American Journal of Applied Mathematics
2025, Vol. 13, No. 6, pp. 393-411
https://doi.org/10.11648/j.ajam.20251306.12

Research Article

Optimality Condition and Numerical Methods of Variable-Order Fractional Optimal Control Problem with Full Free Ends

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Abstract

In this paper, we studied the necessary conditions for seeking optimal trajectory, optimal control, optimal time and optimal state. Then, we applied these qualitative properties to explore two numerical methods of a variable order fractional optimal control problem until full free ends that the initial time, initial state, terminal time and terminal state are free simultaneously. Based on the relations between variable order fractional calculus operators, i.e., the integral formulas by parts, the necessary conditions of optimality of a variable order fractional optimal control problem until full free ends drove from the Euler-Lagrange equation, applying Hamiltonian and variational principle. To develop the solution methods, we proposed the "optimization first, then discretization" (OFTD) method and the "discretization first, then optimization" (DFTO) method. The OFTD method is to solve the variable order fractional optimal control problem until full free ends by transforming the Euler-Lagrange equation into a nonlinear system using the Grünwald-Letnikov definition and manipulating the transversal conditions as an objective function of error quadratic minimization, i.e., a nonlinear programming type problem with equality constraints. The DFTO method is to solve the problem by transforming the variable order fractional calculus into a classical optimal control problem with integer order using the expansion formulas of the variable order fractional calculus operators. Finally, we demonstrated the validity and accuracy of the proposed methods through various types of numerical test problems.

Keywords

Variable-Order Fractional Optimal Control Problem, Variable-Order Fractional Calculus, Variable-Order Fractional Differential Equation, Optimality Condition, Approximation Method

1. Introduction

In the last decade, many scientists have studied the variable order (V-O) fractional calculus more and more actively as it models phenomena more accurately than classical integer and fractional calculus [14]. The V-O fractional calculus have been applied in the fields of viscous elastic mechanics [5], automatic control [3], image processing [20] and signal processing [4], and its efficiency and advantages have been demonstrated scientifically.

The early stage, Samko and Ross [12] proposed the idea of V-O fractional calculus and Sierociuk, Malesza and Macias

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Received: 2 October 2025; **Accepted:** 15 October 2025; **Published:** 26 November 2025

[13] clarified the derivation of V-O fractional derivative, its interpretations by switching scheme and approximate computation method by operational matrices. Moreover, the extension formulas to higher-order derivatives for V-O fractional operators have been developed to enrich the properties of V-O fractional calculus [1, 2, 9]. In particular, Tavares, Almeida and Torres explained the correlation and numerical methods between the Caputo V-O fractional calculus and its variations [17]. Ortigueira and Valerio added new and convenient definitions to various variable-order fractional operators and enriched the expression of transfer function by linear dynamical system and V-O Mittag-Leffler function [10].

By the progressive studies for such V-O fractional calculus, it is interesting to show good results, with the expansion of research into variational method, optimal control and automatic control with applications beyond the constant and fractional order optimal control problem [11, 15]. The authors of [18] studied the necessary conditions of optimality and the computational method of generalized fractional variational problems with a combined Caputo higher order derivative and a state time delay. In addition, they proposed the methods for finding the minimum solution of the isoperimetric problem with combined Caputo derivative of variable fractional order and a new variational problem with holonomic constraints [19]. Heydari [6] derived the necessary condition and numerical solution method of extreme by expressing state variable and control variable as Chebyshev basis functions and transforming the original problem into the nonlinear algebraic system for a type of variable order fractional optimal control problem (V-OFOCP). The operational matrix method and Gauss- Legendre quadrature of the V-O fractional integral (V-OFI) are used in [8] and the wavelet method is applied in [7] to approximate the V-OFOCP to a system of algebraic equation for numerical solutions. The optimal control problem (OCP) of a general V-O fractional delay model for product advertising effect is resolved in [16]. Here, they modeled a V-OFOCP that maximizes the number of purchasers who buy a product, and solved it numerically using a nonstandard difference method. In addition, a few papers on the V-OFOCP described by a partial differential equation introduced, making the V-O field more interested.

In this paper, we propose the theoretical properties of optimality and numerical methods of a variable order fractional optimal control problem until full free ends (FFE). Here, the dynamic system is described as a binomial variable-order fractional differential equation (V-OFDE). The rest of this paper deals with the following sections: Sect. 2 reviews definition and numerical approximation schemes of V-OFDs and proves their relations. The necessary optimality conditions and numerical method of the V-OFOCP are resolved in Sect. 3. In Sect. 4, we demonstrate the validity and accuracy of our theory and methods through some numerical experiments. Finally, conclusion is drawn.

2. Preliminaries

In this section, we review basic definitions and properties of variable order fractional derivative (V-OFDs) and their approximation schemes that be used in this paper.

2.1. Definition

Prior to introducing definitions of the V-O operators, we recall two special functions: Gamma and Psi functions. The Gamma function is defined by $\Gamma(t)=\int_0^\infty \tau^{t-1}\exp(-\tau)d\tau, t>0$ and its fundamental property that will be used in this paper is $\Gamma(t+1)=t\Gamma(t)$. The Psi function defines as the derivative of the logarithm of the Gamma function, i.e. $\Psi(t)=d(\ln(\Gamma(t)))/dt=\Gamma'(t)/\Gamma(t)$.

Definition [10] Given a function $x:[a,b]\to R$ and $\alpha\in C^1([a,b],[0,1])$, then for $t\in[a,b]$,

1. The left Riemann-Liouville variable order fractional integral (V-OFIs, type A and type B) of order $\alpha(t)$ are defined by

$$ {}_a^AI_t^{\alpha(t)}x(t) = {}_aI_t^{\alpha(t)}x(t) = \frac{1}{\Gamma(\alpha(t))}\int_a^t (t-\tau)^{\alpha(t)-1}x(\tau)d\tau, \quad {}_a^BI_t^{\alpha(t)}x(t) = \int_a^t \frac{1}{\Gamma(\alpha(\tau))}(t-\tau)^{\alpha(\tau)-1}x(\tau)d\tau. \tag{1}$$

2. The right Riemann-Liouville V-OFIs (type A and type B) of order $\alpha(t)$ are defined by

$$ {}_tI_b^{\alpha(t)}x(t) = \frac{1}{\Gamma(\alpha(t))}\int_t^b (\tau-t)^{\alpha(t)-1}x(\tau)d\tau, \quad {}_t^BI_b^{\alpha(t)}x(t) = \int_t^b \frac{1}{\Gamma(\alpha(\tau))}(\tau-t)^{\alpha(\tau)-1}x(\tau)d\tau. \tag{2}$$

3. The left Riemann-Liouville V-OFDs (type A and type B) of order $\alpha(t)$ are defined by

$$ {}_aD_t^{\alpha(t)}x(t) = \frac{1}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_a^t (t-\tau)^{-\alpha(t)}x(\tau)d\tau,, \quad {}_a^BD_t^{\alpha(t)}x(t) = \frac{d}{dt}(\int_a^t \frac{1}{\Gamma(1-\alpha(\tau))}(t-\tau)^{-\alpha(\tau)}x(\tau)d\tau). \tag{3}$$

4. The right Riemann-Liouville V-OFDs (type A and type B) of order $\alpha(t)$ are defined by

$$ {}_tD_b^{\alpha(t)}x(t)=\frac{-1}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_t^b(\tau-t)^{-\alpha(t)}x(\tau)d\tau,,\quad {}_t^BD_b^{\alpha(t)}x(t)=-\frac{d}{dt}(\int_t^b\frac{1}{\Gamma(1-\alpha(\tau))}(\tau-t)^{-\alpha(\tau)}x(\tau)d\tau). \tag{4} $$

5. The left Caputo V-OFDs (type A and type B) of order $\alpha(t)$ are defined by

$$ {}_a^CD_t^{\alpha(t)}x(t)=\frac{1}{\Gamma(1-\alpha(t))}\int_a^t(t-\tau)^{-\alpha(t)}x'(\tau)d\tau,,\quad {}_a^{B,C}D_t^{\alpha(t)}x(t)=\int_a^t\frac{1}{\Gamma(1-\alpha(\tau))}(t-\tau)^{-\alpha(\tau)}x'(\tau)d\tau. \tag{5} $$

6. The right Caputo V-OFDs (type A and type B) of order $\alpha(t)$ are defined by

$$ {}_t^CD_b^{\alpha(t)}x(t)=\frac{-1}{\Gamma(1-\alpha(t))}\int_t^b(\tau-t)^{-\alpha(t)}x'(\tau)d\tau,,\quad {}_t^{B,C}D_b^{\alpha(t)}x(t)=\int_t^b\frac{-1}{\Gamma(1-\alpha(\tau))}(\tau-t)^{-\alpha(\tau)}x'(\tau)d\tau. \tag{6} $$

7. The left Grünwald-Letnikov V-OFDs (type A and type B) of order $\alpha(t)$ are defined by

$$ {}_a^{GL}D_t^{\alpha(t)}x(t)=\lim_{\substack{h\to0\\ Nh=t-a}}\sum_{k=0}^N\frac{(-1)^k}{h^{\alpha(t)}}\binom{\alpha(t)}{k}x(t-kh),\quad {}_a^{B,GL}D_t^{\alpha(t)}x(t)=\lim_{\substack{h\to0\\ Nh=t-a}}\sum_{k=0}^N\frac{(-1)^k}{h^{\alpha(kh)}}\binom{\alpha(kh)}{k}x(t-kh). \tag{7} $$

8. The right Grünwald-Letnikov V-OFDs (type A and type B) of order $\alpha(t)$ are defined by

$$ {}_t^{GL}D_b^{\alpha(t)}x(t)=\lim_{\substack{h\to0\\ Nh=b-t}}\sum_{k=0}^N\frac{(-1)^k}{h^{\alpha(t)}}\binom{\alpha(t)}{k}x(t+kh),\quad {}_t^{B,GL}D_b^{\alpha(t)}x(t)=\lim_{\substack{h\to0\\ Nh=b-t}}\sum_{k=0}^N\frac{(-1)^k}{h^{\alpha(kh)}}\binom{\alpha(kh)}{k}x(t+kh). \tag{8} $$

2.2. Expansion Formulas of V-O Fractional Operators and Relations

Here, we denote the approximation of V-O fractional operators of a function as series of terms involving only integer-order derivatives and drive the relations between the operators.

