:

1. $g \circ f = e_{X,Z}$, where $e_{X,Z}$ is the zero morphism of the abelian group $\mathrm{Hom}_{\mathscr{A}}(X,Z)$.
2. The kernel of f, $N(f) = (0, e_{0,X})$, where $e_{0,X} : 0 \to X$ is the zero morphism of the abelian group $\mathrm{Hom}_{\mathscr{A}}(0,X)$.
3. The cokernel of h, $\mathrm{coN}(h) = (0, e_{K,0})$, where $e_{K,0} : K \to 0$ is the zero morphism of the abelian group $\mathrm{Hom}_{\mathscr{A}}(K,0)$.
4. h is the morphism satisfying $i \circ h = f$ and $\mathrm{Ker}(g) = (K,i)$.

$$\begin{tikzcd} & & K \arrow[d, "i"] & \\ 0 \arrow[r] & X \arrow[ur, dashed, "h"] \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Definition 2.3 (Right Exact Sequence in $\mathscr{A}$). Let (S): $X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short sequence of morphisms in $\mathscr{A}$. Then we say that (S) is right exact if:

1. $g \circ f = e_{X,Z}$, where $e_{X,Z}$ is the zero morphism of the abelian group $\mathrm{Hom}_{\mathscr{A}}(X,Z)$.
2. $\mathrm{coN}(g) = (0, e_{Z,0})$, where $e_{Z,0} : Z \to 0$ is the zero morphism of the abelian group $\mathrm{Hom}_{\mathscr{A}}(Z,0)$.
3. $\mathrm{coN}(h) = (0, e_{K,0})$, where $e_{K,0} : K \to 0$ is the zero morphism of the abelian group $\mathrm{Hom}_{\mathscr{A}}(K,0)$.
4. h is the morphism satisfying $i \circ h = f$ and $\mathrm{Ker}(g) = (K,i)$.

$$\begin{tikzcd} & K \arrow[d, "i"] & & \\ X \arrow[ur, dashed, "h"] \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \end{tikzcd}$$

Definition 2.4 (Exact Sequence in $\mathscr{A}$).

Let (S) be:

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

a short sequence of morphisms in $\mathscr{A}$. We say that (S) is exact if it is exact on the left and on the right, that is:

1. $g \circ f = e_{X,Z}$, where $e_{X,Z}$ is the zero morphism of the abelian group $\mathrm{Hom}_{\mathscr{A}}(X,Z)$.
2. $\mathrm{coN}(g) = (0, e_{Z,0})$, where $e_{Z,0} : Z \to 0$ is the zero morphism of the abelian group $\mathrm{Hom}_{\mathscr{A}}(Z,0)$.
3. $\mathrm{coN}(h) = (0, e_{K,0})$, where $e_{K,0} : K \to 0$ is the zero morphism of the abelian group $\mathrm{Hom}_{\mathscr{A}}(K,0)$.
4. h is the morphism satisfying $i \circ h = f$ and $\mathrm{Ker}(g) = (K,i)$.

$$\begin{tikzcd} & & K \arrow[d, "i"] & & \\ 0 \arrow[r] & X \arrow[ur, dashed, "h"] \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \end{tikzcd}$$

Remark 2.1. 0 denotes the zero object of the category $\mathscr{A}$.

Definition 2.5 ($\mathrm{Comp}(\mathscr{A})$).

We call the *category of complexes* of an abelian category $\mathscr{A}$ the category denoted by $\mathrm{Comp}(\mathscr{A})$, defined as follows:

1. The objects are the *complex sequences* of $\mathscr{A}$.
   A complex sequence of $\mathscr{A}$ is a sequence of morphisms of $\mathscr{A}$ $(\alpha_n : X_n \to X_{n+1})_{n\in\mathbb{Z}}$, denoted (X,α), such that
   $$\alpha_{n+1} \circ \alpha_n = 0 = e_{X_n,X_{n+2}} \quad \forall n \in \mathbb{Z},$$
   where $e_{X_n,X_{n+2}}$ is the zero morphism of the abelian group $Hom_{\mathscr{A}}(X_n, X_{n+2})$.
2. The arrows (morphisms) are the *chain maps* of $\mathscr{A}$.
   Let $(X,\alpha) = (\alpha_n : X_n \to X_{n+1})_{n\in\mathbb{Z}}$ and $(Y,\beta) = (\beta_n : Y_n \to Y_{n+1})_{n\in\mathbb{Z}}$ be two complex sequences of $\mathscr{A}$. A chain map of $\mathscr{A}$ $(f_n : X_n \longrightarrow Y_n)_{n\in\mathbb{Z}}$ is the chain map denoted $f : (X,\alpha) \to (Y,\beta)$ such that
   $$f_{n+1} \circ \alpha_n = \beta_n \circ f_n \quad \forall n \in \mathbb{Z}.$$

Proposition 2.1.

Let $\mathscr{A}$ be an abelian category. Then $\mathrm{Comp}(\mathscr{A})$ is an abelian category.

Proof.See page 319 of [5].

