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On the Existence and Uniqueness of Solution of Non-linear Fractional Differential Equations with Integral Boundary Condition

Mtema James Chin*

Department of Mathematics, College of Physical Sciences, Federal University of Agriculture, Makurdi, Nigeria

Email address:
Chin.mtema@uam.edu.ng (Mtema James Chin)
*Corresponding author

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Abstract: In this paper, we investigate the Existence and Uniqueness of solutions of non-linear fractional boundary value differential equations with integral boundary condition using the method of Upper and Lower solutions. We employed the contraction mapping principle and Schauder fixed point theorems. We find out from the analysis that the solution of the boundary value fractional differential equation exists and is unique. An Adomian decomposition method is also used to construct the algorithm for the numerical solution of the nonlinear fractional differential equation. Further, for the implementation of the Adomian decomposition method, several numerical examples are constructed to demonstrate the applicability, accuracy, efficiency, and effectiveness of the method. The results show that the method is accurate and efficient in approximating the exact solution.

Keywords: Existence and Uniqueness, Hadamard Fractional Derivatives, Analytic Solutions, Integral Boundary Condition, Banach Space, Numerical Solution

1. Introduction

The fractional differential equation is popular among many researchers today because it offers something for everyone; that is, different operators are used to model various scenarios, unlike traditional calculus, where variety is lacking. For example, we model using operators such as the Riemann-Liouville fractional operator, the Caputo operator, the Hadamard fractional derivative operator, the Tempered, Grounwald-Letnikov, Prabhakar fractional operator, the Weyl fractional operator, the Cauchy fractional operator, and the Atangana-Baleanu operator, among others. In solving equations, particularly nonlinear fractional differential equations (FrDiEq, it is usually important for us to determine whether the solution exists in the space under consideration. This assertion implies that finding the existence and uniqueness of a solution is crucial to the fact that not all mathematical equations possess solutions. It is equally important to know if the said solution is unique or not.

Various authors have studied several methods for determining the existence and uniqueness of solutions of (FrDiEq). Some of the methods include the following:

The existence and uniqueness of solution to fractional differential equations with fractional boundary conditions using the Atagana-Baleanu derivative were proposed and studied. [1]

The model equation are as follows

$$\begin{cases} {}^{ABC}D_{0^+}^{q} y(x) = & g(x, y(x)) \qquad 1 < q < 2, \qquad x \in [0, b] =: j \\ \\ y(0) = y_0, & {}^{ABC}D_{0^+}^{q-1} y(b) = y_1 \end{cases} \tag{1}$$

where ${}^{ABC}D^{q}_{0+}$ and ${}^{ABC}D^{q-1}_{0+}$ denote Atagana -Baleanu fractional derivative in the sense of Caputo. An implicit numerical scheme based on the trapezoidal method for numerical estimation of the solution was used.

Further more, the existence and uniqueness result for nonlinear coupled systems involving Caputo fractional derivatives with a new kind of coupled boundary condition were given by [2].

$$\begin{cases} {}^{C}D^{\beta}_{0+}u(x) = g(x, u(x), v(x)) \quad x \in J =: 1[0, X], \quad x > 0 \\ {}^{C}D^{\alpha}_{0+}u(x) = f(x, u(x), v(x)) \quad x \in J =: 1[0, X], \\ (u + V)(0) = -(u + v)(x), \int\limits_{\xi}^{\eta}(v - v)(s)ds = A \\ 0 < \xi < \eta < X \end{cases} \tag{2}$$

where ${}^{C}D^{\lambda}_{0+}$ is the Caputo fractional derivatives operator of order $\lambda \in \beta, \alpha, \beta, \alpha \in (0, 1]$, A is non-negative constant and $g, f \in [0, X] \times \mathcal{R}^2 \to \mathcal{R}$ are continuous functions.

Also in a related development an existence, uniqueness, and stability results for a non-linear coupled system using φ -Caputo fractional derivatives were proposed and studied, the proposed results were established using the fixed point theory. [3]

$$\begin{cases} {}^{C}D^{(\rho,\varphi(x))}_{0+}l(x) = \eta(x, l(x), m(x)) \quad x \in J := [0, \varpi] \\ {}^{C}D^{(\tau,\varphi(x))}_{0+}m(x) = \xi(x, l(x), m(x)) \quad x \in J := [0, \varpi] \\ (l + m)(0) = -(l + m)(\varpi), \int\limits_{\rho}^{\xi}(g)dg = B \quad 0 < \rho < \xi < \varpi \end{cases} \tag{3}$$

The operator ${}^{C}D^{\triangle,\varphi}_{0+}$ is $\triangle \in (\rho, \tau)$- order φ- Caputo fractional derivatives and $\varphi(x)$ is the unknown function where $\rho, \tau \in (0, 1]$ B is a non-negative constant.

A fractional Cauchy problem with left generalized Caputo fractional derivatives in the space of continuous differentiable functions is studied by the author of [4],

$$\left({}^{C}_{a}D^{\alpha\rho}_{t}\right)(x) = h(x, t(x), {}^{C}_{a}D^{\alpha_1\rho}_{t}\Big)(x) \cdots {}^{C}_{a}D^{\alpha_m\rho}_{t}\Big)(x)$$

subject with the initial conditions, $\left(y^{i}_{n}\right)(a) = d_i,\ d_i \in \mathcal{R}\ (i = 0, 1, \cdots, n-1)$, where $\rho \in \mathcal{R}^{+}, n = \lceil\alpha\rceil + 1, \alpha_k \in (k-1, k)\ k = 1, 2, \cdots, m < n$ and ${}^{C}_{a}D^{\alpha\rho}_{t}$ denote the generalized Caputo fractional operator of order α_k

Existence and and uniqueness of positive solution for fractional differential equation is given [5]

$$\begin{cases} D^{\beta}_{c}y(t) - D^{\alpha}_{c}y(t) = g(t, y(t) \quad t \in [0, T],\ 0 < \alpha < \beta \\ y(0) = x_0 \end{cases}$$

the Henry Gronwall integral inequalities were utilized for the prove the existence of the solution and the famous fixed contraction mapping principal were also used to established the uniqueness of the solution.

The existence and uniqueness of a solution to damped variable order subdiffusion equation with time delay is studied, under weak assumption on the data. The extended the method of semi discretization to the fractional parabolic problem with delay [6] The authors investigated the existence and uniqueness of the solution for the fractional differential equation with integral boundary condition using the Riemann Liouville fractional derivative operator [7]

In the a similar way, a highly specific non-linear sequential fractional differential equation, were investigated,it was proved that the solution exist and is unique.[8]

A positive solution of non-linear fractional differential equations with integral boundary condition was given. The existence and uniqueness of solution were given via the Guo-Krasnoselskii fixed point theorem. [8]

The analysis of FrDi-Eq is given, the existence, uniqueness and structural stability of solution of non-linear equation were investigated. The dependence of solution of the FrDi-Eq on the order and the initial conditions were also investigated. [9]

Recently a modified Adomian decomposition method with orthogonal Airfoil polynomials for solving non homogeneous differential equations were proposed ,the result of the modified method show that it gives a better approximation to non-linear fractional differential equations than the traditional methods. [10]

A numerical solution of nonlinear multi-term fractional differential equations based on spline Riesz wavelets were given. The method is based on transforming the original equation into an equivalent system of multi-order fractional equations. This system is discretized by fractional derivatives of Riesz spline wavelets and the collocation method. Then the original problem is transformed into a system of algebraic equations and and solved using appropriate software such as Matlab or Mathematica. [11]

Further more the author of [13] developed an efficient Adams-type predictor-corrector method for the numerical solution of fractional differential equations with a power law kernel. Their method is to use a linear approximation to the non-linear problem and then implement finite difference approximations of derivatives. Numerical comparisons with the fractional Adams method are made and simulation results are demonstrated to evaluate the approximation error of the proposed approach.

On solutions of linear and nonlinear fractional differential equations with application to fractional order RC type circuits were proposed. [12].

They solved a class of fractional differential equations of order $\alpha \in (0,1)$ in Caputo-Fabrizio sense. using the exactness of differential equations concept, implicit analytical solutions of fractional differential equations are presented which are extensions and generalizations to the results available in literature.

A novel approach to solve non-linear higher order fractional integro-differential equations using Laplace Adomian decomposition method were given. The higher-order fractional derivative were be expressed in the Caputo derivative sense, and the first-order simple degenerate and the difference kernel were used. Using this approach, the inverse Laplace transform was applied, and the solution of the equation is viewed as the sum of an endless series of components that usually converge to the solution. [14]

Many methods exist in the literature for finding existence and uniqueness of solution for fractional differential equations, the big question is, why then studying the same equations? Our answer been that Scientist are always in search of the best way of solving a problem, which is key in any field of study. In this paper, we seek to fill a methodological research gap, that is to give a method for the existence and uniqueness of solution of the non-linear fractional differential with integral boundary condition with the Hadamard fractional differential operator in the space of condition functions. Why the choice of Hadamard fractional derivative operator? The following are the reasons for the choice of Hadamard fractional derivatives: The Hadamard derivative is better suited for systems or processes with multiplicative scaling or logarithmic time scales, because the Hadamard derivative is defined using logarithmic kernels, which means it's scale-invariant under multiplication; Hadamard derivatives can be use more naturally to model systems with geometric or logarithmic-type behaviour. The operators are often used in solving differential equations with logarithmic solutions or where time or space domains grow exponentially; Hadamard derivatives may lead to simpler boundary conditions in certain contexts, particularly when the domain is $[1,\infty)$ rather than $[0,\infty)$ which is more natural for Hadamard. The Riemann liouville- fractional derivatives often require functions to be well-behaved that is zero or finite at $x = 0$, because of the integral kernel singularity.However, the Hadamard derivative is defined in the interval $x > 0$ and using logarithmic kernels, avoids strong singularities at the origin, making it more stable suitable for near zero in some cases. Finally the Hadamard derivative is defined for on the interval $x > 0$ which align well with systems where negative values don't make sense making it more natural domain for many real-world applications compared to the Riemann Liouville operator which is often used on intervals including zero.