Theorem 2.1 [2] *Fix* $n\in N$ *and* $N\geq n+1$, *and let* $x(\cdot)\in C^{n+1}([a,b],R)$. *Define the (left) moment of x with the order k by*

$$ V_k(t)=(k-n)\int_a^t(\tau-a)^{k-n-1}x(\tau)d\tau. $$

Then,

$$ {}_aD_t^{\alpha(t)}x(t)=S_1(t)-S_2(t)+E_{1,N}(t)+E_{2,N}(t) $$

with

$$ S_1(t)=(t-a)^{-\alpha(t)}\left[\sum_{k=0}^nA(\alpha(t),k)(t-a)^kx^{(k)}(t)+\sum_{k=n+1}^NB(\alpha(t),k)(t-a)^{n-k}V_k(t)\right], $$

$$ A(\alpha(t),k)=\frac{1}{\Gamma(k+1-\alpha(t))}\times\left[1+\sum_{p=n+1-k}^N\frac{\Gamma(p-n+\alpha(t))}{\Gamma(\alpha(t)-k)(p-n+k)!}\right]\quad k=0,\ldots,n, $$

$$ B(\alpha(t),k)=\frac{\Gamma(k-n+\alpha(t))}{\Gamma(-\alpha(t))\Gamma(1+\alpha(t))(k-n)!}\quad k=n+1,\ldots,N, $$

$$ S_2(t)=\frac{x(t)\dot{\alpha}(t)}{\Gamma(1-\alpha(t))}(t-a)^{1-\alpha(t)}\times\left[\frac{\ln(t-a)}{1-\alpha(t)}-\frac{1}{(1-\alpha(t))^2}-\ln(t-a)\sum_{k=0}^N\binom{-\alpha(t)}{k}\frac{(-1)^k}{k+1}\right. $$

$$+\sum_{k=0}^{N}\binom{-\alpha(t)}{k}(-1)^{k}\sum_{p=1}^{N}\frac{1}{p(k+p+1)}\Bigg]+\frac{\dot{\alpha}(t)}{\Gamma(1-\alpha(t))}(t-a)^{1-\alpha(t)}\times\Bigg[\ln(t-a)\sum_{k=n+1}^{N+n+1}\binom{-\alpha(t)}{k-n-1}$$

$$\times\frac{(-1)^{k-n-1}}{k-n}(t-a)^{n-k}V_k(t)-\sum_{k=n+1}^{N+n+1}\binom{-\alpha(t)}{k-n-1}(-1)^{k-n-1}\times\sum_{p=1}^{N}\frac{1}{p(k+p-n)}(t-a)^{n-k-p}V_{k+p}(t)\Bigg].$$

The error of the approximation ${}_aD_t^{\alpha(t)}x(t)\approx S_1(t)-S_2(t)$ is given by $E_{1,N}(t)+E_{2,N}(t)$, where $E_{1,N}(t)$ and $E_{2,N}(t)$ are bounded by

$$\left|E_{1,N}(t)\right|\le L_{n+1}(t)\times\frac{\exp((n-\alpha(t))^2+n-\alpha(t))}{\Gamma(n+1-\alpha(t))(n-\alpha(t))N^{n-\alpha(t)}}\times(t-a)^{n+1-\alpha(t)},$$

$$\left|E_{2,N}(t)\right|\le\frac{L_1(t)\times\left|\alpha^{(1)}(t)\right|(t-a)^{2-\alpha(t)}\exp(\alpha^2(t)-\alpha(t))}{\Gamma(2-\alpha(t))N^{1-\alpha(t)}}\times\left[\left|\ln(t-a)\right|+\frac{1}{N}\right],$$

With

$L_j(t)=\max_{\tau\in[a,t]}\left|x^{(j)}(\tau)\right|,\ j\in\{1,n+1\}.$

Theorem 2.2 [2] *Fix n ∈N and* $N\ge n+1$, *and let* $x(\cdot)\in C^{n+1}([a,b],R)$. *Define the (right) moment of x with the order k by*

$$W_k(t)=(k-n)\int_t^b(b-\tau)^{k-n-1}x(\tau)d\tau.$$

Then,

$${}_tD_b^{\alpha(t)}x(t)=S_1(t)+S_2(t)+E_{1,N}(t)+E_{2,N}(t),$$

Where

$$S_1(t)=(b-t)^{-\alpha(t)}\left[\sum_{k=0}^{n}A(\alpha(t),k)(b-t)^k x^{(k)}(t)+\sum_{k=n+1}^{N}B(\alpha(t),k)(b-t)^{n-k}W_k(t)\right],$$

$$A(\alpha(t),k)=\frac{(-1)^k}{\Gamma(k+1-\alpha(t))}\times\left[1+\sum_{p=n+1-k}^{N}\frac{\Gamma(p-n+\alpha(t))}{\Gamma(\alpha(t)-k)(p-n+k)!}\right]\quad k=0,\ldots,n,$$

$$B(\alpha(t),k)=\frac{(-1)^{n+1}\Gamma(k-n+\alpha(t))}{\Gamma(-\alpha(t))\Gamma(1+\alpha(t))(k-n)!}\quad k=n+1,\ldots,N,$$

$$S_2(t)=\frac{x(t)\dot{\alpha}(t)}{\Gamma(1-\alpha(t))}(b-t)^{1-\alpha(t)}\times\left[\frac{\ln(b-t)}{1-\alpha(t)}-\frac{1}{(1-\alpha(t))^2}-\ln(b-t)\sum_{k=0}^{N}\binom{-\alpha(t)}{k}\frac{(-1)^k}{k+1}\right.$$

$$\left.+\sum_{k=0}^{N}\binom{-\alpha(t)}{k}(-1)^{k}\sum_{p=1}^{N}\frac{1}{p(k+p+1)}\right]+\frac{\dot{\alpha}(t)}{\Gamma(1-\alpha(t))}(b-t)^{1-\alpha(t)}\times\Bigg[\ln(b-t)\sum_{k=n+1}^{N+n+1}\binom{-\alpha(t)}{k-n-1}$$

$$\times\frac{(-1)^{k-n-1}}{k-n}(b-t)^{n-k}W_k(t)-\sum_{k=n+1}^{N+n+1}\binom{-\alpha(t)}{k-n-1}(-1)^{k-n-1}\times\sum_{p=1}^{N}\frac{1}{p(k+p-n)}(b-t)^{n-k-p}W_{k+p}(t)\Bigg].$$

The error of the approximation ${}_tD_b^{\alpha(t)}x(t)\approx S_1(t)+S_2(t)$ is given by $E_{1,N}(t)+E_{2,N}(t)$, where $E_{1,N}(t)$ and $E_{2,N}(t)$ are bounded by

$$\left|E_{1,N}(t)\right|\le L_{n+1}(t)\times\frac{\exp\left((n-\alpha(t))^2+n-\alpha(t)\right)}{\Gamma(n+1-\alpha(t))(n-\alpha(t))N^{n-\alpha(t)}}\times(b-t)^{n+1-\alpha(t)},$$

$$\left|E_{2,N}(t)\right| \le \frac{L_1(t)\times\left|\alpha^{(1)}(t)\right|(b-t)^{2-\alpha(t)}\exp\left(\alpha^2(t)-\alpha(t)\right)}{\Gamma(2-\alpha(t))N^{1-\alpha(t)}}\times\left[\left|\ln(b-t)\right|+\frac{1}{N}\right],$$

With

$$L_j(t)=\max_{\tau\in[a,t]}\left|x^{(j)}(\tau)\right|,\ j\in\{1,n+1\}.$$

It can derive the similar formulas by Theorems 2.1 and 2.2 for the left and right Riemann-Liouville V-OFIs.

Theorem 2.3 [2] Fix n ∈N and $N\ge n+1$*, and let* $x(\cdot)\in C^{n+1}([a,b],R)$*. Then,*

$${}_aI_t^{\alpha(t)}x(t)=(t-a)^{\alpha(t)}\left[\sum_{k=0}^{n}A(\alpha(t),k)(t-a)^k x^{(k)}(t)+\sum_{k=n+1}^{N}B(\alpha(t),k)(t-a)^{n-k}V_k(t)\right]+E_N(t),$$

where

$$A(\alpha(t),k)=\frac{1}{\Gamma(k+1+\alpha(t))}\times\left[1+\sum_{p=n+1-k}^{N}\frac{\Gamma(p-n-\alpha(t))}{\Gamma(-\alpha(t)-k)(p-n+k)!}\right]\quad k=0,\ldots,n,$$

$$B(\alpha(t),k)=\frac{\Gamma(k-n-\alpha(t))}{\Gamma(\alpha(t))\Gamma(1-\alpha(t))(k-n)!}\quad k=n+1,\ldots,N.$$

A bound for the error $E_N(t)$ is as follows:

$$\left|E_N(t)\right|\le L_{n+1}(t)\times\frac{\exp\left((n+\alpha(t))^2+n+\alpha(t)\right)}{\Gamma(n+1+\alpha(t))(n+\alpha(t))N^{n+\alpha(t)}}\times(t-a)^{n+1+\alpha(t)}.$$

Theorem 2.4 [2] Fix n ∈N and $N\ge n+1$*, let* $x(\cdot)\in C^{n+1}([a,b],R)$*. Then,*

$${}_tI_b^{\alpha(t)}x(t)=(b-t)^{\alpha(t)}\left[\sum_{k=0}^{n}A(\alpha(t),k)(b-t)^k x^{(k)}(t)+\sum_{k=n+1}^{N}B(\alpha(t),k)(b-t)^{n-k}W_k(t)\right]+E_N(t),$$

where

$$A(\alpha(t),k)=\frac{(-1)^k}{\Gamma(k+1+\alpha(t))}\times\left[1+\sum_{p=n+1-k}^{N}\frac{\Gamma(p-n-\alpha(t))}{\Gamma(-\alpha(t)-k)(p-n+k)!}\right]\quad k=0,\ldots,n,$$

$$B(\alpha(t),k)=\frac{(-1)^{k+1}\Gamma(k-n-\alpha(t))}{\Gamma(\alpha(t))\Gamma(1-\alpha(t))(k-n)!}\quad k=n+1,\ldots,N.$$

A bound for the error $E_N(t)$ is given by

$$\left|E_N(t)\right|\le L_{n+1}(t)\times\frac{\exp\left((n+\alpha(t))^2+n-\alpha(t)\right)}{\Gamma(n+1+\alpha(t))(n+\alpha(t))N^{n+\alpha(t)}}\times(b-t)^{n+1+\alpha(t)}.$$