Definition 2.6 (Right Exact Sequence in Comp($\mathscr{A}$)).
Let

$$(S):\ (Y,\alpha) \xrightarrow{f} (Z,\beta) \xrightarrow{g} (T,\gamma) \longrightarrow (0)$$

be a short sequence of morphisms in Comp($\mathscr{A}$), where $\mathscr{A}$ is an abelian category. We say that (S) is *right exact* if, for every integer $n \in \mathbb{Z}$, the sequence

$$Y_n \xrightarrow{f_n} Z_n \xrightarrow{g_n} T_n \longrightarrow 0$$

is a short right exact sequence of morphisms in $\mathscr{A}$.

Definition 2.7 (Left Exact Sequence in Comp($\mathscr{A}$)).
Let

$$(S):\ (0) \longrightarrow (Y,\alpha) \xrightarrow{f} (Z,\beta) \xrightarrow{g} (T,\gamma)$$

be a short sequence of morphisms in Comp($\mathscr{A}$), where $\mathscr{A}$ is an abelian category. We say that (S) is *left exact* if, for every integer $n \in \mathbb{Z}$, the sequence

$$0 \longrightarrow Y_n \xrightarrow{f_n} Z_n \xrightarrow{g_n} T_n$$

is a short left exact sequence of morphisms in $\mathscr{A}$.

Definition 2.8 (Exact Sequence in Comp($\mathscr{A}$)).
Let

$$(S):\ (0) \longrightarrow (Y,\alpha) \xrightarrow{f} (Z,\beta) \xrightarrow{g} (T,\gamma) \longrightarrow (0)$$

be a short sequence of morphisms in Comp($\mathscr{A}$), where $\mathscr{A}$ is an abelian category. We say that (S) is *exact* if, for every integer $n \in \mathbb{Z}$, the sequence

$$0 \longrightarrow Y_n \xrightarrow{f_n} Z_n \xrightarrow{g_n} T_n \longrightarrow 0$$

is a short exact sequence of morphisms in $\mathscr{A}$.

Definition 2.9 (*Homological Functor H_n*).
Let A be a ring and Comp(A-Mod) the category of complex sequences of morphisms of A-Mod. We call *homological functor of degree $n \in \mathbb{Z}$* the covariant functor defined by:

$$H_n : \text{Comp(A-Mod)} \longrightarrow \text{Ab}$$

such that:

1. for every complex sequence $(X,\alpha) \in Ob(\text{Comp(A-Mod)})$, we have

$$H_n(X,\alpha) = \mathrm{Ker}(\alpha_{n+1})/\mathrm{Im}(\alpha_n);$$

2. for every complex chain $f : (X,\alpha) \to (Y,\beta)$ in A-Mod, we have:

$$H_n(f) : H_n((X,\alpha)) \longrightarrow H_n((Y,\beta))$$
$$\overline{z_n} \longmapsto \overline{f_n(z_n)}.$$

Example 2.1.
$H_n : \mathrm{Comp}(Ab) = \mathrm{Comp}(\mathbb{Z}-\mathrm{Mod}) \longrightarrow \mathrm{Ab}$ which is a particular case of the homological functor H_n : Comp(A-Mod) $\longrightarrow$ Ab $\forall n \in \mathbb{Z}$.

3. Homological Functor of Degree n: $\tilde{H}_n : \mathbf{Comp}(\mathscr{A}) \to \mathbf{Ab}$

Consider $\tilde{H}_n = H_n \circ \mathrm{Hom}_{\mathrm{Comp}}(\mathscr{A})(X,-)$, where $\mathrm{Hom}_{\mathrm{Comp}}(\mathscr{A})(X,-) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp(Ab)}$ and H_n : Comp(Ab) $\to$ Ab, and X is a projective object of $\mathscr{A}$ so that the functor $\tilde{H}_n$ is exact.

Theorem and Definition 3.1.
Let

$$(0) \longrightarrow (Y,\alpha) \xrightarrow{f} (Z,\beta) \xrightarrow{g} (T,\theta) \longrightarrow (0)$$

be a short exact sequence of morphisms in Comp($\mathscr{A}$), where $\mathscr{A}$ is an abelian category. Then:

1. the *connecting morphism* is defined by $\forall n \in \mathbb{Z}$:

$$\begin{array}{rccc} \phi_n : & \tilde{H}_n((T,\theta)) & \longrightarrow & \dfrac{\tilde{H}_{n+1}((Y,\alpha))}{f^{*-1}_{n+2}(\beta^*_{n+1}(g^{*-1}_{n+1}(k_{n+1})))}; \\ & \overline{k_{n+1}} & \longmapsto & \end{array}$$

see (*figure 1*).

