American Journal of Applied Mathematics
2025; 13(5): 308-319
http://www.sciencepublishinggroup.com/j/ajam
doi: 10.11648/j.ajam.20251305.11

Quadruple Laplace-Sumudu-Aboodh-Elzaki Transform and Its Applications

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To cite this article:
Olaniyan Adegoke Stephen, Omolara Fatimah Bakre, Agbefe Emumena Emmanuel, KazeemMoshood Tolulope. (2025). Quadruple Laplace-Sumudu-Aboodh-Elzaki Transform and Its Applications. *American Journal of Applied Mathematics, 13*(5), 308-319. https://doi.org/10.11648/j.ajam.20251305.11

Received: 28 June 2025; **Accepted:** 9 July 2025; **Published:** 5 September 2025

Abstract: This research introduces an approach that combines four different transforms such as Laplace transform, Sumudu transform, Aboodh transform and Elzaki transform to produce a quadruple transform. We state and apply the quadruple transform for several functions of four variables and went further to proof some fundamental properties and theorems. The existence analysis of the method and partial derivatives theorems were proven. Moreover, we examined how efficient and applicable the transform is by applying it to some integral equations and partial differential equations.

Keywords: Partial Differential Equation, Integral Equation, Laplace Transform, Sumudu Transform, Elzaki Transform, Aboodh Transform

1. Introduction

The application of differential equations in physics, engineering and applied sciences has been subject of active research for years. The importance of differential equations made many researchers to be interested in finding different solutions. Among the solution methods, the integral transform techniques have been the most used in finding solutions to both ordinary and partial differential equations. One dimensional integral transform such as Laplace, Sumudu, Adoodh, Elzaki, Mellin, Shehu and many others have been used to get analytical solutions of ordinary, partial and even fractional differential equations(See [8, 9, 12, 15, 19]). Due to researcher's quest for development, one dimensional transform was extended to two dimensional transform which is a superior version and appeared to be a very efficient method for solving all classes of differential equations (Also See [1–3, 6, 10, 11, 16]). Then subsequently, three dimensional transform such as Triple Shehu [7, 18], Triple Sumudu [13], Triple Aboodh [4] and Triple Laplace [17] were introduced. Recently, three dimensional transforms were extended to four dimensional transform called the Quadruple transform to deal with all kinds of differential equations. In [14], Quadruple Shehu transform was introduced to deal with integral equations and partial differential equations. In 2019, Suliman Alfaqeih [5] introduced quadruple Aboodh transform, its inverse and applied the transform for several functions of four variables. The transform was applied to integral and partial including fractional differential equations to confirm its efficiency and applicability. Very recently, researchers went further to focus attention on the way in which different transform can be combined in two and three dimensional transforms. Ahmed S. A. et al [2] demonstrated an efficient new double transform called the double Laplace-Sumudu transform to solve partial differential equations. Al-Aati et al [3] combined the Laplace transform, Aboodh transform and Sumudu transform to give another transform called the triple Laplace-Adoodh-Sumudu transform to solve some kinds of partial differential equations. However, this paper aims to extend this idea to a four dimensional transform by introducing a special quadruple transform called quadruple Laplace-Aboodh-Sumudu-Elzaki (L-S-A-E) transform which is made up of Laplace transform, Aboodh transform, Sumudu transform and Elzaki transform. This transform will be used to solve integral equations and partial differential equations. Its basic properties, functions and several theorems with some examples will also be presented.

2. Preliminaries

This section presents the basic properties and definitions that are of importance to quadruple Laplace-Sumudu-Aboodh-Elzaki transform.

Definition 2.1. The Laplace transform of a function $h(x)$ is given by

$$L[h(x)] = H(\rho) = \int_0^\infty e^{-\rho x} h(x)dx,$$

while its inverse is

$$h(x) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\rho x} H(\rho)d\rho.$$

Definition 2.2. The Sumudu transform of a function $h(t)$ is given as

$$S[h(t)] = H(\sigma) = \frac{1}{\sigma}\int_0^\infty e^{-\frac{t}{\sigma}} h(t)dt,$$

while its inverse is

$$h(t) = \frac{1}{2\pi i}\int_{\omega - i\infty}^{\omega + i\infty} \frac{1}{\sigma} e^{\frac{t}{\sigma}} H(\sigma)d\sigma.$$