Theorem 2.5 (Relation between the fractional integrals of type A and type B)
Let $0<\alpha(t)<1$ *and* $f(\cdot),g(\cdot)\in C([a,b],R)$*. Then,*

$$\int_a^b g(t)\,{}_aI_t^{\alpha(t)}f(t)dt=\int_a^b f(t)\cdot{}_t^BI_b^{\alpha(t)}g(t)dt. \tag{9}$$

Proof By applying the Fubini's theorem or Dirichelet's formula and changing the independent variable of the integrand, we get

$$\int_a^b g(t)\,{}_aI_t^{\alpha(t)} f(t)dt = \int_a^b (\int_a^t g(t) f(\tau) \frac{1}{\Gamma(\alpha(t))}(t-\tau)^{\alpha(t)-1} d\tau)dt =$$

$$= \int_a^b (\int_\tau^b g(t) f(\tau) \frac{1}{\Gamma(\alpha(t))}(t-\tau)^{\alpha(t)-1} dt)d\tau = \int_a^b f(\tau)\,{}_\tau^B I_b^{\alpha(\tau)} g(\tau)d\tau = \int_a^b f(t)\,{}_t^B I_b^{\alpha(t)} g(t)dt.$$

Theorem 2.6 (Relation between the fractional derivatives of type A and type B)

Let $0<\alpha(t)<1$, *if* $f(\cdot),\alpha(\cdot)$ *and* ${}_t^B I_b^{1-\alpha(t)} g(\cdot) \in AC([a,b],R)$. *Then,*

$$\int_a^b g(t)\,{}_a^C D_t^{\alpha(t)} f(t)dt = f(t)\,{}_t^B I_b^{1-\alpha(t)} g(t)\Big|_a^b + \int_a^b f(t)\cdot {}_t^B D_b^{\alpha(t)} g(t)dt. \tag{10}$$

Proof By definitions, it follows that ${}_a^C D_t^{\alpha(t)} f(t) = {}_aI_t^{1-\alpha(t)} \frac{d}{dt} f(t)$.

Applying theorem 2.7 and integration by parts for classical integer derivatives, we obtain

$$\int_a^b g(t)\,{}_a^C D_t^{\alpha(t)} f(t)dt = \int_a^b g(t)\,{}_aI_t^{1-\alpha(t)} \frac{d}{dt} f(t)dt = \int_a^b \frac{d}{dt} f(t)\,{}_t^B I_b^{1-\alpha(t)} g(t)dt =$$

$$f(t)\,{}_t^B I_b^{1-\alpha(t)} g(t)\Big|_a^b - \int_a^b f(t)\cdot \frac{d}{dt}\,{}_t^B I_b^{1-\alpha(t)} g(t)dt = f(t)\,{}_t^B I_b^{1-\alpha(t)} g(t)\Big|_a^b + \int_a^b f(t)\cdot {}_t^B D_b^{\alpha(t)} g(t)dt.$$

3. Necessary Conditions of Optimality and Approximate Solution Methods of V-Ofocp

3.1. Necessary Conditions of Optimality

In this section, we derive necessary optimality conditions of a type of V-OFOCP with the dynamic system described as a binomial V-OFDE in the sense of Caputo.

Let $\alpha \in C^1([0,\infty),]0,1[)$ and $L, f:[0,+\infty)\times R^2 \to R$ be two differentiable functions. Subject to the control system:

$$M_1\dot{x}(t) + M_2\,{}_a^C D_t^{\alpha(t)} x(t) = f(t,x(t),u(t)) \tag{11}$$

with $(M_1,M_2)\neq(0,0)$ and the variables *a*, *T*, *x*(*a*), *x(T)* with *a*<*T*<∞, we consider the V-OFOCP until FFE of which the cost functional is as the following:

$$J(x,u,a,T) = \int_a^T L(t,x(t),u(t))dt + \Phi(a,x(a),T,x(T)) \to \min \tag{12}$$

We assume that the state variable *x* is differentiable and the control variable *u* is continuous.

Initial time *a*, final time *T*, initial state *x*(*a*) and final state *x*(*T*) are unknown generally.

Theorem 3.1 If (x, u, a, T) is a minimizer of (12) under the dynamic system (11), then there exist a function λ *for which the* (x,u,a,T), $t\in[a,T]$ *satisfies:*

the Hamiltonian system

$$\begin{cases} M_1\dot{\lambda}(t) - M_2\,{}_t^B D_T^{\alpha(t)}\lambda(t) = -\dfrac{\partial H}{\partial x} \\ M_1\dot{x}(t) + M_2\,{}_a^C D_t^{\alpha(t)} x(t) = \dfrac{\partial H}{\partial \lambda}, \end{cases} \tag{13}$$

the stationary condition

$$\frac{\partial H(t,x(t),\lambda(t),u(t))}{\partial u} = 0, \tag{14}$$

and the transversality conditions, that is to say,

the transversality with respect to x

$$\left[M_1\lambda(t) + M_2\,{}_t^B I_T^{1-\alpha(t)}\lambda(t) + \frac{\partial\Phi(t,x(t),T,x(T))}{\partial x}\right]_{t=a} = 0,$$

$$\left[M_1\lambda(t)+M_2\,{}_t^BI_T^{1-\alpha(t)}\lambda(t)-\frac{\partial\Phi(a,x(a),t,x(t))}{\partial x}\right]_{t=T}=0, \tag{15}$$

the transversality with respect to t

$$\left[H(t,x,u,\lambda)-M_2\lambda(t)\,{}_a^CD_t^{\alpha(t)}x(t)+M_1\dot{x}(t)\,{}_t^BI_T^{1-\alpha(t)}\lambda(t)-\frac{\partial\Phi(t,x(t),T,x(T))}{\partial t}\right]_{t=a}=0,$$

$$\left[H(t,x,u,\lambda)-M_2\lambda(t)\,{}_a^CD_t^{\alpha(t)}x(t)+M_1\dot{x}(t)\,{}_t^BI_T^{1-\alpha(t)}\lambda(t)+\frac{\partial\Phi(a,x(a),t,x(t))}{\partial t}\right]_{t=T}=0. \tag{16}$$

Proof: we use Lagrange's indefinite multiplier method to adjoin the dynamic constraint to the cost functional. To start, we define the Hamiltonian H by

$$H(t,x,u,\lambda)=L(t,x,u)+\lambda f(t,x,u),$$

Where λ is a Lagrange multiplier, so that, we can rewrite the original problem as minimum problem without a constraint

$$J(x,u,a,T,\lambda)=\int_a^T H-\lambda\left[M_1\dot{x}+M_2\,{}_a^CD_t^{\alpha(t)}x\right]dt+\Phi(a,x(a),T,x(T)).$$

Now, we consider variations $x+\delta x,\ u+\delta u,\ a+\delta a,\ T+\delta T,\ \lambda+\delta\lambda$.

As the first variation of J must vanish when evaluated along the minimizer, we get

$$0=\int_a^T\left[\frac{\partial H}{\partial x}\delta x+\frac{\partial H}{\partial u}\delta u+\frac{\partial H}{\partial\lambda}\delta\lambda-\delta\lambda\left(M_1\dot{x}(t)+M_2\,{}_a^CD_t^{\alpha(t)}x(t)\right)-\lambda(t)\left(M_1\delta\dot{x}(t)+M_2\,{}_a^CD_t^{\alpha(t)}\delta x(t)\right)\right]dt$$
$$+\delta T\left[H(t,x,u,\lambda)-\lambda(t)(M_1\dot{x}(t)+M_2\,{}_a^CD_t^{\alpha(t)}x(t)\right]_{t=T}-\delta a\left[H(t,x,u,\lambda)-\lambda(t)(M_1\dot{x}(t)+M_2\,{}_a^CD_t^{\alpha(t)}x(t)\right]_{t=a}$$
$$+\frac{\partial\Phi(a,x(a),T,x(T))}{\partial a}\delta a+\frac{\partial\Phi(a,x(a),T,x(T))}{\partial x}\left(\dot{x}(a)\delta a+\delta x(a)\right)$$
$$+\frac{\partial\Phi(a,x(a),T,x(T))}{\partial T}\delta T+\frac{\partial\Phi(a,x(a),T,x(T))}{\partial x}\left(\dot{x}(T)\delta T+\delta x(T)\right).$$

Integral formulas by parts on integer and variable-order fractional operators (Theorem 2.6) give the relations

$$\int_a^T\lambda(t)\delta\dot{x}(t)dt=-\int_a^T\delta x(t)\dot{\lambda}(t)dt+\delta x(T)\lambda(T)-\delta x(a)\lambda(a),$$

$$\int_a^T\lambda(t)\,{}_a^CD_t^{\alpha(t)}\delta x(t)dt=\int_a^T\delta x(t)\,{}_t^BD_T^{\alpha(t)}\lambda(t)dt+\delta x(T)\left[{}_t^BI_T^{1-\alpha(t)}\lambda(t)\right]_{t=T}-\delta x(a)\left[{}_t^BI_T^{1-\alpha(t)}\lambda(t)\right]_{t=a}.$$

Therefore, we deduce the following expression:

$$\int_a^T\left[\delta x\left(\frac{\partial H}{\partial x}+M_1\dot{\lambda}(t)-M_2\,{}_t^BD_T^{\alpha(t)}\lambda(t)\right)+\delta u\frac{\partial H}{\partial u}+\delta\lambda\left(\frac{\partial H}{\partial\lambda}-M_1\dot{x}(t)-M_2\,{}_a^CD_t^{\alpha(t)}x(t)\right)\right]dt$$
$$-\delta x(T)\left[M_1\lambda(t)+M_2\,{}_t^BI_T^{1-\alpha(t)}\lambda(t)-\frac{\partial\Phi(a,x(a),T,x(T))}{\partial x}\right]_{t=T}+\delta x(a)\left[M_1\lambda(t)+M_2\,{}_t^BI_T^{1-\alpha(t)}\lambda(t)+\frac{\partial\Phi(a,x(a),T,x(T))}{\partial x}\right]_{t=a}$$
$$+\delta T\left[H(t,x,u,\lambda)-\lambda(t)\left(M_1\dot{x}(t)+M_2\,{}_a^CD_t^{\alpha(t)}x(t)\right)+\frac{\partial\Phi(a,x(a),T,x(T))}{\partial T}+\frac{\partial\Phi(a,x(a),T,x(T))}{\partial x}\dot{x}(t)\right]_{t=T}$$
$$-\delta a\left[H(t,x,u,\lambda)-\lambda(t)\cdot\left(M_1\dot{x}(t)+M_2\,{}_a^CD_t^{\alpha(t)}x(t)\right)-\frac{\partial\Phi(a,x(a),T,x(T))}{\partial a}-\frac{\partial\Phi(a,x(a),T,x(T))}{\partial x}\dot{x}(t)\right]_{t=a}.$$

Now, we define the new variables $\delta x_a=(x+\delta x)(a+\delta a)-x(a)$, $\delta x_T=(x+\delta x)(T+\delta T)-x(T)$.