2. the sequence

$$\dots \longrightarrow \tilde{H}_n((Y,\alpha)) \xrightarrow{\tilde{H}_n(f)} \tilde{H}_n((Z,\beta)) \xrightarrow{\tilde{H}_n(g)}$$

$$\tilde{H}_n((T,\theta)) \xrightarrow{\phi_n} \tilde{H}_{n+1}((Y,\alpha)) \xrightarrow{\tilde{H}_{n+1}(f)}$$

$$\tilde{H}_{n+1}((Z,\beta)) \xrightarrow{\tilde{H}_{n+1}(g)} \tilde{H}_{n+1}((T,\theta)) \xrightarrow{\phi_{n+1}} \dots$$

is a long exact sequence of abelian group morphisms. That is:

$$\forall n \in \mathbb{Z} : \quad \begin{cases} \mathrm{Im}(\tilde{H}_n(f)) = \mathrm{Ker}(\tilde{H}_n(g)) \\ \mathrm{Im}(\tilde{H}_n(g)) = \mathrm{Ker}(\phi_n) \\ \mathrm{Im}(\phi_n) = \mathrm{Ker}(\tilde{H}_{n+1}(f)). \end{cases}$$

Proof.
(i.)

We show that the connecting morphism ϕ_n is well-defined. We have:

$$\begin{array}{rccc} \phi_n : & \tilde{H}_n((T,\theta)) & \longrightarrow & \dfrac{\tilde{H}_{n+1}((Y,\alpha))}{f^{*-1}_{n+2}(\beta^*_{n+1}(g^{*-1}_{n+1}(k_{n+1})))} \\ & \overline{k_{n+1}} & \longmapsto & \end{array}.$$

First, we show that ϕ_n is a correspondence.
Let $k_{n+1} = g^{*-1}_{n+1}(k_{n+1}) \in \mathrm{Hom}_{Comp(\mathscr{A})}(X, Z_{n+1})$.
For all $n \in \mathbb{Z}$, we have:

$$\beta^*_{n+1}(k_{n+1}) \in \mathrm{Hom}_{Comp(\mathscr{A})}(X, Z_{n+2}).$$

We show that $\beta^*_{n+1}(k_{n+1}) \in \mathrm{Im}(f^*_{n+2})$. From the diagram below (page 6) we have:

$$g^*_{n+2} \circ \beta^*_{n+1} = \theta^*_{n+1} \circ g^*_{n+1} \quad \forall n \in \mathbb{Z}.$$

Hence, for all $n \in \mathbb{Z}$:

$$\begin{aligned} g^*_{n+2}(\beta^*_{n+1}(k_{n+1})) &= \theta^*_{n+1}(g^*_{n+1}(k_{n+1})) \\ &= \theta^*_{n+1}(\theta^*_n(k'_n)), \\ &\textit{since } \exists k'_n \in \mathrm{Hom}_{Comp(\mathscr{A})}(X, T_n) : \\ &\theta^*_n(k'_n) = g^*_{n+1}(k_{n+1})) \\ &= 0 = e_{\mathrm{Hom}_{Comp(\mathscr{A})}(X,T_n),\mathrm{Hom}_{Comp(\mathscr{A})}(X,T_{n+2})} \\ &\textit{because } \theta^*_{n+1} \circ \theta^*_n = 0. \end{aligned}$$

Also,

$$\begin{aligned} g^*_{n+2}(\beta^*_{n+1}(k_{n+1})) &= 0 = e_{\mathrm{Hom}_{Comp(\mathscr{A})}(X,T_n),\mathrm{Hom}_{Comp(\mathscr{A})}(X,T_{n+1})} \\ &\Rightarrow \beta^*_{n+1}(k_{n+1}) \in \mathrm{Ker}(g^*_{n+2}) = \mathrm{Im}(f^*_{n+2}) \\ &\Rightarrow \beta^*_{n+1}(k_{n+1}) \in \mathrm{Im}(f^*_{n+2}). \end{aligned}$$

Since $f^*_{n+2} : \mathrm{Hom}_{Comp(\mathscr{A})}(X, Y_{n+2}) \to \mathrm{Hom}_{Comp(\mathscr{A})}(X, Z_{n+2})$, *we have:*
$f^{*-1}_{n+2}(\beta^*_{n+1}(k_{n+1})) = f^{*-1}_{n+2}(\beta^*_{n+1}(g^{*-1}_{n+1}(k_{n+1})))$ *is well-defined.*

Next, we show that the correspondence ϕ_n is well-defined. It is enough to show that for all $n \in \mathbb{Z}$, the choice of ϕ_n does not depend on k_{n+1}, i.e., $\phi_n(\overline{k_{n+1}}) = \overline{h_{n+1}}$.

Let $\overline{k_{n+1}} = \overline{g^{*-1}_{n+1}(k_{n+1})}$, $\overline{k'_{n+1}} = \overline{g^{*-1}_{n+1}(k'_{n+1})} \in \tilde{H}_n((T,\theta))$ *such that* $\overline{k_{n+1}} = \overline{k'_{n+1}}$.
We show that $\phi_n(\overline{k_{n+1}}) = \phi_n(\overline{k'_{n+1}})$.