Definition 2.3. The Aboodh transform of a function $h(y)$ is given as

$$A[h(y)] = H(\alpha) = \frac{1}{\alpha}\int_0^\infty e^{-\alpha y} h(y)dy,$$

while its inverse is

$$h(y) = \frac{1}{2\pi i}\int_{\xi - i\infty}^{\xi + i\infty} \frac{1}{\alpha} e^{\alpha y} H(\alpha)d\alpha.$$

Definition 2.4. The Elzaki transform of a function $h(z)$ is given as

$$E[h(z)] = H(\beta) = \beta\int_0^\infty e^{-\frac{z}{\beta}} h(z)dz,$$

while its inverse is

$$h(z) = \frac{1}{2\pi i}\int_{\mu - i\infty}^{\mu + i\infty} \beta e^{\frac{z}{\beta}} H(\beta)d\beta.$$

3. Quadruple Laplace-Sumudu-Aboodh-Elzaki (L-S-A-E) of Some Basic Functions and Examples

The definition, inverse and examples of L-S-A-E are presented in this section.

Definition 3.1. The quadruple Laplace-Sumudu-Aboodh-Elzaki transform of the function $h(x,t,y,z)$ of variables $x > 0, t > 0, y > 0$ and $z > 0$ can be defined as

$$\begin{gathered} L_x S_t A_y E_z[h(x,t,y,z)] = H(\rho,\sigma,\alpha,\beta) \\ = \frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x + \frac{t}{\sigma} + \alpha y + \frac{z}{\beta}\right)} \\ h(x,t,y,z)dxdtdydz \end{gathered}$$

The linear integral transformation of quadruple L-S-A-E transform can clearly be shown as expressed below:

$$\begin{aligned} & L_x S_t A_y E_z[\lambda h(x,t,y,z) + \psi g(x,t,y,z)] \\ &= \frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x + \frac{t}{\sigma} + \alpha y + \frac{z}{\beta}\right)}[\lambda h(x,t,y,z) \\ &\quad + \psi g(x,t,y,z)]dxdtdydz \\ &= \frac{\beta\lambda}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x + \frac{t}{\sigma} + \alpha y + \frac{z}{\beta}\right)}dxdtdydz \\ &\quad + \frac{\beta\psi}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x + \frac{t}{\sigma} + \alpha y + \frac{z}{\beta}\right)}dxdtdydz \\ &= \lambda L_x S_t A_y E_z[h(x,t,y,z)] + \psi L_x S_t A_y E_z[g(x,t,y,z)] \end{aligned}$$

Definition 3.2. The inverse of L-S-A-E transform $L_x^{-1}S_t^{-1}A_y^{-1}E_z^{-1}[H(\rho,\sigma,\alpha,\beta)] = h(x,t,y,z)$ can be defined as

$$\begin{gathered} L_x^{-1}S_t^{-1}A_y^{-1}E_z^{-1}[H(\rho,\sigma,\alpha,\beta)] = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\rho x}\left[\frac{1}{2\pi i}\int_{\omega - i\infty}^{\omega + i\infty}\right. \\ \left.\frac{1}{\sigma}e^{\frac{t}{\sigma}}\left[\frac{1}{2\pi i}\int_{\xi - i\infty}^{\xi + i\infty}\frac{1}{\alpha}e^{\alpha y}\left[\frac{1}{2\pi i}\int_{\mu - i\infty}^{\mu + i\infty}\beta e^{\frac{z}{\beta}}d\beta H(\rho,\sigma,\alpha,\beta)\right]d\alpha\right]d\sigma\right]d\rho \end{gathered}$$

where $H(\rho,\sigma,\alpha,\beta)$ and for all ρ,σ,α and β are analytic function defined in the region $Re\rho \geq \gamma$, $Re\sigma \geq \omega$, $Re\alpha \geq \xi$ and $Re\beta \geq \mu$

3.1. Examples

1. Let $h(x,t,y,z)=1$ for $x>0, t>0, y>0$ and $z>0$ then

$$H(\rho,\sigma,\alpha,\beta)=L_xS_tA_yE_z[1]=\frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+\frac{t}{\sigma}+\alpha y+\frac{z}{\beta}\right)}dxdtdydz$$

$$=\int_0^\infty e^{-\rho x}dx\frac{1}{\sigma}\int_0^\infty e^{-\frac{t}{\sigma}}dt\int_0^\infty e^{-\alpha y}dy\beta\int_0^\infty e^{-\frac{z}{\beta}}dz=\frac{\beta^2}{\rho\alpha^2}$$