Because $\dot{\delta}x(a),\dot{\delta}x(T)$ are arbitrary, in particular, we can consider variation functions for which $\dot{\delta}x(a)=0,\dot{\delta}x(T)=0$. By

Taylor's theorem,

$$(x+\delta x)(a+\delta a)-(x+\delta x)(a)=(x+\delta x)'(a)\delta a+O(\delta a^2)=\left[\dot{x}(a)+\delta\dot{x}(a)\right]\delta a+O(\delta a^2)$$
$$=\dot{x}(a)\delta a+\delta\dot{x}(a)\delta a+O(\delta a^2)=\dot{x}(a)\delta a+O(\delta a^2),$$
$$(x+\delta x)(T+\delta T)-(x+\delta x)(T)=(x+\delta x)'(T)\delta T+O(\delta T^2)=\left[\dot{x}(T)+\dot{\delta}x(T)\right]\delta T+O(\delta T^2)$$
$$=\dot{x}(T)\delta T+\dot{\delta}x(T)\delta T+O(\delta T^2)=\dot{x}(T)\delta T+O(\delta T^2).$$

and

$$\delta x_a=(x+\delta x)(a+\delta a)-(x+\delta x)(a)+(x+\delta x)(a)-x(a)=\delta x(a)+\dot{x}(a)\delta a+O(\delta a^2)$$
$$\delta x_T=(x+\delta x)(T+\delta T)-(x+\delta x)(T)+(x+\delta x)(T)-x(T)=\delta x(T)+\dot{x}(T)\delta T+O(\delta T^2)$$

Thus, $\delta x(a)=\delta x_a-\dot{x}(a)\delta a+O(\delta a^2)$, $\delta x(T)=\delta x_T-\dot{x}(T)\delta T+O(\delta T^2)$.

Finally, we arrive at the following expression

$$\int_a^T\left[\delta x\left(\frac{\partial H}{\partial x}+M_1\dot{\lambda}(t)-M_2\,{}_t^BD_T^{\alpha(t)}\lambda(t)\right)+\delta\lambda\left(\frac{\partial H}{\partial\lambda}-M_1\dot{x}(t)-M_2\,{}_a^CD_t^{\alpha(t)}x(t)\right)+\delta u\frac{\partial H}{\partial u}\right]dt$$
$$+\delta x_T\cdot(-1)\left[M_1\lambda(t)+M_2\,{}_t^BI_T^{1-\alpha(t)}\lambda(t)-\frac{\partial\Phi(a,x(a),T,x(T))}{\partial x}\right]_{t=T}$$
$$+\delta x_a\left[M_1\lambda(t)+M_2\,{}_t^BI_T^{1-\alpha(t)}\lambda(t)+\frac{\partial\Phi(a,x(a),T,x(T))}{\partial x}\right]_{t=a}$$
$$+\delta T\left[H(t,x,u,\lambda)-M_2\lambda(t)\,{}_a^CD_t^{\alpha(t)}x(t)+M_1\dot{x}(t)\,{}_t^BI_T^{1-\alpha(t)}\lambda(t)+\frac{\partial\Phi(a,x(a),T,x(T))}{\partial T}\right]_{t=T}$$
$$-\delta a\left[H(t,x,u,\lambda)-M_2\lambda(t)\,{}_a^CD_t^{\alpha(t)}x(t)+M_1\dot{x}(t)\,{}_t^BI_T^{1-\alpha(t)}\lambda(t)-\frac{\partial\Phi(a,x(a),T,x(T))}{\partial a}\right]_{t=a}$$
$$+O(\delta a^2)+O(\delta T^2)=0.$$

From the condition of the theorem, the first variation of J $\delta J(a,T,x,u,\lambda)[\delta a,\delta T,\delta x,\delta u,\delta\lambda]=0$ holds, thus, for arbitrary $\delta a,\delta T,\delta x,\delta\lambda,\delta u$ and $\delta x_a,\delta x_T$, three integrands and four multipliers must be zero. Finally, the necessary conditions of optimality are derived and the theorem is proved.

Remark 1 If $\alpha(t)$=1 and (M_1, M_2)=(1, 0), then we can obtain the necessary optimality conditions for the classical OCP.

Remark 2 From the theorem 3.1, we can simply know that if T is fixed and $x(T)$ is free, then the transversality condition holds with

$$\left[M_1\lambda(t)+M_2\,{}_t^BI_T^{1-\alpha(t)}\lambda(t)-\frac{\partial\Phi(t,x(t))}{\partial x}\right]_{t=T}=0,$$

and if $x(T)$ is fixed and T is free, then the transversality condition holds with

$$\left[H(t,x(t),u(t),\lambda(t))-M_2\lambda(t)\,{}_a^CD_t^{\alpha(t)}x(t)+M_2\dot{x}(t)\,{}_t^BI_T^{1-\alpha(t)}\lambda(t)+\frac{\partial\Phi(t,x(t))}{\partial t}\right]_{t=T}=0.$$

Furthermore, if the a, T, $x(a)$ and $x(T)$ are fixed simultaneously, then theorem 3.1 holds with no transversality conditions.

3.2. Approximation Solution Methods of V-OFOCP

We answer V-OFOCP numerically in two ways. We solve it by utilizing V-O fractional necessary conditions and then applying approximation formulas of V-O operators ("optimize first, then discretize" (OFTD)) or by transforming the problem into a classical integer-order OCP and then deriving the optimality necessary conditions of approximated integer-order OCP ("discretize first, then optimize" (DFTO)). With the second method, we can solve various types of V-OFOCPs in the sense of Caputo or Riemann-Liouville.

In this section, we consider the following V-OFOCP in the sense of Riemann-Liouville (or Caputo) subject to the control system and the initial boundary condition

$$M_1\dot{x}(t)+M_2\,{}_aD_t^{\alpha(t)}x(t)=f(t,x(t),u(t)),$$

$$\text{or}\quad M_1\dot{x}(t)+M_2\,{}_a^CD_t^{\alpha(t)}x(t)=f(t,x(t),u(t)),\tag{17}$$

$$x(a)=x_a\tag{18}$$

with $(M_1, M_2) \neq (0,0)$ and fixed real number x_a, the cost functional is

$$J(x,u,T) = \int_a^T L(t, x(t), u(t))dt + \Phi(T, x(T)) \to \min . \quad (19)$$

First Method (*OFTD*)

The optimality condition (13) is a system of variable-order fractional nonlinear ordinary differential equations. Therefore, if we exchange V-O fractional operators by Grünwald-Letnikov operators and take the fact that the order function of the first terms in (17) is $\alpha(t) \equiv 1$ into account, we get the following finite difference equations.

$$\begin{cases} M_1 \, {}_a^{GL}D_t^1 x(t) + M_2 \, {}_a^{GL}D_t^{\alpha(t)} x(t) = \dfrac{\partial H}{\partial \lambda} = F(t, x(t), \lambda(t), u(t)) \\ M_1 \, {}_t^{B,GL}D_T^1 \lambda(t) - M_2 \, {}_t^{B,Gl}D_T^{\alpha(t)} \lambda(t) = -\dfrac{\partial H}{\partial x} = G(t, x(t), \lambda(t), u(t)) \end{cases}.$$

Here, we defined functions F( ), G( ) as follows:

$$F(t, x(t), \lambda(t), u(t)) := \frac{\partial H}{\partial \lambda}, \ G(t, x(t), \lambda(t), u(t)) := -\frac{\partial H}{\partial x}.$$

Then, the difference equation at each point t_i of the uniform grid points $a=t_0<t_1<t_2<\ldots<t_N=T$ is transformed into the following nonlinear system consisting of the backward difference and the forward difference:

$$\begin{cases} M_1 \sum_{k=0}^{m} \omega_k^1 x(t_m - kh) + M_2 \sum_{k=0}^{m} \omega_k^{\alpha(t)} x(t_m - kh) = F_m(t_m, x(t_m), \lambda(t_m), u(t_m)) \\ M_1 \sum_{k=0}^{N-m} \varpi_k^1 \lambda(t_m + kh) - M_2 \sum_{k=0}^{N-m} \varpi_k^{\alpha(t)} \lambda(t_m + kh) = G_m(t_m, x(t_m), \lambda(t_m), u(t_m)) \end{cases}, \quad m = 0,1,2,\cdots,N.$$

Now, we use Podlubny's matrix approach to get the following

$$\begin{cases} M_1 \begin{pmatrix} \omega_0^1 & & & \\ \omega_1^1 & \omega_0^1 & & \\ \vdots & \vdots & \ddots & \\ \omega_N^1 & \omega_{N-1}^1 & \cdots & \omega_0^1 \end{pmatrix} \begin{Bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{Bmatrix} + M_2 \begin{pmatrix} \omega_0^{\alpha(t)} & & & \\ \omega_1^{\alpha(t)} & \omega_0^{\alpha(t)} & & \\ \vdots & \vdots & \ddots & \\ \omega_N^{\alpha(t)} & \omega_{N-1}^{\alpha(t)} & \cdots & \omega_0^{\alpha(t)} \end{pmatrix} \begin{Bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{Bmatrix} = \begin{Bmatrix} F_0 \\ F_1 \\ \vdots \\ F_N \end{Bmatrix} \\ M_1 \begin{pmatrix} \varpi_0^1 & \varpi_1^1 & \cdots & \varpi_N^1 \\ & \varpi_0^1 & \cdots & \varpi_{N-1}^1 \\ & & \ddots & \vdots \\ & & & \varpi_0^1 \end{pmatrix} \begin{Bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_N \end{Bmatrix} + M_2 \begin{pmatrix} \varpi_0^{\alpha(t)} & \varpi_1^{\alpha(t)} & \cdots & \varpi_N^{\alpha(t)} \\ & \varpi_0^{\alpha(t)} & \cdots & \varpi_{N-1}^{\alpha(t)} \\ & & \ddots & \vdots \\ & & & \varpi_0^{\alpha(t)} \end{pmatrix} \begin{Bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_N \end{Bmatrix} = \begin{Bmatrix} G_0 \\ G_1 \\ \vdots \\ G_N \end{Bmatrix} \end{cases}, \quad (20)$$

Where $\omega_k^{\alpha(t)} = (-1)^k h^{-\alpha(t_m)} \begin{pmatrix} \alpha(t_m) \\ k \end{pmatrix}$, $\varpi_k^{\alpha(t)} = (-1)^k h^{-\alpha(kh)} \begin{pmatrix} \alpha(kh) \\ k \end{pmatrix}$ $m = 0,1,2,\cdots,N$.