Set:

$$\phi_n(\overline{k_{n+1}}) = \overline{h_{n+1}} \textit{ and } \phi_n(\overline{k'_{n+1}}) = \overline{h'_{n+1}}.$$

We show that $\overline{h_{n+1}} = \overline{h'_{n+1}}$.
We have:
$k_{n+1} - k'_{n+1} \in \mathrm{Im}(f^*_{n+1})$ *and so there exists* $j_{n+1} \in \mathrm{Hom}_{Comp(\mathscr{A})}(X, Y_{n+1})$ *such that*

$$f^*_{n+1}(j_{n+1}) = k_{n+1} - k'_{n+1}.$$

Then:

$$\begin{aligned} & f^*_{n+1}(j_{n+1}) = k_{n+1} - k'_{n+1} \\ \Rightarrow\ & \beta^*_{n+1}\big(f^*_{n+1}(j_{n+1})\big) = \beta^*_{n+1}\big(k_{n+1} - k'_{n+1}\big) \\ & = f^*_{n+2}\big(\alpha^*_{n+1}(j_{n+1})\big) \\ & (\textit{since } \beta^*_{n+1} \circ f^*_{n+1} = f^*_{n+2} \circ \alpha^*_{n+1}) \\ \Rightarrow\ & \alpha^*_{n+1}(j_{n+1}) = f^{*-1}_{n+2}\big(\beta^*_{n+1}(k_{n+1} - k'_{n+1})\big) \\ & = f^{*-1}_{n+2}\big(\beta^*_{n+1}(k_{n+1})\big) - f^{*-1}_{n+2}\big(\beta^*_{n+1}(k'_{n+1})\big) \\ & = h_{n+1} - h'_{n+1} \in \mathrm{Im}(\alpha^*_{n+1}) \end{aligned}$$

$$\begin{aligned} &\Rightarrow \overline{h_{n+1} - h'_{n+1}} = \bar{0} \quad \Rightarrow \quad \overline{h_{n+1}} = \overline{h'_{n+1}} \\ &\Rightarrow \phi_n(\overline{k_{n+1}}) = \phi_n(\overline{k'_{n+1}}). \end{aligned}$$

Hence, ϕ_n is well-defined.

(ii.)

Recall that $Hom_{\mathscr{A}}(X, -) : \mathscr{A} \longrightarrow Ab$ is a covariant, additive, left exact functor, and it is exact if and only if X is a projective object of $\mathscr{A}$.

(ii.1) Let's show that $\mathrm{Im}(\tilde{H}_n(f)) = \ker(\tilde{H}_n(g))$

We have:

$$(0) \longrightarrow (Y,\alpha) \xrightarrow{f} (Z,\beta) \xrightarrow{g} (T,\theta)$$

$$\longrightarrow (0)$$

a short exact sequence of morphisms in Comp($\mathscr{A}$), so we have the following diagrams:

$$\begin{array}{ccccccccc} (0): \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ (Y,\alpha): \cdots & \longrightarrow & Y_n & \xrightarrow{\alpha_n} & Y_{n+1} & \xrightarrow{\alpha_{n+1}} & Y_{n+2} & \longrightarrow & \cdots \\ \downarrow f & & \downarrow f_n & & \downarrow f_{n+1} & & \downarrow f_{n+2} & & \\ (Z,\beta): \cdots & \longrightarrow & Z_n & \xrightarrow{\beta_n} & Z_{n+1} & \xrightarrow{\beta_{n+1}} & Z_{n+2} & \longrightarrow & \cdots \\ \downarrow g & & \downarrow g_n & & \downarrow g_{n+1} & & \downarrow g_{n+2} & & \\ (T,\theta): \cdots & \longrightarrow & T_n & \xrightarrow{\theta_n} & T_{n+1} & \xrightarrow{\theta_{n+1}} & T_{n+2} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ (0): \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

(ii.1.a) Let's show that $\mathrm{Im}(\tilde{H}_n(f)) \subset \ker(\tilde{H}_n(g))$.

Let $\bar{h}_n \in \mathrm{Im}(\tilde{H}_n(f))$. For all $n \in \mathbb{Z}$, we have:

$$\begin{aligned} & \bar{h}_n \in \mathrm{Im}(\tilde{H}_n(f)) \\ & \Rightarrow \exists\, \bar{k}_n \in \tilde{H}_n(Y,\alpha) : \tilde{H}_n(f)(\bar{k}_n) = \bar{h}_n \\ & \Rightarrow \overline{f_n(k_n)} = \bar{h}_n \\ & \Rightarrow \tilde{H}_n(g)\big(\overline{f_n(k_n)}\big) = \tilde{H}_n(g)(\bar{h}_n) \\ & \Rightarrow \overline{g_n(f_n(k_n))} = \tilde{H}_n(g)(\bar{h}_n) \\ & \Rightarrow \bar{0} = \tilde{H}_n(g)(\bar{h}_n) \quad \textit{since } g_n \circ f_n = 0 = e_{Y_n,T_n} \\ & \Rightarrow \bar{h}_n \in \mathrm{Ker}(\tilde{H}_n(g)). \end{aligned}$$

Hence, $\mathrm{Im}(\tilde{H}_n(f)) \subset \mathrm{Ker}(\tilde{H}_n(g))$.