2. Let $h(x,t,y,z)=e^{ax+bt+cy+dz}$ then

$$H(\rho,\sigma,\alpha,\beta)=L_xS_tA_yE_z[e^{ax+bt+cy+dz}]=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+\frac{t}{\sigma}+\alpha y+\frac{z}{\beta}\right)}e^{ax+bt+cy+dz}dxdtdydz$$

$$=\frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\rho-a)x}e^{-\left(\frac{1}{\sigma}-b\right)t}e^{-(\alpha-c)y}e^{-\left(\frac{1}{\beta}-d\right)z}dxdtdydz$$

$$=\left(\frac{1}{\rho-a}\right)\left(\frac{1}{1-b\sigma}\right)\left(\frac{1}{\alpha(\alpha-c)}\right)\left(\frac{\beta^2}{1-\beta d}\right)$$

3. Let $h(x,t,y,z)=e^{i(ax+bt+cy+dz)}$ then

$$H(\rho,\sigma,\alpha,\beta)=L_xS_tA_yE_z[e^{i(ax+bt+cy+dz)}]=\frac{\beta^2}{(\rho-ia)(1-ib\sigma)\alpha(\alpha-ic)(1-i\beta d)}$$

$$-\frac{\beta^2(\rho+ia)(1+ib\sigma)(\alpha+ic)(1+i\beta d)}{\alpha(\rho^2+a^2)(1+b^2\sigma^2)(\alpha^2+c^2)(1+\beta^2d^2)}=\frac{\alpha\beta^2(\alpha\rho-ac)(b\beta\sigma d-1)(a\alpha+c\rho)(b\sigma+\beta d)}{\alpha(\rho^2+a^2)(1+b^2\sigma^2)(\alpha^2+c^2)(1+\beta^2d^2)}$$

$$-\frac{i\alpha\beta^2(ac-\alpha\rho)(b\sigma+\beta d)(a\alpha+c\rho)(b\beta\sigma d-1)}{\alpha(\rho^2+a^2)(1+b^2\sigma^2)(\alpha^2+c^2)(1+\beta^2d^2)}$$

Hence, if $h(x,t,y,z)=\sin(ax+bt+cy+dz)$ then

$$H(\rho,\sigma,\alpha,\beta)=L_xS_tA_yE_z[\sin(ax+bt+cy+dz)]=\frac{\alpha\beta^2(\alpha\rho-ac)(b\beta\sigma d-1)(a\alpha+c\rho)(b\sigma+\beta d)}{\alpha(\rho^2+a^2)(1+b^2\sigma^2)(\alpha^2+c^2)(1+\beta^2d^2)}$$

and if $h(x,t,y,z)=\cos(ax+bt+cy+dz)$ then

$$L_xS_tA_yE_z[\cos(ax+bt+cy+dz)]=\frac{\alpha\beta^2(\alpha\rho-ac)(b\beta\sigma d-1)(a\alpha+c\rho)(b\sigma+\beta d)}{\alpha(\rho^2+a^2)(1+b^2\sigma^2)(\alpha^2+c^2)(1+\beta^2d^2)}$$

4. Let $h(x,t,y,z)=\sinh(ax+bt+cy+dz)$ and since

$$\sinh(ax+bt+cy+dz)=\frac{e^{ax+bt+cy+dz}-e^{-(ax+bt+cy+dz)}}{2}$$

then

$$L_xS_tA_yE_z[\sinh(ax+bt+cy+dz)]=\frac{\beta^2}{2}\left(\frac{1}{(\rho-a)(1-b\sigma)\alpha(\alpha-c)(1-\beta d)}\right.$$

$$\left.-\frac{1}{(\rho+a)(1+b\sigma)\alpha(\alpha+c)(1+\beta d)}\right)$$

Also,for $h(x,t,y,z)=\cosh(ax+bt+cy+dz)$ where

$$\cosh(ax+bt+cy+dz)=\frac{e^{ax+bt+cy+dz}+e^{-(ax+bt+cy+dz)}}{2}$$

then

$$L_xS_tA_yE_z[\cosh(ax+bt+cy+dz)]$$

$$= \frac{\beta^2}{2}\left(\frac{1}{(\rho-a)(1-b\sigma)\alpha(\alpha-c)(1-\beta d)} + \frac{1}{(\rho+a)(1+b\sigma)\alpha(\alpha+c)(1+\beta d)}\right)$$