Consequently, we get a system of nonlinear equation with 2N+2 unknown variables

$X = [x_0, x_1, \cdots, x_N, \lambda_0, \lambda_1, \cdots, \lambda_N]^T$, which can solve by Newton method or secant method.

Remark 3 V-OFOCPs, that have single or multi-term dynamic system with not only left V-OFDs but also right V-OFDs and homogeneous or inhomogeneous initial conditions, can solve to this method.

Second Method (DFTO)

Using the theorem 2.1 (or Theorem 2.5) in Section 2, we can transform the V-OFOCP (17) ~ (19) into the following classical OCP:

$$J(x,u,T)=\int_a^T L(t,x(t),u(t))dt+\Phi(T,x(T)),$$

$$[M_1+M_2A(\alpha(t),1)(t-a)^{1-\alpha(t)}]\dot{x}(t)=f(t,x(t),u(t))+\left[\frac{M_2\dot{\alpha}(t)}{\Gamma(1-\alpha(t))}(t-a)^{1-\alpha(t)}K_1\right.$$

$$\left.-M_2A(\alpha(t),0)(t-a)^{-\alpha(t)}\right]x(t)-M_2\sum_{k=2}^{N}B(\alpha(t),k)(t-a)^{1-k-\alpha(t)}V_k(t)$$

$$+\frac{M_2\dot{\alpha}(t)}{\Gamma(1-\alpha(t))}(t-a)^{1-\alpha(t)}\left[\ln(t-a)\sum_{k=2}^{N+2}\binom{-\alpha(t)}{k-2}\frac{(-1)^{k-2}}{k-1}(t-a)^{1-k}V_k(t)-\sum_{k=2}^{N+2}\binom{-\alpha(t)}{k-2}(-1)^{k-2}\right.$$

$$\left.\cdot\sum_{p=1}^{N}\frac{1}{p(k+p-1)}(t-a)^{1-k-p}V_{k+p}(t)\right],$$

$$\dot{V}_k(t)=(k-1)(t-a)^{k-2}x(t),\quad k=2,\cdots,2N+2,$$

$$\begin{cases}x(a)=x_a\\ V_k(a)=0, k=2,\cdots,2N+2\end{cases},$$

Where

$$A(\alpha(t),k)=\frac{1}{\Gamma(k+1-\alpha(t))}\left[1+\sum_{p=2-k}^{N}\frac{\Gamma(p+1-\alpha(t))}{\Gamma(\alpha(t)-k)(p-1+k)!}\right], k=0,1,$$

$$B(\alpha(t),k)=\frac{\Gamma(k-1+\alpha(t))}{\Gamma(-\alpha(t))\Gamma(1+\alpha(t))(k-1)!}, k=2,...,N,$$

$$K_1=\frac{\ln(t-a)}{1-\alpha(t)}-\frac{1}{[1-\alpha(t)]^2}-\ln(t-a)\sum_{k=0}^{N}\binom{-\alpha(t)}{k}\frac{(-1)^k}{k+1}+\sum_{k=0}^{N}\binom{-\alpha(t)}{k}(-1)^k\sum_{k=1}^{N}\frac{1}{p(k+p+1)}.$$

Now, it follows the classical procedure, by defining the Hamiltonian H by

$$\mathrm{H}=L(t,x(t),u(t))+\lambda_1\cdot\left\{f(t,x,u)+\left[\frac{M_2\dot{\alpha}(t)}{\Gamma(1-\alpha(t))}(t-a)^{1-\alpha(t)}K_1-M_2A(\alpha(t),0)(t-a)^{-\alpha(t)}\right]x(t)-\right.$$

$$-M_2\sum_{k=2}^{N}B(\alpha(t),k)(t-a)^{1-k-\alpha(t)}V_k(t)+\frac{M_2\dot{\alpha}(t)}{\Gamma(1-\alpha(t))}(t-a)^{1-\alpha(t)}\left[\ln(t-a)\sum_{k=2}^{N+2}\binom{-\alpha(t)}{k-2}\cdot\frac{(-1)^{k-2}}{k-1}(t-a)^{1-k}V_k(t)\right.$$

$$\left.\left.-\sum_{k=2}^{N+2}\binom{-\alpha(t)}{k-2}(-1)^{k-2}\sum_{p=1}^{N}\frac{1}{p(k+p-1)}(t-a)^{1-k-p}V_{k+p}(t)\right]\right\}\Big/[M_1+M_2A(\alpha(t),1)(t-a)^{1-\alpha(t)}]$$

$$+\sum_{p=2}^{2N+2}\lambda_p(p-1)(t-a)^{p-2}x(t).$$

Let $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_{2N+2})$ and $X=(x,V_2,\cdots,V_{2N+2})$. Then, the necessary conditions of optimality are

$$\left\{\dot{X}=\frac{\partial H}{\partial\lambda},\quad \dot{\lambda}=-\frac{\partial H}{\partial X},\quad \frac{\partial H}{\partial u}=0.\right.$$

In addition, the transversality conditions for T and x(T) are added. Thus, we can solve the variable order fractional optimal control problem by solving the classical optimal control problem, using the necessary conditions of optimality.

4. Numerical Examples and Results

Example 1 Consider the following V-OFOCP

$$\min J(x,u)=\int_0^1\left[e^t(x(t)-t^4)^2+(1+t^2)^2\left(u(t)+t^4-\frac{24t^{4-\alpha(t)}}{\Gamma(5-\alpha(t))}\right)^2\right]dt \tag{21}$$

subject to the dynamic system

$${}_0^C D_t^{\alpha(t)}x(t)=x(t)+u(t)\,, \tag{22}$$

and the boundary conditions

$$x(0)=0,\;\; x(1)=1. \tag{23}$$

The exact solution that minimize the cost functional J is given by $\left(x^*(t),u^*(t)\right)=\left(t^4, 24t^{4-\alpha(t)}/\Gamma(5-\alpha(t))-t^4\right)$ and the minimum value of the cost functional is $J^*=0$.

We work out this problem with N=3 for the following V-Os

$\alpha_1(t)=0.75-0.25\cos(\pi t)$, $\alpha_2(t)=0.85+0.15t^3$.

First method (OFTD): First, we define the Hamiltonian H by

$$H=e^t(x(t)-t^4)^2+(1+t^2)^2\left(u(t)+t^4-\frac{24t^{4-\alpha(t)}}{\Gamma(5-\alpha(t))}\right)^2+\lambda(t)(x(t)+u(t))\,. \tag{24}$$

According to the theorem 3.1, we obtain the following Hamiltonian system

$$\begin{cases} {}_0^C D_t^{\alpha(t)}x(t)=\dfrac{\partial H}{\partial\lambda}=x(t)+\dfrac{24t^{4-\alpha(t)}}{\Gamma(5-\alpha(t))}-t^4-\dfrac{\lambda(t)}{2(1+t^2)^2} \\ -{}_t^B D_1^{\alpha(t)}\lambda(t)=-\dfrac{\partial H}{\partial x}=-2e^t(x(t)-t^4)-\lambda(t)\,. \end{cases} \tag{25}$$

Using 7 and 8 of the definition and the initial condition, (25) yields the following 2N linear system

$$\begin{cases} \displaystyle\sum_{k=0}^{m}\omega_k^{\alpha(t)}x(t_m-kh)=x(t_m)+\frac{24t_m^{4-\alpha(t_m)}}{\Gamma(5-\alpha(t_m))}-t_m^{\,4}-\frac{\lambda(t_m)}{2(1+t_m^{\,2})^2} \\ \displaystyle\sum_{k=0}^{N-m}\varpi_k^{\alpha(t)}\lambda(t_m+kh)=2e^{t_m}(x(t_m)-t_m^{\,4})+\lambda(t_m) \end{cases},\quad m=0,1,2,\cdots,N. \tag{26}$$

Then, we denote it by Podlubny's matrix approach

$$\begin{pmatrix} \left(\omega_k^{\alpha(t)}\right)_{(N+1)\times(N+1)} & (0)_{(N+1)\times(N+1)} \\ (0)_{(N+1)\times(N+1)} & \left(\varpi_k^{\alpha(t)}\right)_{(N+1)\times(N+1)} \end{pmatrix}_{(2N+2)\times(2N+2)}\times\begin{Bmatrix} x_0\\ \vdots\\ x_N\\ \lambda_0\\ \vdots\\ \lambda_N \end{Bmatrix}=\begin{Bmatrix} F_0\\ \vdots\\ F_N\\ G_0\\ \vdots\\ G_N \end{Bmatrix}, \tag{27}$$

Where $x(t_m) = x_m,\ \lambda(t_m) = \lambda_m,\quad m = 0,1,\cdots,N$.

We solve (27) by using Matlab's *fsolve* built-in function and evaluate the numerical solutions with different step sizes. The maximum absolute errors in the state variable $e_x(t)=|x(t)-x^*(t)|$ and control variable $e_u(t)=|u(t)-u^*(t)|$ are reported in Table 1 and the behaviors of the numerical solutions for state variable $x(t)$ and control variable $u(t)$ are demonstrated in Figures 1 and 2.