$$\begin{tikzcd}
(0)_{\mathrm{Ab}} : \dots \arrow[r] \arrow[d] & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0_{\mathscr{A}})} \arrow[r] \arrow[d] & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0_{\mathscr{A}})} \arrow[r] \arrow[d] & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0_{\mathscr{A}})} \arrow[r] \arrow[d] & \dots \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)(Y,\alpha) : \dots \arrow[r] \arrow[d, "g^*"'] & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Y_n) \arrow[r, "\theta_n^*"] \arrow[d, "g_n^*"'] & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Y_{n+1}) \arrow[r, "\theta_{n+1}^*"] \arrow[d, "g_{n+1}^*"'] & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Y_{n+2}) \arrow[r] \arrow[d, "g_{n+2}^*"'] & \dots \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)(Z,\beta) : \dots \arrow[r] \arrow[d, "f^*"'] & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Z_n) \arrow[r, "\beta_n^*"] \arrow[d, "f_n^*"'] & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Z_{n+1}) \arrow[r, "\beta_{n+1}^*"] \arrow[d, "f_{n+1}^*"'] & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Z_{n+2}) \arrow[r] \arrow[d, "f_{n+2}^*"'] & \dots \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)(T,\theta) : \dots \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,T_n) \arrow[r, "\alpha_n^*"] \arrow[d] & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,T_{n+1}) \arrow[r, "\alpha_{n+1}^*"] \arrow[d] & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,T_{n+2}) \arrow[r] \arrow[d] & \dots \\
(0)_{\mathrm{Ab}} : \dots \arrow[r] & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0_{\mathscr{A}})} \arrow[r] & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0_{\mathscr{A}})} \arrow[r] & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0_{\mathscr{A}})} \arrow[r] & \dots
\end{tikzcd}$$

Figure 1. *Commutative diagram of induced homomorphisms in* $\mathrm{Comp}(\mathscr{A})$.

(ii.1.b) Next, we show that $\mathrm{Ker}(\tilde{H}_n(g)) \subset \mathrm{Im}(\tilde{H}_n(f))$.

Let $\bar{h}_n \in \mathrm{Ker}(\tilde{H}_n(g)) \subset \tilde{H}_n(Z,\beta) = \mathrm{Ker}(\beta_n^*)/\mathrm{Im}(\beta_{n-1}^*)$. Then:

$$\begin{aligned}
&\bar{h}_n \in \mathrm{Ker}(\tilde{H}_n(g)) \\
&\quad \Rightarrow\ \overline{g_n(h_n)} = \bar{0} \in \tilde{H}_n(T,\theta) \\
&\quad \Rightarrow\ g_n(h_n) \in \mathrm{Im}(\theta_{n-1}^*) \\
&\quad \Rightarrow\ \exists\, k_{n-1} \in \mathrm{Hom}(X,T_{n-1}) : g_n^*(h_n) = \theta_{n-1}^*(k_{n-1}) \\
&\quad \Rightarrow\ \exists\, k_{n-1} \in \mathrm{Hom}(X,T_{n-1}) : g_n(h_n) = \theta_{n-1}(k_{n-1}).
\end{aligned}$$

Since g_{n-1}^* is an epimorphism:

$$\begin{aligned}
\exists\, h'_{n-1} \in \mathrm{Hom}(X,Z_{n-1}) \text{ such that} \\
g_{n-1}^*(h'_{n-1}) &= g_{n-1}(h'_{n-1}) \\
&= k_{n-1}.
\end{aligned}$$

Thus:

$$h_n - \beta_{n-1}^*(h'_{n-1}) \in \mathrm{Hom}(X,Z_n)$$

and

$$\begin{aligned}
g_n^*\big(h_n - \beta_{n-1}^*(h'_{n-1})\big) &= g_n^*(h_n) - g_n^*\big(\beta_{n-1}^*(h'_{n-1})\big) \\
&= g_n^*(h_n) - \theta_{n-1}^*\big(g_{n-1}^*(h'_{n-1})\big) \\
&\qquad (\text{since } g_n^* \circ \beta_{n-1}^* = \theta_{n-1}^* \circ g_{n-1}^*) \\
&= g_n^*(h_n) - \theta_{n-1}^*(k_{n-1}) \\
&\qquad (\text{since } g_{n-1}^*(h'_{n-1}) = k_{n-1}) \\
&= g_n(h_n) - g_n(h_n) \\
&= 0 = e_{X,T_n}.
\end{aligned}$$

That is, $h_n - \beta_{n-1}(h'_{n-1}) \in \mathrm{Ker}(g_n^*) = \mathrm{Im}(f_n^*)$. Therefore:

$$\begin{aligned}
\exists\, \ell_n \in \mathrm{Hom}(X,Y_n) \text{ such that } f_n^*(\ell_n) &= f_n(\ell_n) \\
&= h_n - \beta_{n-1}(h'_{n-1}).
\end{aligned}$$