We consider the follow,

$$\begin{cases} {}_1D_x^p y(x) - {}_1D_x^q f(x,y(x)) = g(x,y(t)), \qquad a<x<T \\ y(1)=0, y(e) = \frac{1}{\Gamma(p-q)} \int\limits_1^e \left(log\frac{e}{s} \right)^{p-q-1} f(s,y(s))\frac{1}{s}ds, \end{cases} \tag{4}$$

where $a = 1, T = e$, $1 < p < 2, 0 < q \leq p-1$, $f, g : [a,T] \times [0,\infty)$ are given continuous functions defined on the given the interval. The function f is non-decreasing on y and such assumption is not required for g.

In order to give the existence and uniqueness of the solution for the non-linear fractional differential equation(N-FrDiEq), we transform the N-FrDiEq into an integral equation and applied the Lower and Upper method of solution and make use of the Banach and Schauder fixed point theorems.

The upper and lower solutions are a powerful method for analyzing nonlinear boundary value problems, especially when direct analytical solutions are difficult or impossible to obtain. If you can find an upper and a lower solution such that $\underset{-1}{x}(t) \leq \bar{x_2}(t)$ for all t under certain monotonicity and continuity conditions on the function f it can be shown that a solution x(t) exists between them. A priori bounds: They provide bounds on the solution, which is useful in numerical approximations and in preventing divergence of iterations. Method of monotone iterations: Starting from upper and lower solutions, one can often construct sequences that converge monotonically to the actual solution.

The Banach fixed point theory is a strong hypothesis (contraction) often used to find the existence, uniqueness, and constructive method. It used With the idea of problems with Lipschitz continuity. Schauder fixed point theory is a weaker hypothesis (no contraction, but it needs compactness and convexity), it gives existence only, not uniqueness. It is often used for nonlinear PDEs and boundary value problems where Banach fixed point theorem fails.

The Green function and its properties Let L be a linear differential operator on a domain Ω with boundary operator B. The Green's function $G(x, \xi)$ for L (with boundary conditions B) is defined by

$$L_x G(x, \xi) = \delta(x - \xi), \qquad B[G(\cdot, \xi)] = 0,$$

where δ is the Dirac delta distribution. The Green's function represents the impulse response of the linear operator under the given boundary conditions.

The properites of the green function are as follows

Linearity: If L is linear, the solution of

$$Lu(x) = f(x)$$

can be represented using G by

$$u(x) = \int_\Omega G(x, \xi)\, f(\xi)\, d\xi,$$

assuming the integral is well-defined. This reflects the superposition principle.

Symmetry (Self-adjoint case): If L is self-adjoint (and boundary conditions are symmetric / compatible), then

$$G(x, \xi) = G(\xi, x).$$

Symmetry may fail for non-self-adjoint operators.

Boundary condition satisfaction: For each fixed source point ξ, the function $x \mapsto G(x, \xi)$ satisfies the same boundary conditions as the original problem with respect to x. Consequently, the solution built from G adheres to the prescribed boundary conditions.

Singularity structure: The Green's function has a singularity at $x = \xi$ determined by the fundamental solution of L. For second-order ordinary differential operators the singularity is typically mild (e.g. a jump in derivatives); in higher dimensions the singularity strength depends on the spatial dimension and the order of L.

Continuity and jump conditions: While $G(x, \xi)$ is usually continuous in x away from $x = \xi$, it exhibits specific jump conditions across $x = \xi$. For example, for a second-order operator of the form $(a(x)u')' + \dots$, one has

$$\partial_x G(x, \xi)\big|_{x=\xi^+} - \partial_x G(x, \xi)\big|_{x=\xi^-} = \frac{1}{a(\xi)}$$

(with sign conventions depending on the operator).

Positivity (for certain elliptic problems): For some elliptic operators with Dirichlet boundary conditions the Green's function is positive in the interior,

$$G(x, \xi) > 0 \quad \text{for } x, \xi \in \Omega,\ x \neq \xi.$$

This property is tied to the maximum principle.

Uniqueness (for fixed boundary conditions): For a given operator L and specified boundary conditions, the Green's function is unique. Different boundary conditions produce different Green's functions.

The rest of the paper is organized as follows, in Section (1) definitions and some basic lemmas and theorems are given, in Sections (2) we give the existence and uniqueness of the (N-FrDiEq) . In Section (3) the numerical solution of the is given and solution problems are considered and in Section (5) we given the summary and conclude based on our findings.

Preliminaries

Under this section we give some basic definitions that are useful for the subsequent chapters:

Definition 1.1. [15] The left and right Riemann Liouville fractional integral operator are defined respectively as:

$$\begin{aligned} {}_aI^p g(t) &= \frac{1}{\Gamma(p)} \int_a^t (t-\lambda)^{p-1} g(\lambda) d\lambda, \\ I_b^p g(t) &= \frac{1}{\Gamma(p)} \int_t^b (\lambda - t)^{p-1} g(\lambda) d\lambda, \end{aligned}$$

where $p > 0$ $m - 1 < p < m$, $\Gamma(p)$ represent the gamma function.

Definition 1.2. [15] The left and right Riemann Liouville fractional derivative are defined

$$ {}_aD^p g(t) = \frac{1}{\Gamma(m-p)} \left(\frac{d}{dt}\right)^m \int_a^t (t-\lambda)^{m-p-1} g(\lambda) d\lambda, \tag{5}$$

$$ D_b^p g(t) = \frac{1}{\Gamma(m-p)} \left(\frac{d}{dt}\right)^m \int_t^b (\lambda - t)^{m-p-1} g(\lambda) d\lambda, \tag{6}$$

Definition 1.3. [16] The derivative of a constant K is defined as:

$$ {}_aD^p K = K \frac{(t-a)^p}{\Gamma(1-p)},$$

and the derivative of a power function is given by

$$ {}_aD^p (t-a)^\gamma = \frac{\Gamma(p+1)(t-a)^{\gamma - p}}{\Gamma(\gamma - p + 1)},$$

for $\gamma > -1, p \geqslant 0$.

For the real world problems, the initial conditions interpretations using the Riemann Liouville fractional derivative operator poses a significant difficulty, in order to circumvent the challenge, Michele Caputo proposed the left and right fractional derivative which the initial conditions are similar to the classical derivatives.

Definition 1.4. [16] The left and right Caputo fractional derivative are defined as follows:

$$ {}_a^C D^p g(t) = \frac{1}{\Gamma(m-p)} \int_a^t (t-\lambda)^{m-p-1} \left(\frac{d}{dt}\right)^m g(\lambda) d\lambda, $$
$$ {}^C D_b^p g(t) = \frac{1}{\Gamma(m-p)} \int_t^b (\lambda-t)^{m-p-1} \left(\frac{d}{dt}\right)^m g(\lambda) d\lambda, $$

where p represent the order of the derivative $m-1 < p < m$

The Riemann Liouville fractional derivative and the Caputo fractional derivatives are related by the following formulae.

$$ {}_a^C D^p g(t) = {}_a D^p \Bigg(g(t) - \sum_{i=0}^{m-1} \frac{g^i(a)}{\Gamma(i-p+1)} (t-a)^{i-p} \Bigg) $$

$$ {}^C D_b^p g(t) = D_b^p \Bigg(g(t) - \sum_{i=0}^{m-1} \frac{(-1)^i g^i(b)}{\Gamma(i-p+1)} (b-t)^{i-p} \Bigg) $$

The fractional power $\left(\frac{d}{dt}\right)^p$ of differentiation $\frac{d}{dt}$ is the Riemann Liouville fractional derivative and is invariant with respect to translation on the whole axis [16]

Hadamard gave a modification to the fractional power as follows

$$ \left(t\frac{d}{dt}\right)^p $$

Hadamard definition was based on the nth integral generalization.

$$ {}_a I^m(t) = \int_a^t \frac{dx_1}{x_1} \int_a^{t_1} \frac{dx_2}{x_2} \cdots \int_a^{t_{m-1}} g(t_m) \frac{dx_m}{x_m} $$
$$ = \frac{1}{(m-1)!} \int_a^x \left(log\frac{t}{x} \right)^{m-1} g(x) \frac{dx}{x} $$

Definition 1.5. [17] The left and right Hadamard fractional integrals of order $p \in \mathcal{C}, \mathcal{R}, p > 0$ are defined respectively as follow:

$$ {}_a J^p g(t) = \frac{1}{\Gamma(p)} \int_a^t \left(log\frac{t}{x} \right)^{p-1} g(x) \frac{dx}{x}, $$

$$ J_b^p g(t) = \frac{1}{\Gamma(p)} \int_t^b \left(log\frac{x}{t} \right)^{p-1} g(x) \frac{dx}{x}. $$

Definition 1.6. [17, 18] The Hardamard fractional derivatives are defined as follow:

$$ {}_a D^p g(t) = \delta^m \Big({}_a J^{m-p} g \Big)(t) $$
$$ = \left(t\frac{d}{dt}\right)^m \frac{1}{\Gamma(m-p)} \int_a^t \left(log\frac{t}{x} \right)^{m-p-1} g(x) \frac{dx}{x}, $$

and

$$ D_b^p g(t) = \Big(-\delta \Big)^m \Big(J_b^{m-p} g \Big)(t) $$
$$ = \left(-t\frac{d}{dt}\right)^m \frac{1}{\Gamma(m-p)} \int_t^b \left(log\frac{x}{t} \right)^{m-p-1} g(x) \frac{dx}{x}, $$

where $\delta = t\frac{d}{dt}$, $\delta^0 = g(t)$, $p \in \mathbb{C}, \mathbb{R}(p) \geqslant 0$.