5. Let $h(x,t,y,z) = (xtyz)^n$ then,

$$H(\rho,\sigma,\alpha,\beta) = L_xS_tA_yE_z[(xtyz)^n] = \int_0^\infty x^n e^{-\rho x}dx\frac{1}{\sigma}\int_0^\infty t^n e^{-\frac{t}{\sigma}}dt\int_0^\infty y^n e^{-\alpha y}dy\beta$$

$$= \int_0^\infty z^n e^{-\frac{z}{\beta}}dz\frac{\beta^{n+2}\sigma^n(n!)^4}{\rho^{n+1}\alpha^{n+2}}$$

6. Let $h(x,t,y,z) = x^a t^b y^c z^d$ then,

$$H(\rho,\sigma,\alpha,\beta) = L_xS_tA_yE_z[x^a t^b y^c z^d] = \int_0^\infty x^a e^{-\rho x}dx\frac{1}{\sigma}\int_0^\infty t^b e^{-\frac{t}{\sigma}}dt\int_0^\infty y^c e^{-\alpha y}dy\beta\int_0^\infty z^d e^{-\frac{z}{\beta}}dz$$

$$= \frac{\Gamma(a+1)}{\rho^{a+1}}\sigma^b\Gamma(b+1)\frac{\Gamma(c+1)}{\alpha^{c+2}}\beta^{d+2}\Gamma(d+1)$$

7. Let $h(x,t,y,z) = d(x)e(t)f(y)g(z)$ then,

$$H(\rho,\sigma,\alpha,\beta) = L_xS_tA_yE_z[d(x)e(t)f(y)g(z)] = \frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+\frac{t}{\sigma}+\alpha y+\frac{z}{\beta}\right)}$$

$$d(x)e(t)f(y)g(z)dxdtdydz = \left[\int_0^\infty e^{-\rho x}d(x)dx\right]\left[\frac{1}{\sigma}\int_0^\infty e^{-\frac{t}{\sigma}}e(t)dt\right]\left[\int_0^\infty e^{-\alpha y}f(y)dy\right]$$

$$\left[\beta\int_0^\infty e^{-\frac{z}{\beta}}g(z)dz\right] = L_x[d(x)]S_t[e(t)]A_y[f(y)]E_z[g(z)]$$

4. Existence and Uniqueness of the Quadruple Laplace-Sumudu-Aboodh-Elzaki Transform

Definition 4.1. A function $h(x,t,y,z)$ is said to be of exponential order $a>0, b>0, c>0$ and $d>0$ on $0\leq x,t,y,z\leq\infty$ if there exists a positive constant A such that for all $x>X, t>T, y>Y$ and $z>Z$; we have

$$h(x,t,y,z)\leq Ae^{ax+bt+cy+dz}$$

Hence, $h(x,t,y,z) = O[e^{ax+bt+cy+dz}]$ as $x,t,y,z\to\infty$

Theorem 4.1. Let $h(x,t,y,z)$ be continuous in every finite interval $(0,X),(0,T),(0,Y),(0,Z)$ and of exponential order $e^{ax+bt+cy+dz}$, then the quadruple Laplace-Sumudu-Aboodh-Elzaki transform of $h(x,t,y,z)$ exists for all ρ,σ,α,β provided $Re\rho>a$, $Re\sigma>b$, $Re\alpha>c$ and $Re\beta>d$.

Proof:

$$|H(\rho,\sigma,\alpha,\beta)| = \left|\frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+\frac{t}{\sigma}+\alpha y+\frac{z}{\beta}\right)}h(x,t,y,z)dxdtdydz\right|$$

$$\leq A\int_0^\infty e^{-(\rho-a)x}\frac{1}{\sigma}\int_0^\infty e^{-\left(\frac{1}{\sigma}-b\right)t}\frac{1}{\alpha}\int_0^\infty e^{-(\alpha-c)y}\beta\int_0^\infty e^{-\left(\frac{1}{\beta}-d\right)z}dxdtdydz = \frac{A\beta^2}{(\rho-a)(1-b\sigma)\alpha(\alpha-c)(1-\beta d)}$$

for $Re\rho>a$, $Re\frac{1}{\sigma}>b$, $Re\alpha>c$ and $Re\frac{1}{\beta}>d$. It follows from (2) that the limiting property of quadruple Laplace-Sumudu-Aboodh-Elzaki transform can be expressed as

$$\lim_{\rho\to\infty,\sigma\to\infty,\alpha\to\infty,\beta\to\infty} H(\rho,\sigma,\alpha,\beta) = 0$$

Theorem 4.2. If $f(x,t,y,z)$ and $g(x,t,y,z)$ be continuous functions defined for $x,t,y,z\geq 0$ having the quadruple Laplace-Sumudu-Aboodh-Elzaki transform $F(\rho,\sigma,\alpha,\beta)$ and $G(\rho,\sigma,\alpha,\beta)$ respectively. Let $F(\rho,\sigma,\alpha,\beta) = G(\rho,\sigma,\alpha,\beta)$, then $f(\rho,\sigma,\alpha,\beta) = g(\rho,\sigma,\alpha,\beta)$.