Moreover, $\overline{\beta_{n-1}(h'_{n-1})} \in \tilde{H}_n(Z,\beta) = \mathrm{Ker}(\beta_n^*)/\mathrm{Im}(\beta_{n-1}^*)$,
and since $\beta_{n-1}^*(h'_{n-1}) = \beta_{n-1}(h'_{n-1}) \in \mathrm{Im}(\beta_{n-1}^*)$, we have $\overline{\beta_{n-1}(h'_{n-1})} = \bar{0}_{\tilde{H}_n(Z,\beta)}$.
Thus:

$$\bar{h}_n = \overline{f_n(\ell_n)} + \bar{0}_{\tilde{H}_n(Z,\beta)} = \overline{f_n(\ell_n)} \in \mathrm{Im}(\tilde{H}_n(f))$$

Hence, $\mathrm{Ker}(\tilde{H}_n(g)) \subset \mathrm{Im}(\tilde{H}_n(f))$.
(ii.1.a) and (ii.1.b) imply that: $\mathrm{Im}(\tilde{H}_n(f)) = \mathrm{Ker}(\tilde{H}_n(g))$.

(ii.2) Let's show that $\mathrm{Im}(\tilde{H}_n(g)) = \mathrm{Ker}(\phi_n)$.

(ii.2.a) Show that $\mathrm{Im}(\tilde{H}_n(g)) \subset \mathrm{Ker}(\phi_n)$.
Let $\bar{h}_{n+1} \in \mathrm{Im}(\tilde{H}_n(g))$. Then for all $n \in \mathbb{Z}$:

$$\begin{aligned}
&\bar{h}_{n+1} \in \mathrm{Im}(\tilde{H}_n(g)) \\
&\quad \Rightarrow\ \exists\, \bar{k}_{n+1} \in \tilde{H}_n(Z,\beta) : \tilde{H}_n(g)(\bar{k}_{n+1}) = \bar{h}_{n+1} \\
&\quad \Rightarrow\ \overline{g_{n+1}(k_{n+1})} = \bar{h}_{n+1} \in \tilde{H}_n(T,\theta).
\end{aligned}$$

By definition of ϕ_n we have:

$$\begin{aligned}
\phi_n(\bar{h}_{n+1}) &= \phi_n\big(\overline{g_{n+1}(k_{n+1})}\big) \\
&= \overline{f_{n+2}^{*-1}\big(\beta_{n+1}^*(g_{n+1}^{*-1}(h_{n+1}))\big)} \\
&\in \tilde{H}_{n+1}(Y,\alpha) = \mathrm{Ker}(\alpha_{n+1}^*)/\mathrm{Im}(\alpha_n^*).
\end{aligned}$$

where
$h_{n+1} \in \mathrm{Hom}(X,T_{n+1})$, $w_{n+1} \in \mathrm{Hom}(X,Z_{n+1})$ such that $g_{n+1}(w_{n+1}) = h_{n+1}$, $\beta_{n+1}(w_{n+1}) \in \mathrm{Hom}(X,Z_{n+2})$.
Set $w_{n+1} = k_{n+1}$. Then:

$$g_{n+1}(k_{n+1}) = g_{n+1}(w_{n+1}) = h_{n+1}$$

and thus

$$\overline{k_{n+1}} \in \tilde{H}_n(Z,\beta) = \mathrm{Ker}(\beta_{n+1}^*)/\mathrm{Im}(\beta_n^*)$$

We have:

$$\beta_{n+1}^*(k_{n+1}) = \beta_{n+1}(k_{n+1}) = 0 = e_{X,Z_{n+2}}$$

so

$$\overline{\beta_{n+1}(k_{n+1})} = \bar{0}$$

Hence,

$$\beta^*_{n+1}(k_{n+1}) = \beta_{n+1}(k_{n+1}) \in \mathrm{Im}(\beta^*_n)$$

and therefore there exists $z_n \in \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, Z_n)$ such that

$$\beta^*_n(z_n) = \beta_n(z_n) = \beta_{n+1}(k_{n+1}).$$

Thus:

$$f^{*-1}_{n+2}(\beta^*_{n+1}(k_{n+1})) = f^{*-1}_{n+2}(\beta^*_n(z_n)) = \alpha^*_n(f^{-1}_{n+1}(z_n))$$

That is:

$$\begin{aligned} f^{-1}_{n+2}\big(\beta_{n+1}(k_{n+1})\big) &= f^{-1}_{n+2}\big(\beta_n(z_n)\big) \\ &= \alpha^*_n\big(f^{-1}_{n+1}(z_n)\big) \in \mathrm{Im}(\alpha^*_n). \end{aligned}$$

which implies

$$\overline{f^{*-1}_{n+2}(\beta^*_{n+1}(k_{n+1}))} = \overline{0} \quad \in \tilde{H}_{n+1}(Y, \alpha).$$