Definition 1.7. [17] The Caputo- type Hadamard fractional derivative C-HFrD

Let $\mathbb{R}(p) \geqslant 0$ and $m = \lfloor \mathbb{R}(p) \rfloor + 1$, if $g(t) \in AC_\delta^m[a,b]$, where $0 < a < b < \infty$ and $AC_\delta^m[a,b] = \{ f : [a,b] \to \mathbf{C}, \delta^{m-1} f(t) \in AC[a,b], \delta = t\frac{d}{dt} \}$.

The left and right Caputo- type Hardamard fractional derivative (C-HFrD) is defined respectively as follow:

$$ {}_a^C D^p g(t) = {}_a D^p \Bigg[g(t) - \sum_{i=0}^{m-1} \frac{\delta^i g(a)}{i!} \left(log\frac{x}{a} \right)^i \Bigg](t) \tag{7} $$

and

$$ {}^C D_b^p g(t) = {}_a D^p \Bigg[g(t) - \sum_{i=0}^{m-1} \frac{(-1)^i \delta^i g(b)}{i!} \left(log\frac{b}{x} \right)^i \Bigg](t) \tag{8} $$

The Caputo-type Hadamard fractional derivatives have some group properties:

Let $\mathbb{R}(p) \geqslant 0$, $\lfloor \mathbb{R}(p) \rfloor + 1$ and $\mathbb{R}(\gamma) > 0$, then

$$ {}_a^C D^p \left(log\frac{t}{a} \right)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma-p)} \left(log\frac{t}{a} \right)^{\gamma-p-1}, $$

$$ {}^C D_b^p \left(log\frac{b}{t} \right)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma-p)} \left(log\frac{b}{t} \right)^{\gamma-p-1}, $$

where $\mathbb{R}(p) > m$.

$$ {}_{a}^{C}D^{p}\left(log\frac{t}{a}\right)^{k} = 0 $$

Definition 1.8. [19] The hypergeometric function is defined by a locally uniformly convergent series as

$$ {}_{2}F_{1}(a,b:c;z) = \sum_{i=0}^{\infty}\frac{\Gamma(a+i)}{\Gamma(a)} $$

$$ \frac{\Gamma(b+i)}{\Gamma(b)}.\frac{\Gamma c}{\Gamma(c+i)}.\frac{z^{i}}{i!}. $$

where a, b, c $\in \mathcal{C}$, $|z| < 1$, or it can also be defined by finite integral involving power function.

$$ {}_{2}F_{1}(a,b:c;z) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} $$

$$ \int_{0}^{1} x^{b-1}(1-x)^{c-b-1}(1-zx)^{a}dx, $$

$$ Re(c) > Re(b) > 0, \quad z \in \mathcal{C}\backslash(1,\infty) $$

Definition 1.9. [15] Euler gamma function $\Gamma(x)$ is defined by the Euler integral of the second kind:

$$ \Gamma(s) = \int_{0}^{\infty} y^{s-1}e^{-y}dy \quad (\mathbb{R}(s) > 0) $$

where $y^{s-1} = e^{(s-1)log(y)}$

Definition 1.10. [15] The gamma functions has both the upper and the lower incomplete gamma function $\gamma(v,y)$, We first define the lower incomplete gamma function: The lower incomplete gamma function $\gamma(v,y)$ is defined as:

$$ \gamma(v,y) = \int_{0}^{y} e^{-s}s^{v-1}ds. \quad Re(v) > 0. $$

and the upper incomplete gamma function $\Gamma(v,y)$ is defined as

$$ \Gamma(v,y) = \int_{y}^{\infty} e^{-s}s^{v-1}ds, \quad Re(v) > 0. $$

Definition 1.11. [15] The generalized Mittag-Liffler function is defined for a complex $x \in \mathbb{C}$, $\varpi, \Psi, \eta, \mathbb{C}$ and $\mathbb{R}(\varpi) > 0$ by

$$ \mathbf{E}^{\eta}_{\varpi,p}(x) := \sum_{k=1}^{\infty}\frac{(\eta)_{k}\ x^{k}}{\Gamma(\varpi k+p)k!} = \frac{1}{\Gamma(\eta)}{}_{1}\Psi_{1}\left[\begin{array}{c}(\eta,1)\\(p,\varpi)\end{array}\middle| x\right] $$

where $(\eta)_{k}$ is the pochhammer symbol, ${}_{1}\Psi_{1}$ defines the Fox wright function.

2. Solution to the Linear Fractional Differential Equation

In this section we give a method for establishing the existence and uniqueness of solution of the non-linear fractional differential equation (Fr-DiEq), here and throughout in the paper we let $a = 1$ and $T = e = log_{e}(.)$, we begin be defining some reinvent lemmas.

Let Y= C([a, T]) be the Banach space of all real-valued continuous functions defined on the compact interval [a, T], endowed with the maximum norm. Define the subspace

$B = y \in Y : y(x) \geqslant 0 \ x \in [a,T]$ of y. For a solution $y \in Y$, we mean a function y(x) 0 , $1 \leqslant x \leqslant e$

Let $a, b \in \mathcal{R}^{+}$ such that $b > a$. For any $y \in [a,b]$, we define the upper control function $Q(x,y)$= $sup\{g(x,\eta) : a \leqslant \eta \leqslant y\}$ and the lower -control function

$Q_{1}(x,y)$= $inf\{g(x,\eta) : y \leqslant \eta \leqslant b\}$

Clearly, we see that $Q(x,y)$ and $Q_{1}(x,y)$ are non decreasing on the argument y and $Q_{1}(x,y) \leqslant g(x,y) \leqslant Q(x,y)$.

Lemma 2.1. [20] Let $m-1 < p \leqslant m, m \in \mathcal{N}$. The equality $\left(I_{1}^{p}D_{1}^{p}y\right)(x) = 0$ is valid if and only if

$$ y(x) = \sum_{j=1}^{m} c_{j}(logx)^{p-j}, \quad for\ \ each\ \ x \in [1,e] \tag{9} $$

where $A_{j} \in \mathcal{R}, j = 1, \cdots, m$ are arbitrary constants

Lemma 2.2. [21] For all $p > 0$ and $q > -1$

$$ \frac{1}{\Gamma(p)}\int_{1}^{x}\left(log\frac{x}{s}\right)^{p-1}\left(logs\right)^{q}\frac{1}{s}ds = \frac{\Gamma(q+1)}{\Gamma(p+q+1)}\left(logx\right)^{p+q} \tag{10} $$

Theorem 2.1. Let $y \in C^1([a,T])$ $y^{(2)}$ and $\frac{\partial f}{\partial x}$ exist then y is a solution of (4) if and only if

$$y(x) = \int_a^T G(x,s)g(s,y(s))\frac{1}{s}ds + \frac{1}{\Gamma(p+q)}\int_0^x \left(log\frac{x}{s}\right)^{p-q-1} f(s,y(s))\frac{1}{s}ds \tag{11}$$

$$G(x,s) = \begin{cases} \left(logxlog\frac{T}{s}\right)^{p-1} - \left(log\frac{x}{s}\right)^{p-1} & a < s < x < T \\ \frac{\left(logxlog\frac{T}{s}\right)^{p-1}}{\Gamma(p)} & a \le x \le s \le T \end{cases} \tag{12}$$

Proof. Let y be a solution of (4). Equation (4) is rewritten as follows

$$I_a^p D_a^p y(x) = I_a^p\bigg(- g(x,y(x)) + D_a^q f(x,y(x))\bigg) \quad a < x < T \tag{13}$$

Using lemma 2.1 we get

$$\begin{aligned} &y(x) - A_1(logx)^{p-1} - A_2(logx)^{p-2} = -I_a^p g(x,y(x)) + I_a^p D_a^q f(x,y(x)) \\ &= -I_a^p g(x,y(x)) + I_a^{p-q} I_a^p D_a^q f(x,y(x)) = -I_a^p g(x,y(x)) + I_a^{p-q}\left(I_a^p D_a^q f(x,y(x)) - A_3\left(logx\right)^{q-1}\right) \end{aligned}$$

$$= -\frac{1}{\Gamma p}\int_a^x \left(\log\frac{x}{s}\right)^{p-1} g(s,y(s))\frac{1}{s}ds + \frac{1}{\Gamma(p-q)}\int_a^x \left(\log\frac{x}{s}\right)^{p-q-1} g(s,y(s))\frac{1}{s}ds - \left[A_3\Gamma q\left(logx\right)^{p-1}\right]/\Gamma(p)$$

this result to

$$y(x) = A_3 + \left(A_1\frac{\Gamma p}{\Gamma q}\right)(logx)^{p-1} - \frac{1}{\Gamma p}\int_a^x \left(\log\frac{x}{s}\right)^{p-1} g(s,y(s))\frac{1}{s}ds + \frac{1}{\Gamma(p-q)}\int_a^x \left(\log\frac{x}{s}\right)^{p-q-1} f(s,y(s))\frac{1}{s}ds$$

applying the boundary conditions $y(a) = 0$ and

$$y(T) = \frac{1}{\Gamma(p-q)}\int_a^T \left(\log\frac{x}{s}\right)^{p-q-1} f(s,y(s))\frac{1}{s}ds$$

this gives

$$A_3 = 0, \quad A_1 = \frac{A_3}{p-1} = \frac{1}{\Gamma p}\int_a^T \left(\log\frac{x}{s}\right)^{p-1} g(s,y(s))\frac{1}{s}ds$$