Proof: Since

$$h(x,t,y,z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\rho x}\left[\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{1}{\sigma}e^{\frac{t}{\sigma}}\ \left[\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}\frac{1}{\alpha}e^{\alpha y}\left[\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\beta e^{\frac{z}{\beta}}d\beta \quad H(\rho,\sigma,\alpha,\beta)\right] d\alpha\right] d\sigma\right] d\rho$$

then, we can infer that

$$f(x,t,y,z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\rho x}\left[\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{1}{\sigma}e^{\frac{t}{\sigma}}\left[\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}\frac{1}{\alpha}e^{\alpha y}\left[\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\beta e^{\frac{z}{\beta}}F(\rho,\sigma,\alpha,\beta)d\beta\right] d\alpha\right] d\sigma\right] d\rho$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\rho x}\left[\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{1}{\sigma}e^{\frac{t}{\sigma}}\left[\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}\frac{1}{\alpha}e^{\alpha y}\left[\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\beta e^{\frac{z}{\beta}}G(\rho,\sigma,\alpha,\beta)d\beta\right] d\alpha\right] d\sigma\right] d\rho = g(x,t,y,z)$$

5. Some Basic Properties of Quadruple Laplace-Sumudu-Aboodh-Elzaki Transform

5.1. Shifting Property

$$L_xS_tA_yE_z[e^{(ax+bt+cy+dz)}h(x,t,y,z)] = \frac{1-\beta d}{\alpha(1-\sigma b)}H\left(\rho-a, \frac{\sigma}{1-\sigma b}, \alpha-c, \frac{\beta}{1-\beta d}\right)$$

Proof:

$$L_xS_tA_yE_z[e^{(ax+bt+cy+dz)}h(x,t,y,z)] = \frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+\frac{t}{\sigma}+\alpha y+\frac{z}{\beta}\right)}e^{ax+bt+cy+dz}h(x,t,y,z)dxdtdydz$$

$$= \frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\rho-a)x}e^{-\left(\frac{1}{\sigma}-b\right)t}e^{-(\alpha-c)y}e^{-\left(\frac{1}{\beta}-d\right)z}h(x,t,y,z)dxdtdydz$$

Using $u=\frac{\sigma}{1-\sigma b}$ and $v=\frac{\beta}{1-\beta d}$, then

$$L_xS_tA_yE_z[e^{(ax+bt+cy+dz)}h(x,t,y,z)] = \frac{v(1-\beta d)}{u\alpha(1-\sigma b)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\rho-a)x}e^{-\frac{t}{u}}$$
$$e^{-(\alpha-c)y}e^{-\frac{z}{v}}h(x,t,y,z)dxdtdydz = \frac{1-\beta d}{\alpha(1-\sigma b)}H\left(\rho-a, \frac{\sigma}{1-\sigma b}, \alpha-c, \frac{\beta}{1-\beta d}\right)$$

Theorem 5.1. If $H(\rho,\sigma,\alpha,\beta) = L_xS_tA_yE_z[h(x,t,y,z)]$, then

$$L_xS_tA_yE_z[h(x-a,t-b,y-c,z-d)\mathbf{H}(x-a,t-b,y-c,z-d) = e^{-\left(\rho a+\frac{b}{\sigma}+\alpha c+\frac{d}{\beta}\right)}H(\rho,\sigma,\alpha,\beta)$$

where the Heaviside unit step function denoted by $\boldsymbol{H}(x,t,y,z)$ is given as

$$\boldsymbol{H}(x-a,t-b,y-c,z-d) = \begin{cases} 1 & x>a, t>b, y>c, z>d \\ 0 & x<a, t<b, y<c, z<d \end{cases}$$

Given by the definition, we have

$$L_xS_tA_yE_z[h(x-a,t-b,y-c,z-d)\boldsymbol{H}(x-a,t-b,y-c,z-d)] = \frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+\frac{t}{\sigma}+\alpha y+\frac{z}{\beta}\right)}$$
$$\begin{bmatrix} h(x-a,t-b,y-c,z-d) \\ \boldsymbol{H}(x-a,t-b,y-c,z-d)\end{bmatrix} dxdtdydz = \frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x+\frac{t}{\sigma}+\alpha y+\frac{z}{\beta}\right)}$$
$$h(x-a,t-b,y-c,z-d)dxdtdydz$$