Thus

$$\phi_n(\overline{h_{n+1}}) = \overline{f^{*-1}_{n+2}(\beta^*_{n+1}(k_{n+1}))} = \overline{0} \quad \in \tilde{H}_{n+1}(Y, \alpha)$$

which shows that $\overline{h_{n+1}} \in \mathrm{Ker}(\phi_n)$.
Hence

$$\mathrm{Im}(\tilde{H}_n(g)) \subset \mathrm{Ker}(\phi_n).$$

(ii.2.b) Show that $\mathrm{Ker}(\phi_n) \subset \mathrm{Im}(\tilde{H}_n(g))$.
Let $\overline{k_{n+1}} \in \mathrm{Ker}(\phi_n) \subset \tilde{H}_n(T, \theta)$. Then for all $n \in \mathbb{Z}$:

$$\overline{k}_{n+1} \in \mathrm{Ker}(\phi_n) \Rightarrow \phi_n(\overline{k}_{n+1}) = \overline{0} \in \tilde{H}_{n+1}(Y, \alpha)$$

Let $w_{n+1} \in \mathrm{Hom}(X, Z_{n+1})$ such that:

$$g^*_{n+1}(w_{n+1}) = g_{n+1}(w_{n+1}) = k_{n+1}$$

The connecting morphism gives:

$$\phi_n(\overline{k}_{n+1}) = \overline{f^{*-1}_{n+2}(\beta^*_{n+1}(w_{n+1}))} = \overline{0}$$

which implies:

$$\begin{aligned} &\exists\, y_{n+1} \in \mathrm{Hom}(X, Y_{n+1}) \text{ such that} \\ &\qquad \beta^*_{n+1}(w_{n+1}) = \beta_{n+1}(w_{n+1}) \\ &\qquad\qquad = f_{n+2}\big(\alpha_{n+1}(y_{n+1})\big) \\ &\qquad\qquad = f^*_{n+2}\big(\alpha_{n+1}(y_{n+1})\big). \end{aligned}$$

Consider:

$$w'_{n+1} = w_{n+1} - f_{n+1}(y_{n+1}) \in \mathrm{Hom}(X, Z_{n+1})$$

Then:

$$g^*_{n+1}(w'_{n+1}) = g_{n+1}(w'_{n+1}), \quad \beta^*_{n+1}(w'_{n+1}) = \beta_{n+1}(w'_{n+1})$$

so:

$$\begin{aligned} g_{n+1}(w'_{n+1}) &= g_{n+1}(w_{n+1} - f_{n+1}(y_{n+1})) \\ &= g_{n+1}(w_{n+1}) - g_{n+1}(f_{n+1}(y_{n+1})) \\ &= k_{n+1} \quad \text{since } g_{n+1} \circ f_{n+1} = 0 \end{aligned}$$

$$\begin{aligned} \beta_{n+1}(w'_{n+1}) &= \beta_{n+1}(w_{n+1} - f_{n+1}(y_{n+1})) \\ &= \beta_{n+1}(w_{n+1}) - \beta_{n+1}(f_{n+1}(y_{n+1})) \\ &= f_{n+2}(\alpha_{n+1}(y_{n+1})) - \beta_{n+1}(f_{n+1}(y_{n+1})) \\ &= 0 \quad \text{since } \beta_{n+1} \circ f_{n+1} = f_{n+2} \circ \alpha_{n+1} \end{aligned}$$

Thus:

$$\overline{k_{n+1}} = \overline{g_{n+1}(w'_{n+1})} \in \mathrm{Im}(\tilde{H}_n(g))$$

Hence $\mathrm{Ker}(\phi_n) \subset \mathrm{Im}(\tilde{H}_n(g))$.

(ii.2.a) and (ii.2.b) imply that: $\mathrm{Ker}(\phi_n) = \mathrm{Im}(\tilde{H}_n(g))$.
(ii.3) Show that $\mathrm{Im}(\phi_n) = \mathrm{Ker}(\tilde{H}_{n+1}(f))$.
(ii.3.a) Show that $\mathrm{Im}(\phi_n) \subset \mathrm{Ker}(\tilde{H}_{n+1}(f))$.
Let $\overline{y}_{n+1} \in \mathrm{Im}(\phi_n)$. Then for all $n \in \mathbb{Z}$:

$$\overline{y}_{n+1} \in \mathrm{Im}(\phi_n) \Rightarrow \exists \overline{t}_{n+1} \in \tilde{H}_n(T, \theta) : \phi_n(\overline{t}_{n+1}) = \overline{y}_{n+1}$$

By definition of the connecting morphism ϕ_n:

$$\phi_n(\overline{t}_{n+1}) = \overline{f^{*-1}_{n+2}(\beta^*_{n+1}(g^{*-1}_{n+1}(t_{n+1})))} \in \tilde{H}_{n+1}(Y, \alpha)$$

Since g^*_{n+1} is an epimorphism:

$$\begin{aligned} &\forall\, t_{n+1} \in \mathrm{Hom}(X, T_{n+1}), \\ &\exists\, z_{n+1} \in \mathrm{Hom}(X, Z_{n+1}) : \ g^*_{n+1}(z_{n+1}) = t_{n+1}. \end{aligned}$$