Hence,

$$\begin{aligned} y(x) &= \frac{1}{\Gamma p}\int_a^T \left(\log\frac{x}{s}\right)^{p-1} g(s,y(s))\frac{1}{s}ds - \frac{1}{\Gamma p}\int_a^x \left(\log\frac{x}{s}\right)^{p-1} g(s,y(s))\frac{1}{s}ds + \frac{1}{\Gamma(p-q)}\int_a^x \left(\log\frac{x}{s}\right)^{p-q-1} f(s,y(s))\frac{1}{s}ds \\ &= \int_a^T G(x,s)g(s,y(s))\frac{1}{s}ds + \frac{1}{\Gamma(p-q)}\int_a^x \left(\log\frac{x}{s}\right)^{p-q-1} f(s,y(s))\frac{1}{s}ds \end{aligned}$$

The converse of this theorem is also true. The proof is complete.

Lemma 2.3. [22] write as corollary
i. $G(x,s) > 0$ for $x, s \in (a,T)$
ii. $G(x,s) = G(s,s) > 0$ for $s \in (a,T)$

In proving the existence and uniqueness of the solution of the non-linear FrDiEq (4) we make use of the contraction mapping principle. We therefore, state the following Banach fixed point theorem as follows:

Definition 2.1. Let $\phi : Y \longrightarrow Y$. and $(Y, ||.||)$ be a Banach space. The operator ϕ is called a contraction operator if $\exists$ an $\eta \in (0,1)$ such that $y, y_1 \in Y$, then

$$||\phi_y - \phi_{y_1}|| \leq \eta||y - y_1||$$

Theorem 2.2. [23] Let ϖ be a non-empty closed convex subset of a Banach space Y and $\phi : \varpi \longrightarrow \varpi$ be a contraction operator. Then there is a unique $yin\varpi$ with $\phi_y = y$

Theorem 2.3. [23] Let ϖ be a non-empty closed convex subset of a Banach space Y and $\phi : \varpi \longrightarrow \varpi$ be a continuous compact operator operator, then ϕ has a fixed point in ϖ

In order to apply the Schauder fixed to determine the existence of the solution for FrDiEq,we first transform equation (12) and define an operator $\tau \longrightarrow Y$ by

$$(\phi_y)(x) = \int_a^T G(x,s)g(s,y(s))\frac{1}{s}ds \qquad (14)$$

$$+\frac{1}{\Gamma(p-q)}\int_a^x \left(\log\frac{x}{s}\right)^{p-q-1} f(s,y(s))\frac{1}{s}ds \qquad (15)$$

where the said fixed point has to be necessarily satisfy $\phi_y = y$ which is the identity operator.

We make some assumptions to enable us prove the require results as follow: (G_1) let $y^*, y_* \in \tau$, such that $a \leq y_* \leq y^* \leq b$ such that,

$$\begin{cases} D_a^p y^*(x) + Q(x, y^*(x) \geq D_a^p f(x, y^*(x)) \\ \\ D_a^p y^*(x) + Q_1(x, y_*(x) \leq D_a^p f(x, y_*(x)) \end{cases}$$

for any $x \in [a,T]$ and $y, y_1 \in Y$, there exist positive real numbers $\eta_1, \eta_2 < 1$ such that

$$|f(x,y_1) - f(x,y)| \leq \eta_1||y_1 - y||,$$
$$|g(x,y_1) - g(x,y)| \leq \eta_2||y_1 - y||.$$

The functions y^* and y_* are called the the upper and lower solutions for the FrDiEq (4)

Theorem 2.4. Assume that (G_1) is satisfied, the the FrDiEq (4) has atleast one solution $y \in Y$ satisfying the relation

$$y_*(x) \leq y(x) \leq y^*(x), x \in [a,T].$$

Proof. Let $\varpi = \{y \in \tau, y_*(x) \leq y(x) \leq y^*(x), x \in [a,T]\}$ equipped with the $||y|| = max_{x\in[a,T]}|y(x)|$, then we get $||y|| \leq b.$. Therefore, ϖ is a convex, bounded, and closed subset of the Banach space Y. intuitively , the continuity of the functions f and g implies the continuity of the operator ϕ on ϖ which is given by (14). Next if $y \in \varpi$, there exist positive constants ϖ_g and ϖ_f such that

$$max\{g(x,y(x)) : x \in [a,T], y(x) \leq b\} < \varpi_g$$

and

$$max\{f(x,y(x)) : x \in [a,T], y(x) \leq b\} < \varpi_f$$

therefore,

$$|(\phi_y(x))| \leq \int_a^T G(s,s)|g(s,y(s))|\frac{1}{s}ds + \frac{1}{\Gamma(p-q)}\int_a^T (log\frac{x}{s})^{p-q-1}|f(s,y(s))|\frac{1}{s}ds$$

$$\leq \varpi_g \int_a^T \frac{\left(logslog\frac{T}{s}\right)^{p-1}}{\Gamma p}\frac{1}{s}ds + \varpi_f\frac{(logx)^{p-q}}{\Gamma(p-q+1)} \leq \frac{\Gamma(p)\varpi_g}{\Gamma(2p)} + \frac{\varpi_f}{\Gamma(p-q+1)}$$

Hence,

$$||\phi_y|| \leq \frac{\Gamma(p)\varpi_g}{\Gamma(2p)} + \frac{\varpi_f}{\Gamma(p-q+1)}$$

Therefore, $\phi(\varpi)$ is uniformly bounded. We now prove the equicontinuity of $\phi(\varpi)$. Let $y \in \varpi$, then for any $x_1, x_2 \in [a,T], x_2 > x_1$ we get

$$|(\phi_y)(x_2) - (\phi_y)(x_1)| \leq \frac{1}{\Gamma p}|\int_a^T \left(logx_2 \log\frac{T}{s}\right)^{p-1} g(s,y(s))\frac{1}{s}ds - \int_a^T \left(logx_1 \log\frac{T}{s}\right)^{p-1} g(s,y(s))\frac{1}{s}ds| \qquad (16)$$

$$+\frac{1}{\Gamma p}|\int_a^{x_2}\left(log\frac{T}{s}\right)^{p-1}g(s,y(s))\frac{1}{s}ds+\int_a^{x_1}\left(\log\frac{x_2}{s}\right)^{p-1}g(s,y(s))\frac{1}{s}ds|$$
$$+\frac{1}{\Gamma(p-q)}|\int_a^{x_2}\left(\log\frac{x_1}{s}\right)^{p-q-1}f(s,y(s))\frac{1}{s}ds-\frac{1}{\Gamma(p-q)}\int_a^{x_1}\left(\log\frac{x_1}{s}\right)^{p-q-1}f(s,y(s))\frac{1}{s}ds|$$

$$\leq\frac{(logx_2)^{p-1}-(logx_1)^{p-1}}{\Gamma p}\int_a^{T}\left(\log\frac{T}{s}\right)^{p-1}|g(s,y(s))|\frac{1}{s}ds \tag{17}$$

$$\frac{1}{\Gamma p}\int_a^{x_1}\left(\left(\log\frac{x_2}{s}\right)^{p-1}-\left(\log\frac{x_1}{s}\right)^{p-1}\right)|g(s,y(s))|\frac{1}{s}ds+\frac{1}{\Gamma p}\int_{x_1}^{x_2}\left(\log\frac{x_2}{s}\right)^{p-1}|g(s,y(s))|\frac{1}{s}ds$$
$$+\frac{1}{\Gamma(p-q)}\int_a^{x_1}\left(\left(\log\frac{x_2}{s}\right)^{p-q-1}\right. \tag{18}$$

$$\left.-\left(\log\frac{x_1}{s}\right)^{p-q-1}\right)|f(s,y(s))|\frac{1}{s}ds+\frac{1}{\Gamma(p-q)}\int_{x_1}^{x_2}\left(\log\frac{x_2}{s}\right)^{p-q-1}|f(s,y(s))|\frac{1}{s}ds\leq\frac{\varpi_g}{\Gamma(p+1)}\left(\left(\log x_2\right)^{p-1}\right. \tag{19}$$

$$\left.-\left(\log x_1\right)^{p-1}-\left(\log x_2\right)^{p}-\left(\log x_1\right)^{p}\right)+\frac{\varpi_f}{\Gamma(p-q-1)}\left(\left(\log x_2\right)^{p-q}-\left(\log x_1\right)^{p-q}\right. \tag{20}$$

The right hand side of equation (20) is independent of y, as $x_1\to x_2$, y tends to zero, Hence, $\phi(\varpi)$ is equicontinuous. From Arzele Ascoli theorem we deduce that $\phi:\varpi\to Y$ is compact.