Table 1. *Maximum error obtained by first method with different orders and step sizes for example 1.*

Step size (h)	$\alpha_1(t)$=0.75-0.25cos(πt)		$\alpha_2(t)$=0.85+0.15t^3	
	$e_x(t)$	$e_u(t)$	$e_x(t)$	$e_u(t)$
0.02	1.095413e-02	1.332969e-02	1.135268e-02	1.569175e-02
0.01	5.441278e-03	6.938590e-03	5.638499e-03	7.858686e-03
0.005	2.712173e-03	3.538104e-03	2.809468e-03	3.930923e-03

Second method (DFTO): Following the procedure discussed in Sect. 3.2, we obtain the following classical integer-order BVP

$$\begin{cases} \dot{x} = -\dfrac{\varphi^2(t)}{2(1+t^2)^2}\lambda_1 + \varphi(t)x(t) + \left(\dfrac{24t^{4-\alpha(t)}}{\Gamma(5-\alpha(t))} - t^4\right)\varphi(t) - \varphi(t)A(\alpha(t),0)t^{-\alpha(t)}x(t) \\ \qquad\qquad -\varphi(t)\displaystyle\sum_{k=2}^{N} B(\alpha(t),k)(1-t)^{1-k-\alpha(t)}V_k(t) \\ \dot{V}_p = (p-1)t^{p-2}x(t),\quad p = 2,\dots,N \\ \dot{\lambda}_1 = -2e^t(x(t)-t^4) - \varphi(t)\lambda_1(t) + \varphi(t)A(\alpha(t),0)t^{-\alpha(t)}\lambda_1(t) - \varphi(t)\displaystyle\sum_{k=2}^{N}(k-1)t^{k-2}\lambda_k(t) \\ \dot{\lambda}_p = B(\alpha(t),p)t^{1-p-\alpha(t)}\varphi(t)\lambda_1(t),\quad p = 2,\dots,N \end{cases}$$

with the boundary conditions

$$\begin{cases} x(0) = 0 \\ V_p(0) = 0,\quad p = 2,\dots,N, \end{cases} \qquad \begin{cases} x(1) = 1 \\ \lambda_p(1) = 0,\quad p = 2,\dots,N, \end{cases}$$

Where $\varphi(t) = 1/(A(\alpha(t),1)t^{1-\alpha(t)})$. The results reported in Table 2 and Figures 1 and 2.

Table 2. *Absolute errors obtained by second method with N=3 and α(t) in some values of t for example 1.*

T	$\alpha_1(t)$=0.75-0.25cos(πt)		$\alpha_2(t)$=0.85+0.15t^3	
	$e_x(t)$	$e_u(t)$	$e_x(t)$	$e_u(t)$
0.1	3.567367e-03	1.297942e-02	2.119784e-03	1.783134e-02
0.5	8.736848e-03	1.262192e-02	6.175192e-03	1.185386e-02
0.9	1.686719e-02	1.371493e-03	2.093246e-02	1.914894e-03

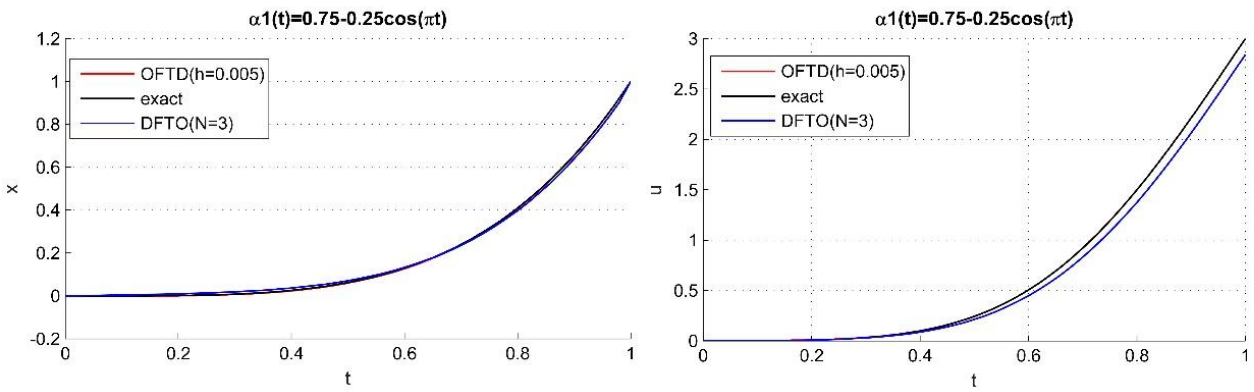

Figure 1. *Comparison between the exact optimal solution and numerical solutions with h=0.005(OFTD), N=3(DFTO) and $\alpha_1(t)$ for example 1.*

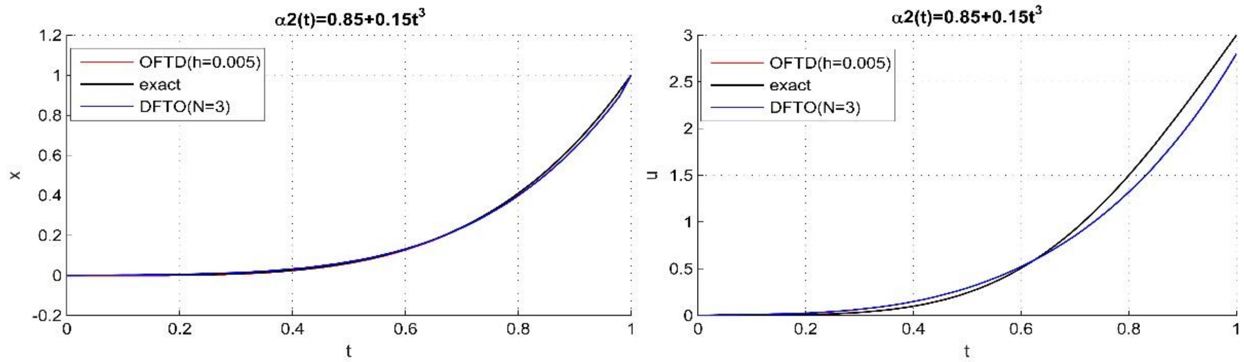

Figure 2. *Comparison between the exact optimal solution and numerical solutions with h=0.005(OFTD), N=3(DFTO) and $\alpha_2(t)$ for example 1.*

Example 2 Consider the following V-OFOCP

$$\min J(x,u)=\int_0^1\left[(x(t)-t^{5/2})^4+(1+t^2)\left(u(t)+t^6-\frac{15\sqrt{\pi}}{8\Gamma(7/2-\alpha(t))}t^{5/2-\alpha(t)}\right)^2\right]dt\,,$$

subject to the dynamic system

$${}_0^C D_t^{\alpha(t)}x(t)=tx^2(t)+u(t)\,,$$

moreover, the boundary conditions $x(0)=0$, $x(1)=1$.

The exact solution is given by $\left(x^*(t),u^*(t)\right)=\left(t^{5/2},\frac{15\sqrt{\pi}}{8\Gamma(7/2-\alpha(t))}t^{5/2-\alpha(t)}-t^6\right)$ and the minimum value of the cost function is $J^*=0$. We will solve this problem with N=3 for some different V-Os which are

$$\alpha_1(t)=0.5+0.25\sin(t),\ \alpha_2(t)=0.5+0.25t^2.$$

First method: Define Hamiltonian

$$H=(x(t)-t^{5/2})^4+(1+t^2)\left(u(t)+t^6-\frac{15\sqrt{\pi}}{8\Gamma(7/2-\alpha(t))}t^{5/2-\alpha(t)}\right)^2+\lambda(t)(tx^2(t)+u(t))\,.$$

According to the theorem 3.1, we obtain the following Hamiltonian system

$$\begin{cases} {}_0^C D_t^{\alpha(t)} x(t) = \dfrac{\partial H}{\partial \lambda} = t x^2(t) + \dfrac{15\sqrt{\pi}}{8\Gamma(7/2-\alpha(t))} t^{5/2-\alpha(t)} - t^6 - \dfrac{\lambda(t)}{2(1+t^2)}, \\ -{}_t^B D_1^{\alpha(t)} \lambda(t) = -\dfrac{\partial H}{\partial x} = -4(x(t)-t^{5/2})^3 - 2t\lambda(t)x(t) \end{cases}$$

$$x(0)=0, \quad x(1)=1.$$

$$\begin{cases} \sum_{k=0}^{m} \omega_k^{\alpha(t)} x(t_m - kh) = t_m x^2(t_m) + \dfrac{15\sqrt{\pi}}{8\Gamma(7/2-\alpha(t_m))} t_m{}^{5/2-\alpha(t_m)} - t_m^6 - \dfrac{\lambda(t_m)}{2(1+t_m{}^2)} \\ \sum_{k=0}^{N-m} \varpi_k^{\alpha(t)} \lambda(t_m + kh) = 4(x(t_m) - t_m{}^{5/2})^3 + 2t_m \lambda(t_m) x(t_m) \end{cases}, \quad m = 0,1,2,\cdots,N.$$

The absolute errors in the state variable $e_x(t)$ and control variable $e_u(t)$ are reported in Table 3 and the behaviors of the numerical solutions for $x(t)$ and $u(t)$ are demonstrated in Figures 3 and 4.

Table 3. Maximum error obtained by first method with different orders and step sizes for example 2.

Step size (h)	$\alpha_1(t)$=0.5+0.25sin(t)		$\alpha_2(t)$= 0.5+0.25t^2	
	$e_x(t)$	$e_u(t)$	$e_x(t)$	$e_u(t)$
0.02	4.671456e-03	3.070821e-03	2.909017e-02	1.097773e-05
0.01	2.327704e-03	1.475230e-03	1.482769e-02	1.357548e-06
0.005	1.158586e-03	7.224332e-04	7.474894e-03	1.691460e-07

Table 4. Absolute errors obtained by second method with N=3 and α(t) in some values of t for example 2.

T	$\alpha_1(t)$=0.5+0.25sin(t)		$\alpha_2(t)$=0.5+0.25t^2	
	$e_x(t)$	$e_u(t)$	$e_x(t)$	$e_u(t)$
0.1	5.591295e-03	1.601909e-04	4.633698e-03	2.318421e-04
0.5	2.620952e-02	1.601974e-04	2.333812e-02	5.533690e-04
0.9	1.331096e-02	1.463594e-04	8.058294e-03	1.050266e-03

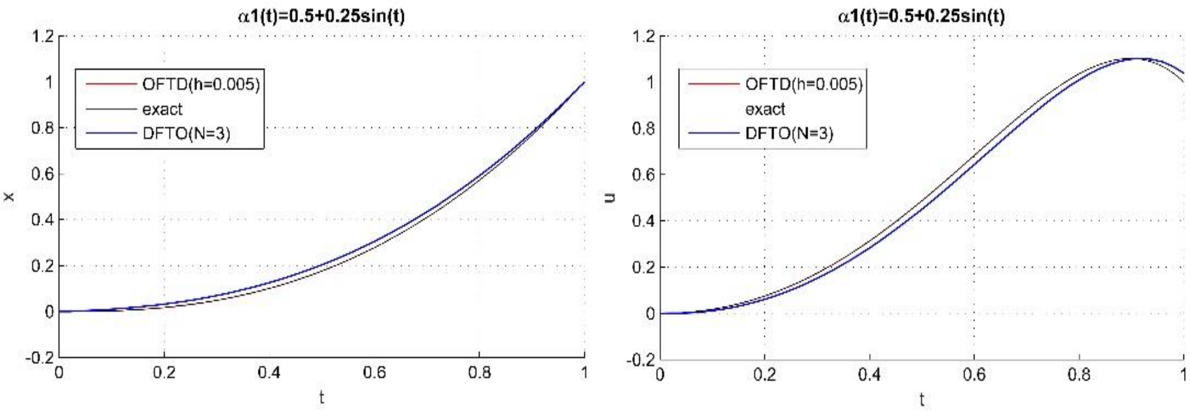

Figure 3. Comparison between the exact optimal solution and numerical solutions with h=0.005(OFTD), N=3(DFTO) and $\alpha_1(t)$ for example 2.