Moreover:

$$\theta^*_{n+1} \circ g^*_{n+1} = g^*_{n+2} \circ \beta^*_{n+1}$$

Thus:

$$\begin{aligned} \theta_{n+1}\big(g_{n+1}(z_{n+1})\big) &= g_{n+2}\big(\beta_{n+1}(z_{n+1})\big) \\ &= g_{n+2}\big(f_{n+2}(y_{n+1})\big) = 0. \end{aligned}$$

i.e.,

$$\beta^*_{n+1}(z_{n+1}) \in \mathrm{Ker}(g^*_{n+2}) = \mathrm{Im}(f^*_{n+2})$$

Hence, there exists $y_{n+1} \in \mathrm{Hom}(X, Y_{n+2})$ such that $f^*_{n+2}(y_{n+1}) = \beta_{n+1}(z_{n+1})$.
Set $t_{n+1} = g_{n+1}(z_{n+1})$, with $z_{n+1} = g^{-1}_{n+1}(t_{n+1})$. Then $\phi_n(\overline{t_{n+1}}) = \overline{y_{n+1}}$, which implies

$$\tilde{H}_{n+1}(f)(\overline{y}_{n+1}) = \overline{f_{n+2}(y_{n+1})} = \overline{0}$$

Hence $\overline{y}_{n+1} \in \mathrm{Ker}(\tilde{H}_{n+1}(f))$.
So $\mathrm{Im}(\phi_n) \subset \mathrm{Ker}(\tilde{H}_{n+1}(f))$.
(ii.3.b) Show that $\mathrm{Ker}(\tilde{H}_{n+1}(f)) \subset \mathrm{Im}(\phi_n)$.

Let $\overline{y_{n+1}} \in \mathrm{Ker}(\tilde{H}_{n+1}(f))$. Then:

$$\tilde{H}_{n+1}(f)(\overline{y_{n+1}}) = \overline{f_{n+2}(y_{n+1})} = \overline{0}$$
$$\Rightarrow f_{n+2}(y_{n+1}) \in \mathrm{Im}(\beta_{n+1}).$$

Hence $\exists z_{n+1} \in \mathrm{Hom}(X, Z_{n+1})$ such that $f^*_{n+2}(y_{n+1}) = \beta_{n+1}(z_{n+1})$.
Set $t_{n+1} = g_{n+1}(z_{n+1})$ with $z_{n+1} = g^{-1}_{n+1}(t_{n+1})$. Then $\overline{t_{n+1}} \in \tilde{H}_n(T, \theta)$ and

$$\phi_n(\overline{t_{n+1}}) = \overline{y_{n+1}}$$

Hence $\overline{y_{n+1}} \in \mathrm{Im}(\phi_n)$, i.e., $\mathrm{Ker}(\tilde{H}_{n+1}(f)) \subset \mathrm{Im}(\phi_n)$.
(ii.3.a) and (ii.3.b) imply that: $\mathrm{Ker}(\tilde{H}_{n+1}(f)) = \mathrm{Im}(\phi_n)$. Therefore,(ii.1), (ii.2), and (ii.3) imply that the sequence

$$\ldots \longrightarrow \tilde{H}_n((Y,\alpha)) \xrightarrow{\tilde{H}_n(f)} \tilde{H}_n((Z,\beta)) \xrightarrow{\tilde{H}_n(g)} \tilde{H}_n((T,\theta))$$

$$\xrightarrow{\phi_n} \tilde{H}_{n+1}((Y,\alpha)) \xrightarrow{\tilde{H}_{n+1}(f)} \tilde{H}_{n+1}((Z,\beta)) \xrightarrow{\tilde{H}_{n+1}(g)} \tilde{H}_{n+1}((T,\theta))$$

$$\xrightarrow{\phi_{n+1}} \ldots$$

is a long exact sequence of abelian group morphisms.

4. Conclusion

In this work the connecting morphism is well defined for $\forall n \in \mathbb{Z}$ $\phi_n : \begin{array}{ccc} \tilde{H}_n((T,\theta)) & \longrightarrow & \tilde{H}_{n+1}((Y,\alpha)) \\ \overline{k_{n+1}} & \longmapsto & \overline{f^{*-1}_{n+2}(\beta^*_{n+1}(g^{*-1}_{n+1}(k_{n+1})))} \end{array}$
and the sequence

$$\ldots \longrightarrow \tilde{H}_n((Y,\alpha)) \xrightarrow{\tilde{H}_n(f)} \tilde{H}_n((Z,\beta)) \xrightarrow{\tilde{H}_n(g)} \tilde{H}_n((T,\theta))$$

$$\xrightarrow{\phi_n} \tilde{H}_{n+1}((Y,\alpha)) \xrightarrow{\tilde{H}_{n+1}(f)} \tilde{H}_{n+1}((Z,\beta)) \xrightarrow{\tilde{H}_{n+1}(g)} \tilde{H}_{n+1}((T,\theta))$$

$$\xrightarrow{\phi_{n+1}} \ldots$$

is a long exact sequence of abelian group morphisms.

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Author Contributions

Ablaye Diallo is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The author declares no conflicts of interest.

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