Next we want to prove that $\phi(\varpi)\subseteq\varpi$, to do this we make use of the Schauder fixed point. Let $y\in\varpi$ then by Theorem 2.1 we have the following

$$(\phi_y)(x)=\int_a^T G(x,s)g(s,y(s))\frac{1}{s}ds+\frac{1}{\Gamma(p+q)}\int_0^x\left(log\frac{x}{s}\right)^{p-q-1}f(s,y(s))\frac{1}{s}ds$$
$$\leq\int_a^T G(x,s)Q(s,y(s))\frac{1}{s}ds+\frac{1}{\Gamma(p+q)}\int_a^x\left(log\frac{x}{s}\right)^{p-q-1}f(s,y^*(s))\frac{1}{s}ds$$
$$\leq\int_a^T G(x,s)Q(s,y^*(s))\frac{1}{s}ds+\frac{1}{\Gamma(p+q)}\int_a^x\left(log\frac{x}{s}\right)^{p-q-1}f(s,y^*(s))\frac{1}{s}ds\leq y^*(x)$$

similarly,

$$(\phi_y)(x)=\int_a^T G(x,s)g(s,y(s))\frac{1}{s}ds+\frac{1}{\Gamma(p+q)}\int_0^x\left(log\frac{x}{s}\right)^{p-q-1}f(s,y(s))\frac{1}{s}ds$$
$$\geq\int_a^T G(x,s)Q(s,y(s))\frac{1}{s}ds+\frac{1}{\Gamma(p+q)}\int_a^x\left(log\frac{x}{s}\right)^{p-q-1}f(s,y_*(s))\frac{1}{s}ds$$

$$\geq \int_a^T G(x,s)Q(s,y_*(s))\frac{1}{s}ds + \frac{1}{\Gamma(p+q)}\int_a^x \left(log\frac{x}{s}\right)^{p-q-1} f(s,y_*(s))\frac{1}{s}ds \geq y^*(x)$$

Therefore, $y_*(x) \leq (\phi_y)(x) \leq y^*(x)$, $x \in [a,T]$ this implies that $\phi(\varpi) \subseteq \varpi$. In accordance with Schauder fixed theorem, the operator ϕ has at least one fixed point $y \in \varpi$. Hence, the non- linear FrDiEq (4) has at least one solution $y \in Y$ and $y_*(x) \leq y(x) \leq y^*(x), x \in [a,T]$. This complete the proof.

Using the Banach fixed point theorem(contraction mapping principle) we present the uniqueness of the solution of the non-linear FrDiEq (4).

Theorem 2.5. Assume that, assumption G_1 and G_2 hold and

$$\left(\frac{\Gamma(p)\eta_2}{\Gamma(2p)} + \frac{\eta_1}{\Gamma(p-q+1)}\right) < 1, \tag{21}$$

then the non- linear FrDiEq (4) has a unique solution $y \in \varpi$

Proof. By Theorem 2.4 it implies that, the non- linear FrDiEq (4) has at least one solution in ϖ , next we need to prove that the operator defined in Theorem 2.4 is a contraction on ϖ

$$|(\phi_y)(x) - (\phi_{y_1})(x)| \leq \int_a^T G(x,s)|g(s,y(s)) - g(s,y_1(s))|\frac{1}{s}ds$$
$$+ \frac{1}{\Gamma(p-q)}\int_a^T \left(log\frac{x}{s}\right)^{p-q-1} |f(s,y(s)) - f(s,y_1(s))|\frac{1}{s}ds$$
$$\leq \left(\frac{\Gamma(p)\eta_2}{\Gamma(2p)} + \frac{\eta_1}{\Gamma(p-q+1)}\right)||y-y_1||,$$

Therefore, the operator ϕ is a contraction mapping by theorem 2.5. Hence the non- linear FrDiEq (4) has a unique solution $y \in \varpi$.

3. Numerical Method

3.1. Definition of the Abbreviations Used

We defined the abbreviation used in this research paper in this section. AE= Absolute error; $y(t)$= exact solution; $u_h(t)$= Approximate solution; ADM= Adomian Decomposition Method; M = Non-linear operator; H= know source; $\varpi v(x)$= Linear operator; A_m= Adomian polynomials; D_h^α = Hadamard fractional derivative of order α ; I_H^α = Hadamard integral operator of order α ; $\Phi(s)$, Ψ= functions; ${}_1\Psi_1$ = Fox Wright function

3.2. The Derivation of the Method

Adomian Decomposition method is a very powerful numerical method developed for solving linear and non-linear differential equations. It involves the division of the unknown function $v(x)$ into components which are infinite and expressed in the form $v_0, v_1, v_2 \cdots$. For the non-linear terms the Adomian polynomials, denoted by A_m, are calculated in-terms of the non-linearity. To be more precise, we assume that, equation (11) be written as

$$v(x) = \varpi v(x) + M(v(x)) + H(x) \tag{22}$$

where $\varpi v(x)$ is a linear operator to be inverted. M is a non-linear operator assume to be analytic, and H is a known given function.

The solution v(x) is decompose into a rapidly convergent series of solution components, and then we decomposed the analytic non-linearity Mv into a series of Adomian polynomials.

$$vx = \sum_{m=0}^{\infty} v_m(x) \tag{23}$$

$$Mvx = \sum_{m=0}^{\infty} A_m \tag{24}$$

where $A_m = A_m\Big(v_0, v_1, v_2\Big)$ are the well known Adomian Polynomial defined by [25] and are given by

$$A_m = \frac{1}{m!}\frac{d^m}{d\eta^m}M\left(\sum_{i=0}^{m} v_{i\eta^i}\right)|_{\eta=0}, \qquad m \geq 0. \tag{25}$$

for reference and ease of understanding , we list some few Adomian polynomial for a general analytic non-linearity,

$$\begin{aligned} M[v] &= g(v) \\ A_0 &= g(v_0 \\ A_1 &= g^{'}(v_0).v_1 \end{aligned}$$

$$A_2 = g^{'}(v_0).v_2 + g^{''}(v_0).\frac{v_1^2}{2!}$$
$$A_3 = g^{'}(v_0).v_3 + g^{''}(v_0)v_1.v_2 + g^{'''}(v_0).\frac{v_1^3}{3!}$$
$$A_4 = g^{'}(v_0).v_4 + g^{''}(v_0)\left(v_1.v_3 + \frac{v_2^2}{2!}\right) + g^{'''}(v_0).\frac{v_1^2 v_2}{2!} + g^{iv}v_0.\frac{v_1^4}{4!}$$

Hence, by equation (23), (24) and equation (11) becomes

$$\sum_{m=0}^{\infty} v_m(x) = \varpi\left(\sum_{m=0}^{\infty} v_m(x)\right) + \sum_{m=0}^{\infty} A_m(x) + H_x \quad (26)$$

From equation (26), the v_m are determined by the following recursion

$$\begin{cases} v_0(x) = H(x) \\ v_{m+1} = \varpi\Big(v_m(x)\Big) + A_m, m = 0, 1, 2\cdots \end{cases}$$

If we defined the M-term approximation to the solution to be

$$\phi M(x) = \sum_{m=0}^{M-1} v_m(x), \quad (27)$$

then the Adomian decomposition solution asserts that the exact solution v(x) is given by $v(x) = \underset{M\to+\infty}{lim} \phi M(x)$

4. Numerical Examples

Under this section, we consider some numerical examples to validate the applicability, accuracy and effectiveness of the Adomian decomposition method.

Example 4.1. Consider the non-linear fractional differential equation

$$\begin{cases} {}_aD_x^{3/2}y(x) - {}_aD_x^{1/2}(y(x)) = sin(x), \qquad a < x < T \\ y(1) = 0, y(e) = \int\limits_a^T \frac{1}{s}y(s)^2 ds, \end{cases} \quad (28)$$

To fin the solution of the fractional differential equation, we first convert the boundary value problem (BVP) into a voltera -type integral equation and solve via a successive approximation scheme. We rewrite the fractional differential equation as

$${}_aD_x^{3/2}y(x) - {}_aD_x^{1/2}(y(x)) = sin(x)$$

we denote

$$\varpi[v](x) = {}_aD_x^{3/2}y(x) - {}_aD_x^{1/2}(y(x)) = sin(x)$$

applying ${}_aI_x^{3/2}$ on both sides, making use of the the linearity property and inversion formula, this gives,

$$v(x) = {}_aI_x^{3/2}\Big[sin(x)\Big] + {}_aI_x^{3/2}\Big[{}_aI_x^{1/2}(v(x))\Big]$$

Applying the identity,

$${}_aI_x^{\alpha}\,{}_aD_x^{\beta}v(x) = {}_aI_x^{\alpha-\beta}, \qquad if(\alpha > \beta),$$

this yield the expression

$$v(x) = {}_aI_x^{3/2}\Big[sin(x)\Big] + {}_aI_x^{1}v(x),$$

further evaluation gives

$$v(x) = g(x) + {}_aI_x^{1}v(x),$$

where

$$g(x) = \frac{1}{\Gamma(3/2)}\int\limits_a^x \left(log\frac{x}{t}\right)\frac{sin(t}{t}dt,$$

We now decompose into operator form as follows.