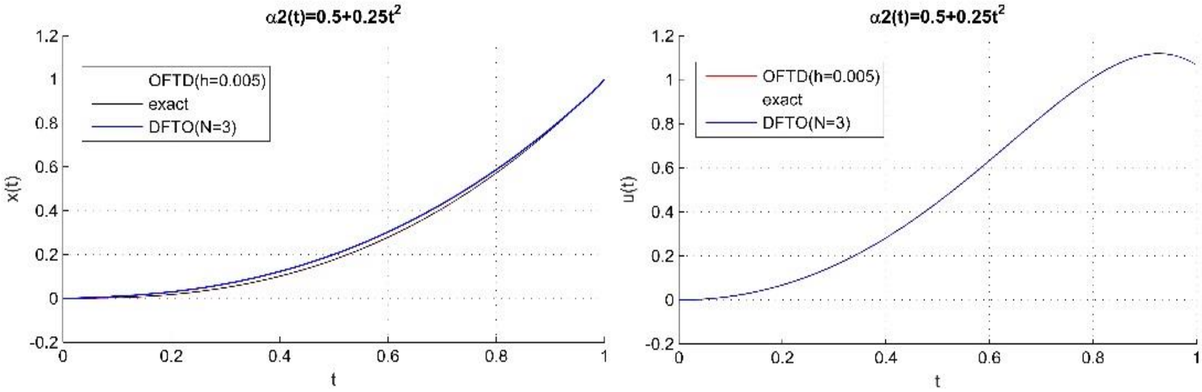

Figure 4. *Comparison between the exact optimal solution and numerical solutions with h=0.005(OFTD), N=3(DFTO) and $\alpha_2(t)$ for example 2.*

Second method: Following the procedure discussed in section 3.2, we obtain the following classical integer-order BVP

$$\begin{cases} \dot{x} = -\dfrac{\varphi^2(t)}{2(1+t^2)}\lambda_1 + t\varphi(t)x^2(t) + \left(\dfrac{15\sqrt{\pi}}{8\Gamma(7/2-\alpha(t))}t^{5/2-\alpha(t)} - t^6\right)\varphi(t) - \varphi(t)A(\alpha(t),0)t^{-\alpha(t)}x(t) \\ \qquad -\displaystyle\sum_{k=2}^{N} B(\alpha(t),k)(1-t)^{1-k-\alpha(t)}V_k(t) \\ \dot{V}_p = (p-1)t^{p-2}x(t), \quad p=2,\dots,N \\ \dot{\lambda}_1 = -4(x(t)-t^{5/2})^3 - 2t\lambda_1(t)\varphi(t)x(t) + \varphi(t)A(\alpha(t),0)t^{-\alpha(t)}\lambda_1(t) - \varphi(t)\displaystyle\sum_{k=2}^{N}(k-1)t^{k-2}\lambda_k(t) \\ \dot{\lambda}_p(t) = B(\alpha(t),p)t^{1-p-\alpha(t)}\varphi(t)\lambda_1(t), \quad p=2,\dots,N \end{cases}$$

with the boundary conditions

$$\begin{cases} x(0)=0 \\ V_p(0)=0, \quad p=2,\dots,N, \end{cases} \qquad \begin{cases} x(1)=1 \\ \lambda_p(1)=0, \quad p=2,\dots,N, \end{cases}$$

Where $\varphi(t) = 1/(A(\alpha(t),1)t^{1-\alpha(t)})$.

The absolute errors in the state and control variables are reported in Table 4 and the behaviors of the numerical solutions for $x(t)$ and $u(t)$ are shown in Figures 3 and 4.

Example 3 [Comparison to 11, 15] Consider the following V-OFOCP

$$\min J(x,u) = \int_0^1 \left[tu(t) - (\alpha(t)+2)x(t)\right]^2 dt, \tag{28}$$

subject to the dynamic system

$$\dot{x}(t) + {}_0^C D_t^{\alpha(t)} x(t) = u(t) + t^2, \tag{29}$$

moreover, the boundary conditions

$$x(0)=0, \quad x(1) = 2/\Gamma(3.5). \tag{30}$$

We will solve this problem with N=3 for the following V-O $\alpha(t)=(t+1)/4$, $0 \le t \le 1$ according to the second method.

According to the procedure discussed in Section 3.2, we obtain the following BVP

$$\begin{cases} \dot{x} = -\dfrac{\varphi^2(t)}{2t^2}\lambda_1 + \dfrac{\varphi(t)(\alpha(t)+2)}{t}x + \varphi(t)A(\alpha(t),0)t^{-\alpha(t)}x + \varphi(t)t^2 - \varphi(t)\displaystyle\sum_{k=2}^{N} B(\alpha(t),k)(1-t)^{1-k-\alpha(t)}V_k(t) \\ \dot{V}_p = (p-1)t^{p-2}x(t), \quad p=2,\dots,N \\ \dot{\lambda}_1 = -\dfrac{\varphi(t)(\alpha(t)+2)}{t}\lambda_1(t) - \varphi(t)A(\alpha(t),0)t^{-\alpha(t)}\lambda_1(t) - \varphi(t)\displaystyle\sum_{k=2}^{N}(k-1)t^{k-2}\lambda_k(t) \\ \dot{\lambda}_p(t) = B(\alpha(t),p)t^{1-p-\alpha(t)}\varphi(t)\lambda_1(t), \quad p=2,\dots,N \end{cases}$$

with the boundary conditions

$$\begin{cases} x(0)=0 \\ V_p(0)=0, \ \ p=2,\dots,N, \end{cases} \qquad \begin{cases} x(1)=2t^{2.5}/\Gamma(3.5) \\ \lambda_p(1)=0, \ \ p=2,\dots,N, \end{cases}$$

Where, $\varphi(t)=1/\left(1+A(\alpha(t),1)t^{1-\alpha(t)}\right)$. The results reported in Figure 5.

Let us consider the case when $\alpha(t)$=1/2 or 1/4. In this case, we can simply know that the exact solution is $\left(\bar{x}(t),\bar{u}(t)\right)=\left(2t^{\alpha+2}/\Gamma(\alpha+3),\ 2t^{\alpha+1}/\Gamma(\alpha+2)\right)$. If $\alpha(t)$=1/2, the maximum error of state variable is 0.0172 (0.03105 in [15]) and that of control variable is 0.0640 (0.2041 in [15]). Also, if $\alpha(t)$=1/4, those errors are 0.0062 and 0.0377 respectively.

On the other hand, let us consider the case when the dynamic system involves Riemann-Liouville V-OFD instead of Caputo V-OFD in (29). Then, the dynamic system is

$$\dot{x}(t)+{}_0D_t^{\alpha(t)}x(t)=u(t)+t^2 .$$

In this case, we get the following BVP

$$\begin{cases} \dot{x}=\dfrac{\alpha(t)+2}{t}\varphi_0(t)x(t)-\dfrac{\varphi_0(t)^2}{2t^2}\lambda_1(t)+\varphi_0(t)\varphi_1(t)x(t)+\varphi_0(t)\cdot t^2+\varphi_0(t)\left[-\displaystyle\sum_{k=2}^{N}B(\alpha(t),k)t^{1-k-\alpha(t)}V_k(t)\right. \\ \qquad \left.+\varphi_2(t)\ln t\cdot\displaystyle\sum_{k=2}^{N+2}\binom{-\alpha(t)}{k-2}\cdot\frac{(-1)^{k-2}}{k-1}t^{1-k}V_k(t)-\varphi_2(t)\sum_{k=2}^{N+2}\binom{-\alpha(t)}{k-2}(-1)^{k-2}\sum_{p=1}^{N}\frac{1}{p(k+p-1)}t^{1-k-p}V_{k+p}(t)\right] \\ \dot{V}_p=(p-1)t^{p-2}x(t) \\ \dot{\lambda}_1=-\dfrac{\alpha(t)+2}{t}\varphi_0(t)\lambda_1(t)-\varphi_0(t)\varphi_1(t)\lambda_1(t)-\displaystyle\sum_{p=2}^{2N+2}\lambda_p(p-1)t^{p-2} \\ \dot{\lambda}_p=\delta_{\left\lceil\frac{N+1-p}{N}\right\rceil,\,1}\cdot B(\alpha(t),p)t^{1-p-\alpha(t)}\varphi_0(t)\lambda_1(t)-\delta_{\left\lceil\frac{N+3-p}{N+2}\right\rceil,\,1}\cdot\varphi_0(t)\varphi_2(t)\lambda_1(t)\ln t\cdot\dbinom{-\alpha(t)}{p-2}\dfrac{(-1)^{p-2}}{p-1}t^{1-p} \\ \qquad +\varphi_0(t)\varphi_2(t)\lambda_1(t)\displaystyle\sum_{k+l=p}\binom{-\alpha(t)}{k-2}(-1)^{k-2}\frac{1}{l(k+l-1)}t^{1-k-l}, \quad p=2,\dots,2N+2 \end{cases}$$

with the boundary conditions

$$\begin{cases} x(0)=0 \\ V_p(0)=0, \quad k=2,\dots,2N+2, \end{cases} \qquad \begin{cases} x(1)=2/\Gamma(3.5) \\ \lambda_p(1)=0, \quad k=2,\dots,2N+2, \end{cases}$$

where

$$\delta_{\lceil m\rceil,\,1}=\begin{cases} 1, & \lceil m\rceil=1 \\ 0, & \lceil m\rceil\neq 1, \end{cases}$$

and $\lceil m\rceil$ denotes the minimum integer that is greater than m. The behaviors of the numerical solutions for $x(t)$ and $u(t)$ of these both case problems are drawn in Figure 5.