Let

$$\varpi[v](x) := {}_aI_x^{1}v(x),$$

then,

$$[v](x) = g(x) + \varpi[v](x).$$

We now define the iteration sequence,

$$v(x) = \begin{cases} v_0(x) = g(x) \\ y_{m+1}(x) = g(x) + {}_aI_x^1(v_m(x)) = g(x) + \int\limits_a^1 v_m(t)dt. \end{cases} \tag{29}$$

Each v_m converges under suitable conditions, (bounded kernel to the solution v(x)). Incorporating the integral boundary conditions we now enforce.

$$v(e) = \int\limits_a^T \frac{1}{s} v(s)^2 ds. \tag{30}$$

So after computing the approximate solution $v^m(x)$, we adjust it such that

$$v^m(e) = \int\limits_a^T \frac{1}{s} v^m(s)^2 ds. \tag{31}$$

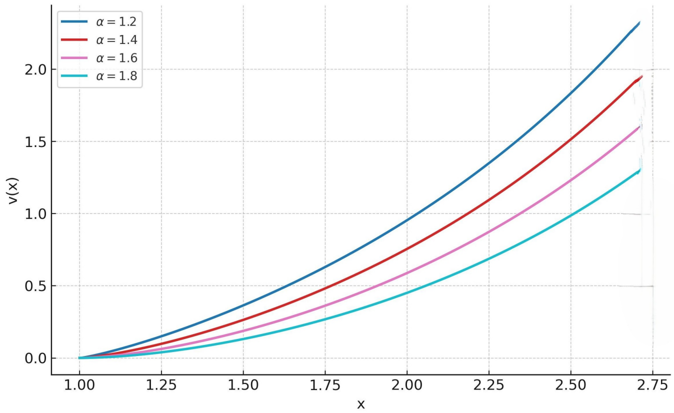

***Figure 1.** A graph of the approximate solution for multiple values of alpha for Example (4.1).*

A software (MALAB) is used to evaluate the approximate solution function, and the graph below illustrates the numerical solution of a fractional boundary value problem with integral boundary conditions for various values of $\alpha \in \{1.2, 1.4, 1.6, 1.8\}$.

From Figure 1 of the graph, we see that the value of α has a significant effect on the solution. For a small value of α the solution function approaches the value 2, but when the value of α approaches 2 the function approaches zero. This shows that, the Adomian decomposition give a good approximation of the exact solution.(let the the approximate solution be $u_h(t)$ here and through out the paper.)

Table 1, show the approximate solution $(u_h(t))$, the exact solution $(y(t))$ and the absolute error AE for Example 4.1. The results from the table show that, the Adomian decomposition method gives a good approximation to exaction solution.

***Table 1.** 1 shown the exact solution$y(t)$ and approximate Solutions $u_h(t)$ and the with Absolute Errors (AE).*

x	y	$u_h(0)$	AE($u_h(0)$)	$u_h(1)$	AE($u_h(1)$)
1.0000	0.000000	0.000000	0.00e+00	0.000000	0.00e+00
1.2455	0.078172	0.068577	9.59e-03	0.076994	1.18e-03
1.4909	0.220378	0.174142	4.62e-02	0.212348	8.03e-03
1.7364	0.412362	0.288467	1.24e-01	0.383452	2.89e-02
1.9819	0.656442	0.401371	2.55e-01	0.581022	7.54e-02
2.2273	0.960985	0.507415	4.54e-01	0.798606	1.62e-01
2.4728	1.339265	0.603428	7.36e-01	1.030958	3.08e-01
2.7183	1.809770	0.687607	1.12e+00	1.273591	5.36e-01

Figure 2 presents the graph of the absolute error (AE). From the graph, we observe that the AE for x value between 1 and 1.5 increases, however, for x values $1.5 < x < 2.3$ error decreases steadily. This error graph shows that the Adomian decomposition method gives a good approximation to the exact solution.

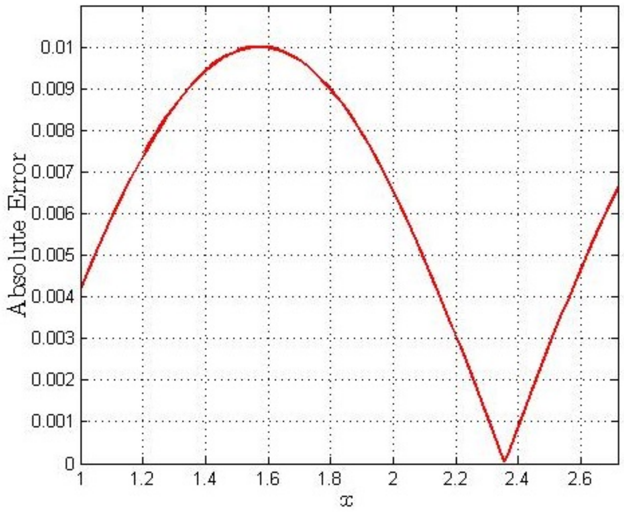

Figure 2. *A graph of the the absolute error AE for Example 4.1.*

Example 4.2. Consider the non-linear fractional differential equation with integral boundary condition

$$\begin{cases} D^{\alpha}u(t) + D^{\beta}u(t) + \mu F(t, u(t)) = 0, \ t \in (1, e), \\ u(1) = 1, \ u'(1) = -1, \ u(e) = \lambda \int_1^e u(s)ds, \end{cases} \tag{32}$$

where $1 < \alpha \le 2$, $0 < \beta < 1$, $\mu = \frac{1}{2}$, $\lambda = \frac{3}{2}$, and

$$F(t, u(t)) = t(u^3(t) + \cos(t) + 1).$$

We decompose the solution $u(t)$ as:

$$u(t) = \sum_{n=0}^{\infty} u_n(t),$$

and the nonlinear term $F(t, u(t))$ as:

$$F(t, u(t)) = \sum_{n=0}^{\infty} A_n(t),$$

where A_n are Adomian polynomials defined recursively and the first few Adomian polynomials are presented below:

$$\begin{aligned} A_0(t) &= u_0^3(t), \\ A_1(t) &= 3u_0^2(t)u_1(t), \\ A_2(t) &= 3u_0^2(t)u_2(t) + 6u_0(t)u_1^2(t), \\ A_3(t) &= 3u_0^2(t)u_3(t) + 6u_0(t)u_1(t)u_2(t) + 6u_1^3(t). \end{aligned}$$

We make use of the boundary conditions to construct the approximate solution function as follows

$$u_0(t) = 1 - \log t + \frac{3}{4}(\log t)^2.$$

For $n \ge 1$, the terms are defined by:

$$u_n(t) = -\frac{1}{2}\left[I^{\alpha}(F_n(t)) + I^{\beta}(F_n(t))\right],$$

where $F_n(t) = tA_{n-1}(t) + \cos(t) + 1$, and I^{α}, I^{β} are Hadamard fractional integrals:

$$I^{\alpha}[f](t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds.$$

we give the approximate solution function as follows

$$u(t) \approx u_0(t) + u_1(t) + u_2(t) + u_3(t) + u_4(t),$$

Using a computer software (Matlab), various terms of the approximate solution function $u(t)$ are evaluated. The result is as shown on the graph below.

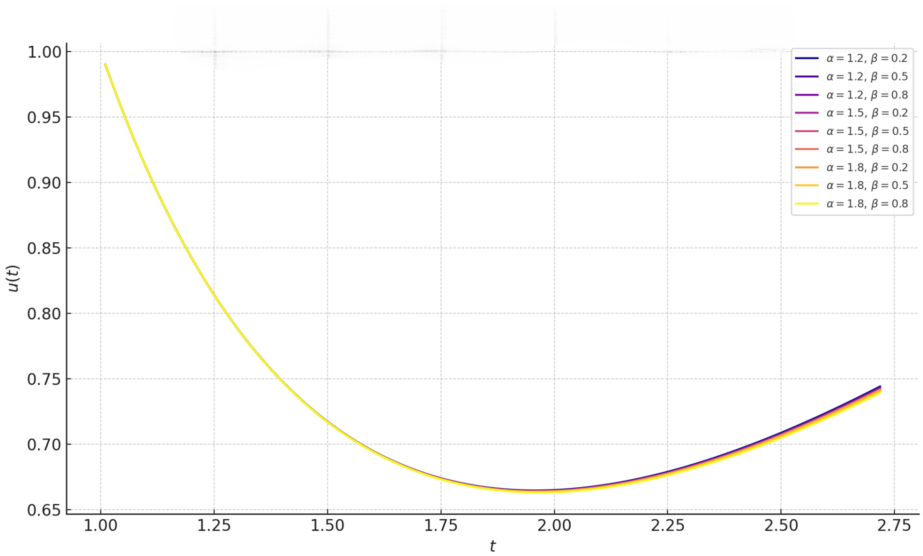

Figure 3. *A graph of the approximate solution for multiple values of α and β for Example 4.2.*

applying the Adomian decomposition method, we obtain a series solution for a non-linear Hadamard-type fractional boundary value problem with integral boundary condition.

The graph shows that, the approximate solution for Example 4.2. The graph shows that increasing the value of α and β does not have much effect on the approximate solution.critical look at the graph indicated that the Adomian decomposition method gives a good approximation to the exact solution.

Example 4.3.

$$D_H^{\alpha}u(t) \;-\; \frac{1}{5}\big(u(t)+1\big) \;-\; D_H^{\gamma}u(t) \;=\; f(t,u), \qquad 1<\alpha\le 2,\; 0<\gamma<1, \tag{33}$$

with integral boundary condition

$$u(1)=0, \qquad u(e)=\int_1^e u(s)\,ds. \tag{34}$$

where f(t,u) is given as

$$f(t,u)=\begin{cases}\sin t, & 1\le t<\tau,\\ u^2, & \tau\le t\le e,\end{cases} \qquad \tau := \frac{1+e}{2}. \tag{35}$$

Applying the Hadamard fractional integral I_H^{α} to (33). For $1<\alpha\le 2$ we have

$$I_H^{\alpha}D_H^{\alpha}u(t)=u(t)-u(1)-u'(1)log_e t=u(t)-Alog_e t, \quad A:=u'(1).$$

Hence, the integral equation becomes

$$u(t)=Alog_e t+I_H^{\alpha}\big[D_H^{\gamma}u\big](t)+\frac{1}{5}I_H^{\alpha}[u](t)+\frac{1}{5}I_H^{\alpha}[1](t)+I_H^{\alpha}\big[f(t,u)\big](t), \tag{36}$$

where,

$$I_H^{\alpha}[1](t)=\frac{(log_e t)^{\alpha}}{\Gamma(\alpha+1)}.$$

equation (36) is written as

$$u(t)=G(t)+\mathcal{L}[u](t)+\mathcal{N}[u](t),$$

with

$$\begin{aligned}G(t) &:= Alog_e t+\frac{1}{5}\frac{(log_e t)^{\alpha}}{\Gamma(\alpha+1)},\\ \mathcal{L}[u](t) &:= I_H^{\alpha}\big[D_H^{\gamma}u\big](t)+\frac{1}{5}I_H^{\alpha}[u](t),\\ \mathcal{N}[u](t) &:= I_H^{\alpha}\big[f(t,u)\big](t).\end{aligned}$$

We, assume the Adomian decomposition of the form

$$u(t)=\sum_{n=0}^{\infty}u_n(t), \qquad f(t,u)=\sum_{n=0}^{\infty}A_n(t),$$

with Adomian polynomials defined as

$$A_n(t)=\frac{1}{n!}\,\frac{\partial^n}{\partial\lambda^n}f\Big(t,\sum_{k=0}^{\infty}\lambda^k u_k(t)\Big)\bigg|_{\lambda=0}.$$

we choose the equation as the initial approximation,

$$u_0(t)=Alog_e t+\frac{1}{5}\frac{(log_e t)^{\alpha}}{\Gamma(\alpha+1)}. \tag{37}$$

The recurrence relation for $n\ge 0$ is

$$u_{n+1}(t)=I_H^{\alpha}\big[D_H^{\gamma}u_n\big](t)+\frac{1}{5}I_H^{\alpha}[u_n](t)+I_H^{\alpha}[A_n](t). \tag{38}$$

For powers of $log_e t$ the Hadamard operators act as follows (valid for $p > -1$):

$$I_H^\alpha\left[(log_e t)^p\right] = \frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}(log_e t)^{p+\alpha}, \quad D_H^\gamma\left[(log_e t)^p\right] = \frac{\Gamma(p+1)}{\Gamma(p+1-\gamma)}(log_e t)^{p-\gamma},$$

and therefore the composition

$$I_H^\alpha D_H^\gamma\left[(log_e t)^p\right] = \frac{\Gamma(p+1)}{\Gamma(p+1+\alpha-\gamma)}(log_e t)^{p+\alpha-\gamma}. \tag{39}$$

Using (37) we compute u_1, u_2 exactly in terms of Gamma-ratios and Hadamard integrals of the Adomian polynomials.

$$u_(0)(t) = A log_e t + \frac{1}{5}\frac{(log_e t)^\alpha}{\Gamma(\alpha+1)}, \qquad A := u'(1).$$

From (38),

$$u_(1)(t) = I_H^\alpha\left[D_H^\gamma u_0\right](t) + \frac{1}{5}I_H^\alpha[u_0](t) + I_H^\alpha[A_0](t).$$

Applying the identities (53) termwise to the log_e-monomials in u_0 gives the following exact algebraic part:

$$u_1(t) = A\frac{(log_e t)^{1+\alpha-\gamma}}{\Gamma(2+\alpha-\gamma)} + \frac{1}{5}\frac{(log_e t)^{2\alpha-\gamma}}{\Gamma(2\alpha+1-\gamma)} + \frac{1}{5}A\frac{(log_e t)^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{1}{25}\frac{(log_e t)^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{40}$$

The remaining part is the Hadamard integral of $A_0(t)$ which is piecewise.
For the piecewise A_0 given by (35) (recall $A_0(t) = f(t, u_0(t))$): If $t < \tau$ then $A_0(t) = \sin t$ and

$$I_H^\alpha[A_0](t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(log_e\frac{t}{s}\right)^{\alpha-1}\frac{\sin s}{s}\,ds.$$

If $t \geq \tau$ then the integral splits,

$$I_H^\alpha[A_0](t) = \frac{1}{\Gamma(\alpha)}\left[\int_1^\tau \left(log_e\frac{t}{s}\right)^{\alpha-1}\frac{\sin s}{s}\,ds + \int_\tau^t \left(log_e\frac{t}{s}\right)^{\alpha-1}\frac{u_0(s)^2}{s}\,ds\right].$$

Thus, the initial iterate is

$$u_1(t) = u_1(t) + I_H^\alpha[A_0](t),$$

with u_1 given in (40).
The recurrence gives

$$u_2(t) = I_H^\alpha\left[D_H^\gamma u_1\right](t) + \frac{1}{5}I_H^\alpha[u_1](t) + I_H^\alpha[A_1](t).$$

If $u_1(t) = \sum_j c_j (log_e t)^{p_j}$ then, by linearity and (53),

$$u_2(t) = \sum_j c_j\frac{\Gamma(p_j+1)}{\Gamma(p_j+1+\alpha-\gamma)}(log_e t)^{p_j+\alpha-\gamma}$$
$$+\frac{1}{5}\sum_j c_j\frac{\Gamma(p_j+1)}{\Gamma(p_j+1+\alpha)}(log_e t)^{p_j+\alpha}. \tag{41}$$

The coefficients c_j, p_j are the explicit coefficients/exponents appearing in (40); the remaining contributions are $I_H^\alpha[A_1](t)$ and the Hadamard integrals arising from the terms contained in $I_H^\alpha[A_0]$, integral condition, we truncate the ADM series at order N to obtain the partial sum,

$$U_N(t; A) = \sum_{n=0}^{N} u_n(t; A).$$

Solve for A from the scalar equation coming from the integral boundary condition in equation (43):

$$\Phi_N(A) := U_N(e; A) - \int_1^e U_N(s; A)\,ds = 0.$$

Using a computer software (MATLAB) the approximate solution is evaluated at different grid points. The result of the evaluation is presented in Table 2.

Table 2. *A Table shown the approximate solution and the absolute error some selected grid points Example 4.3.*

t	$u_h(t)$	AE
1.000000	0.0000000000	0.0000000000
1.245469	-4.5951960000	4.4695490000
1.490938	-9.1188670000	8.9321170000
1.736406	-13.6269150000	13.4096110000
1.981875	-15.2906760000	15.0487300000
2.227344	-9.6791270000	9.3906770000

t	$u_h(t)$	AE
2.472813	-7.1891850000	6.8484000000
2.718282	-16.3871850000	15.9922130000

Table 2 presents the the approximate solution and the absolute error for Example 4.3. from the Table 2, we see that the Adomian decomposition method gives a good approximation to the non- linear fractional boundary value problem with integral boundary condition. This result is consistent with the early presentation in Section 2. This result demonstrated that the proposed method is accurate and efficient in approximating the exact solution.

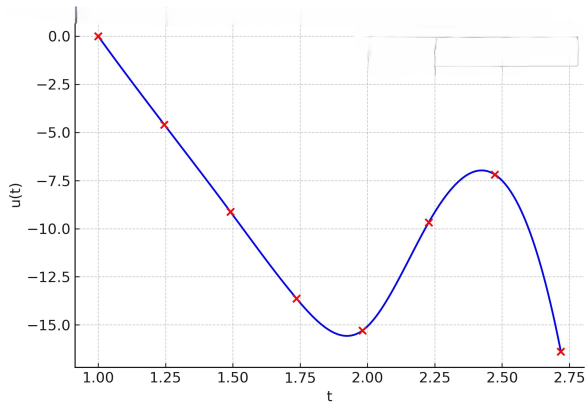

Figure 4. *Graph of the approximate solution $u_h(t)$ for Example 4.3.*

Figure 4 shows the approximate solution and Figure 5 present the absolute error. The result indicated that the Adomian decomposition method is efficient in approximating the exact. It also proof the applicability, accuracy and efficiency of the method.