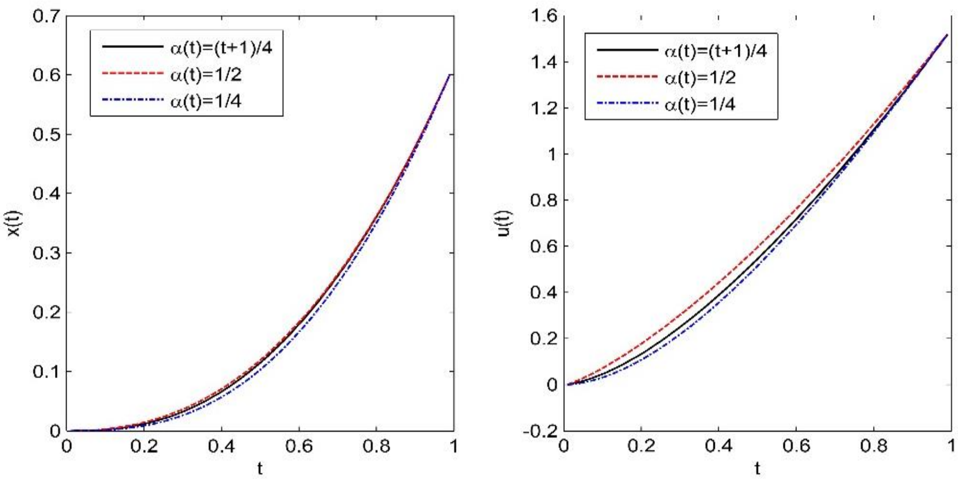

Figure 5. The graphs of the numerical solutions of state variable x(t) and control variable u(t) with N=3 for example 3.

Example 4 Consider the following V-OFOCP with free terminal time

$$\min J(x,u)=\int_0^T\left[tu(t)-(\alpha(t)+2)x(t)\right]^2dt$$

subject to the dynamic system

$$\dot{x}(t)+{}_0^CD_t^{\alpha(t)}x(t)=u(t)+t^2,$$

and boundary conditions $x(0)=0$, $x(T)=1$.

According to the theorem 3.1, we obtain the following Hamiltonian system

$$\begin{cases} x'(t)+{}_0^CD_t^{\alpha(t)}x(t)=\dfrac{\partial H}{\partial\lambda}=x(t)\dfrac{(\alpha(t)+2)}{t}+t^2-\dfrac{\lambda(t)}{2t^2} \\ \lambda'(t)-{}_t^BD_1^{\alpha(t)}\lambda(t)=-\dfrac{\partial H}{\partial x}=-\lambda(t)\dfrac{(\alpha(t)+2)}{t}. \end{cases} \tag{31}$$

We can get the nonlinear system of equations (NSEs) according to the first method.

As terminal time is free, we can consider transversality condition as the following error equation

$$e(T)=H(T,x(T),u(T),\lambda(T))-M_2\lambda(T){}_a^CD_T^{\alpha(T)}x(T)=0. \tag{32}$$

Calculation Algorithm is as follows:

Step 1 Set arbitrary T_0>0.

Step 2 Solve NSEs (31)~(32) exchanged T for T_k.

Step 3 Estimate $E(T_k)=[e(T_k)]^2$ and if $E(T_k)<\varepsilon$, then stop calculation, else determine $T_k\rightarrow T_{k+1}$ by using shooting method type, chord or Newton method.

Step 4 Repeat step 2 and 3.

We have solved NSEs (31) and error equation $e(T)$ (32) by using *fsolve* and *fminsearch* built-in functions. To compare with the result of [15], we solved the problem with α=1/2, h=0.01 and then, its results are T=1.0901 and J=1.8351e-12. when $\alpha(t)=(t+1)/4$, $0<t<3$ and $\alpha(t)=0.5/(1+t^2)$, $0\le t\le\infty$, the results are shown in Figure 6.

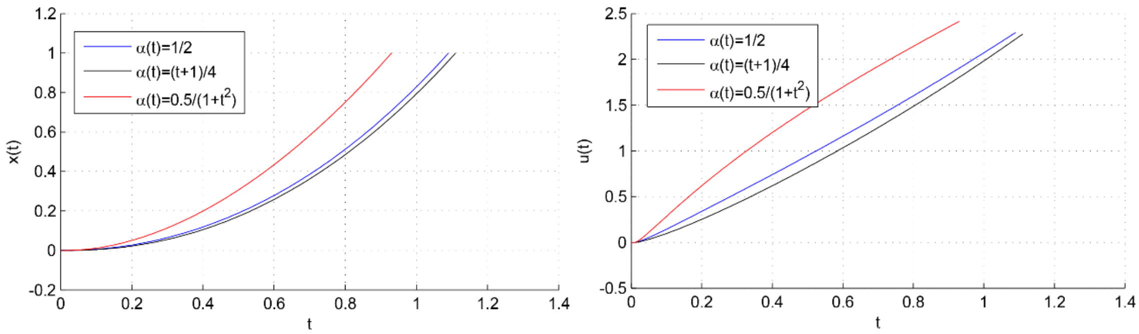

Figure 6. *The graphs of the numerical solutions of state variable x(t) and control variable u(t) for example 4 where terminal time is free.*

Example 5 Consider the following V-OFOCP where both terminal time and state are free

$$\min J(x,u,T,x(T))=\int_0^T\left[tu(t)-(\alpha(t)+2)x(t)\right]^2dt+(x(T)-1)^2$$

subject to the dynamic system

$$\dot{x}(t)+{}_0^CD_t^{\alpha(t)}x(t)=u(t)+t^2,$$

and the initial condition $x(0)=0$.

NSEs are composed of (31) in example 4, and the following simultaneous equation for transversal conditions is accepted as additional condition

$$E(T,x(T))=\begin{cases} e_1(T,x(T))=H(T,x(T),u(T),\lambda(T))-M_2\lambda(T){}_a^CD_T^{\alpha(T)}x(T)+\dfrac{\partial\Phi(T,x(T))}{\partial t} \\ e_2(T,x(T))=\lambda(T)-\dfrac{\partial\Phi(T,x(T))}{\partial x}, \end{cases}$$

Calculation Algorithm is as follows:

Step 1 Set arbitrary T_0>0, $x(T_0)=x_0$>0.

Step 2 Solve NSEs (31) that added $x(T_k)=x_k$.

Step 3 Estimate $\left\|E(T_k,x_k)\right\|_2$ and if $\left\|E(T_k,x_k)\right\|_2<\varepsilon$, then stop calculation, else determine

$(T_k, x_k)\rightarrow(T_{k+1}, x_{k+1})$ by using shooting method type, chord or Newton method.

Step 4 Repeat steps 2 and 3.

Using this algorithm, we get the numerical results of V-OFOCP where both terminal time and state are free as shown in Figure 7.

Table 5 shows numerical results of examples 4 and 5 synthetically.

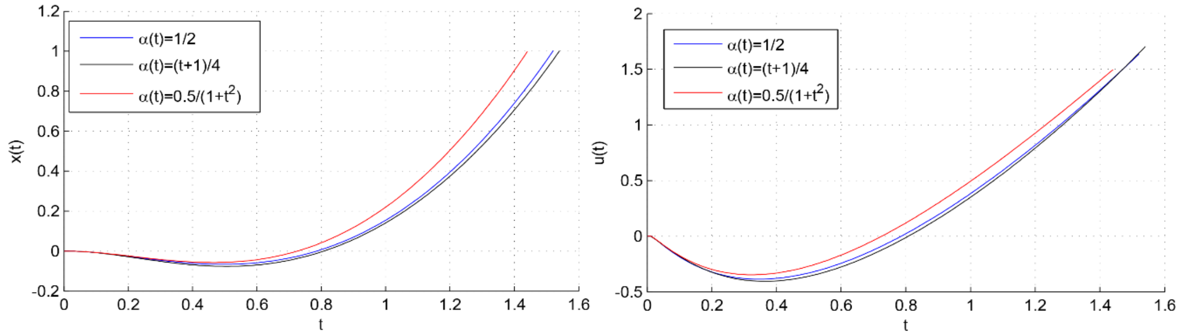

Figure 7. The graphs of the numerical solutions of state variable x(t) and control variable u(t) for example 5 where both terminal time and state are free.

Table 5. Numerical results of examples 4 and 5.

$\alpha(t)$	T	$x(T)$	J	$\|E\|_2$	Color	Position
$\alpha(t)=1/2$	1.0901	Fixed	1.8351e-12	4.6563e-17	blue	Figure 6
	1.5302	1.0017	3.0229e-06	0.0088	blue	Figure 7
$\alpha(t)=(t+1)/4$	1.1103	Fixed	6.1594e-13	4.7458e-17	black	Figure 6
	1.5509	1.0022	5.0506e-06	0.0112	black	Figure 7
$\alpha(t)=0.5/(1+t^2)$	0.9384	Fixed	7.4178e-12	6.5834e-17	red	Figure 6
	1.4410	0.9981	3.7756e-06	0.0100	red	Figure 7

The V-OFOCP with FFE can resolve by the proposed method also.

5. Conclusion

We have investigated the optimality conditions and the numerical methods to solve V-OFOCP with the dynamic system described as a binomial V-OFDE and full free ends. We have worked out V-OFOCPs where both time and state are free simultaneously at the initial and terminal points by considering transversality conditions as error equations. The proposed method is directly based on the approximation of V-OFDs and able to solve numerically various types of V-OFOCP involving Caputo V-OFDs or Riemann-Liouville V-OFDs. We have provided some illustrative test problems to show the efficiency and accuracy of the proposed methods. These methods can apply to multi term V-OFOCPs and it involves integer or fractional order problems as special cases. The theoretical research, computational method, analog implementation and application of V-O fractional automatic control and optimal control in closed system are important as open system and are now open.

Abbreviations

V-O	Variable Order
V-OFOCP	Variable Order Fractional Optimal Control Problem
V-OFDE	Variable Order Fractional Differential Equation
V-OFD	Variable Order Fractional Derivative
V-OFI	Variable Order Fractional Integral
OCP	Optimal Control Problem
OFTD	Optimization First, Then Discretization
DFTO	Discretization First, Then Optimization
FFE	Full Free End
NSE	Nonlinear System of Equation
BVP	Boundary Value Problem

Acknowledgments

The authors would like to thank the associate editor and the anonymous referees for insightful comments, their painstaking efforts and interests for improving the presentation and the contents. This work carried out without any support.

Conflicts of Interest

The authors declare no conflicts of interest.

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