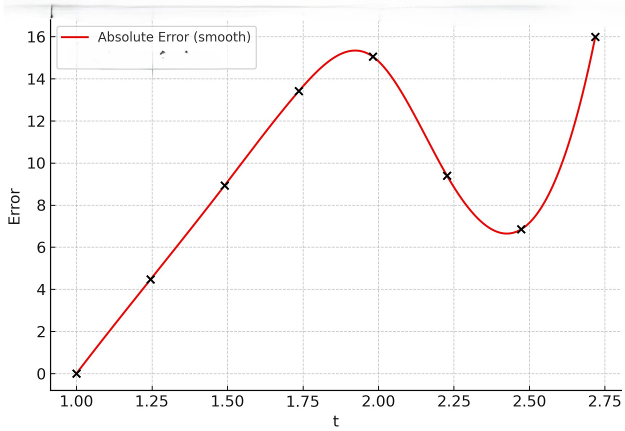

Figure 5. *Graph of the absolute error AE for Example 4.3.*

Example 4.4. Consider the non-linear fractional differential equation with integral boundary condition

$$D_H^\alpha u(t) + D_H^\gamma u(t) - \frac{1}{5}\big(u(t) + t^3 - 4t\big) = 0, \qquad 1 < \alpha \leq 2, \;\; 2 < \gamma < 3, \tag{42}$$

with boundary conditions

$$u(1) = 0, \qquad u(e) = \int_1^e u(s)\, ds. \tag{43}$$

where, D_H^α denotes the Hadamard fractional derivative of order α, D_H^β denote the Hadamard fractional derivative of order β and the Hadamard fractional integral of order α is

$$(I_H^\alpha g)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \Big(log_e \frac{t}{\tau}\Big)^{\alpha-1} \frac{g(\tau)}{\tau}\, d\tau.$$

Apply I_H^α to both sides of (42). Using the identity valid for $1 < \alpha \leq 2$,

$$I_H^\alpha D_H^\alpha u(t) = u(t) - u(1) - u'(1) log_e t = u(t) - A log_e t, \qquad A := u'(1),$$

and $u(1) = 0$, we obtain the exact integral equation

$$u(t) - A log_e t + I_H^\alpha \big[D_H^\gamma u\big](t) - \frac{1}{5} I_H^\alpha \big[u\big](t) - \frac{1}{5} I_H^\alpha \big[t^3 - 4t\big](t) = 0. \tag{44}$$

Rearrange to isolate the unknown u terms on the left

$$u(t) + \underbrace{I_H^\alpha \big[D_H^\gamma u\big](t) - \frac{1}{5} I_H^\alpha \big[u\big](t)}_{=:-K[u](t)} = A log_e t + \frac{1}{5} I_H^\alpha \big[t^3 - 4t\big](t). \tag{45}$$

we introduce the operator K by

$$(Kv)(t) := -\, I_H^{\alpha}\left[D_H^{\gamma} v\right](t) + \frac{1}{5} I_H^{\alpha}[v](t). \tag{46}$$

Then equation (45) is exactly

$$(I-K)u = Alog_e t + G(t), \qquad G(t) := \frac{1}{5} I_H^{\alpha}\left[t^3 - 4t\right](t). \tag{47}$$

The resolvent representation and splitting is as follows.
Assuming $(I-K)^{-1}$ exists (Neumann series convergence), the exact solution is the resolvent action

$$u(t) = (I-K)^{-1}\left[Alog_e t + G\right](t) = \sum_{n=0}^{\infty} K^n\big(Alog_e t + G\big)(t). \tag{48}$$

since the right-hand side of the equation is affine in A, we define

$$\Phi(t) := (I-K)^{-1}[log_e t] = \sum_{n=0}^{\infty} K^n[log_e t], \qquad \Psi(t) := (I-K)^{-1}[G] = \sum_{n=0}^{\infty} K^n[G]. \tag{49}$$

Linearity yields the decomposition

$$u(t) = A\Phi(t) + \Psi(t). \tag{50}$$

Applying the boundary conditions we have Substitute (50) into the boundary condition $u(e) = \int_1^e u(s)\,ds$ to obtain the exact linear scalar equation for A:

$$A\Phi(e) + \Psi(e) = \int_1^e \big(A\Phi(s) + \Psi(s)\big)\,ds.$$

Rearranging to make A the subject, we have:

$$A\Big(\Phi(e) - \int_1^e \Phi(s)\,ds\Big) = \int_1^e \Psi(s)\,ds - \Psi(e). \tag{51}$$

Therefore, provided the denominator is nonzero, the exact value of A is

$$A \;=\; \frac{\displaystyle\int_1^e \Psi(s)\,ds \;-\; \Psi(e)}{\displaystyle\Phi(e) \;-\; \int_1^e \Phi(s)\,ds}\,. \tag{52}$$

Once A is computed by equation (52), the solution is given by equation (50).

When iterates remain finite linear combinations of powers of $log_e t$, the Hadamard operators act simply. For $p > -1$ we have

$$I_H^{\alpha}\left[(log_e t)^p\right] = \frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}(log_e t)^{p+\alpha},$$

$$D_H^{\mu}\left[(log_e t)^p\right] = \frac{\Gamma(p+1)}{\Gamma(p+1-\mu)}(log_e t)^{p-\mu},$$

and thus (when the Gamma arguments are finite)

$$I_H^{\alpha} D_H^{\gamma}\left[(log_e t)^p\right] = \frac{\Gamma(p+1)}{\Gamma(p+1+\alpha-\gamma)}(log_e t)^{p+\alpha-\gamma}. \tag{53}$$

Therefore, starting from the functions $log_e t$ and $G(t)$, each K-iterate applied to a log_e-power yields another explicit log_e-power with known Gamma prefactor. This allows closed-form computation of the first several terms of Φ and Ψ.

$$\Phi(t) = log_e t + K[log_e t] + K^2[log_e t] + \cdots,$$
$$K[log_e t] = -\frac{1}{5} I_H^{\alpha}[log_e t] + I_H^{\alpha} D_H^{\gamma}[log_e t] \;= -\frac{1}{5}\frac{(log_e t)^{1+\alpha}}{\Gamma(2+\alpha)} \;+\; \frac{\Gamma(2)}{\Gamma(2+\alpha-\gamma)}(log_e t)^{1+\alpha-\gamma},$$

and so on. Each further K simply applies the linear map (46) to the previous log_e-power result and produces higher or lower powers of $log_e t$ depending on α, γ.

The known source is

$$G(t) = \frac{1}{5} I_H^{\alpha}[t^3 - 4t](t) = \frac{1}{5\Gamma(\alpha)} \int_1^t \Big(log_e \frac{t}{\tau}\Big)^{\alpha-1} \big(\tau^3 - 4\tau\big)\frac{d\tau}{\tau}.$$

With the substitution $x = log_e(t/\tau)$ this reduces to one-dimensional integrals of the form

$$\int_0^{log_e t} x^{\alpha-1} e^{m(log_e t - x)}\,dx = e^{m log_e t} \int_0^{log_e t} x^{\alpha-1} e^{-mx}\,dx = t^m\, \gamma\big(\alpha,\; m log_e t\big),$$

where $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function and $m \in \{2, 0\}$ (after simplification). Thus $G(t)$ can be expressed in closed form in terms of t^m times incomplete-gamma functions; it is entirely explicit and smooth on $[1, e]$.

We consider the Hadamard fractional boundary value problem on $t \in (1, e)$:

$$D_H^{\alpha} u(t) + D_H^{\gamma} u(t) - \frac{1}{5}\left(u(t) + t^3 - 4t\right) = 0, \quad 1 < \alpha \leq 2,\ 2 < \gamma < 3. \tag{54}$$

with boundary conditions:

$$u(1) = 0, \quad u(e) = \int_1^e u(s)\, ds, \tag{55}$$

Using a computer software (MATLAB), we evaluate the approximate solution function and present the result in Table 3.

Table 3. *The approximate solution and the exact solution and the absolute error for Example 4.4.*

t	$u_h(t)$	$y(t)$	AE
1.0	0.000000	0.000000	0.000000
1.2	0.041532	0.041534	0.000002
1.4	0.117840	0.117842	0.000002
1.6	0.228915	0.228917	0.000002
1.8	0.374748	0.374750	0.000002
2.0	0.555332	0.555334	0.000002
2.4	1.020164	1.020166	0.000002
2.7	1.477821	1.477823	0.000002

Figure 6 shows the graph of the absolute error. a close look at the graph indicated that the absolute error increased steadily for the value $1 < t < 2$ and then decreases down word . The implication is that, the error decreases as t approaches $t = 3$ and increases as t approaches 2 from the left side. This is an indication that for $1 < t < 2$ the error is high and for $2 < t < 3$ the absolute error is low. The conclusion is that for lower values of the fractional operator, the method give high error, however, for higher values of the fractional derivative the method gives minimal errors. We also see from the absolute error graph that the Adomia decomposition method use in the research paper is efficient and accurate.

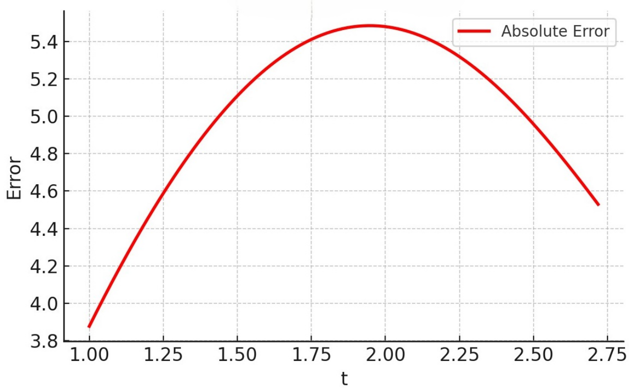

Figure 6. *Graph of the absolute error for Example 4.4.*

5. Summary and Conclusion

We give the existence and uniqueness of solution for the non-linear fractional differential equation with integral boundary conditions via the Upper and Lower solution approach. The fixed point theorem is use to establish the uniqueness of the solution of non-linear fractional differential equation with integral boundary condition. For the numerical solution, we make use of the Adomian decomposition method, some Examples are constructed to illustrate the applicability and effectiveness of the method. From the graphs and tables presented in Section 3 we see that, the method gives a good approximation to the exact solution.

Author Contributions

Mtema James Chin is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The author declares no conflicts of interest.

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