Putting $r = x - a, u = t - b, v = y - c$ and $w = z - d$, we get

$$= \frac{\beta}{\sigma\alpha}\left[e^{-\left(\rho a+\frac{b}{\sigma}+\alpha c+\frac{d}{\beta}\right)}\right]\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho r+\frac{u}{\sigma}+\alpha v+\frac{w}{\beta}\right)}f(r,u,v,w)drdudvdw$$

$$= e^{-(\rho a+\frac{b}{\sigma}+\alpha c+\frac{d}{\beta})}H(\rho,\sigma,\alpha,\beta)$$

Theorem 5.2. Let function $h(x,t,y,z)$ be periodic on periods a, b, c, d and $L_xS_tA_yE_z[h(x,t,y,z)]$ exists then:

$$L_xS_tA_yE_z[h(x,t,y,z)] = \frac{(1-e^{-\rho a-\frac{b}{\sigma}-\alpha c-\frac{d}{\beta}})^{-1}\beta}{\sigma\alpha}\int_0^a\int_0^b\int_0^c\int_0^d e^{-\rho x-\frac{t}{\sigma}-\alpha y-\frac{z}{\beta}}h(x,t,y,z)dxdtdydz$$

Given by the definition,

$$L_xS_tA_yE_z[h(x,t,y,z)] = \frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\rho x-\frac{t}{\sigma}-\alpha y-\frac{z}{\beta}}h(x,t,y,z)dxdtdydz$$

$$= \frac{\beta}{\sigma\alpha}\int_0^a\int_0^b\int_0^c\int_0^d e^{-\rho x-\frac{t}{\sigma}-\alpha y-\frac{z}{\beta}}h(x,t,y,z)dxdtdydz$$

$$+\frac{\beta}{\sigma\alpha}\int_a^\infty\int_b^\infty\int_c^\infty\int_d^\infty e^{-\rho x-\frac{t}{\sigma}-\alpha y-\frac{z}{\beta}}h(x,t,y,z)dxdtdydz$$

Again putting $x = r + a, t = u + b, y = v + c$ and $z = w + d$, we have

$$L_xS_tA_yE_z[h(x,t,y,z)] = \frac{\beta}{\sigma\alpha}\int_0^a\int_0^b\int_0^c\int_0^d e^{-\rho x-\frac{t}{\sigma}-\alpha y-\frac{z}{\beta}}h(x,t,y,z)dxdtdydz$$

$$+e^{-\rho a-\frac{b}{\sigma}-\alpha c-\frac{d}{\beta}}\frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\rho r-\frac{u}{\sigma}-\alpha v-\frac{w}{\beta}}h(r+a,u+b,v+c,w+d)drdudvdw$$

$$= \frac{\beta}{\sigma\alpha}\int_0^a\int_0^b\int_0^c\int_0^d e^{-\rho x-\frac{t}{\sigma}-\alpha y-\frac{z}{\beta}}h(x,t,y,z)dxdtdydz + e^{-\rho a-\frac{b}{\sigma}-\alpha c-\frac{d}{\beta}}\frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\rho r-\frac{u}{\sigma}-\alpha v-\frac{w}{\beta}}$$

$$h(r,u,v,w)drdudvdw = \frac{\beta}{\sigma\alpha}\int_0^a\int_0^b\int_0^c\int_0^d e^{-\rho x-\frac{t}{\sigma}-\alpha y-\frac{z}{\beta}}h(x,t,y,z)dxdtdydz + e^{-\rho a-\frac{b}{\sigma}-\alpha c-\frac{d}{\beta}}L_xS_tA_yE_z[h(x,t,y,z)]$$

As a consequence of equation (3), we have

$$L_xS_tA_yE_z[h(x,t,y,z)] = \frac{(1-e^{-\rho a-\frac{b}{\sigma}-\alpha c-\frac{d}{\beta}})^{-1}\beta}{\sigma\alpha}\int_0^a\int_0^b\int_0^c\int_0^d e^{-\rho x-\frac{t}{\sigma}-\alpha y-\frac{z}{\beta}}h(x,t,y,z)dxdtdydz$$

5.2. Convolution Theorem of Quadruple Laplace-Sumudu-Aboodh-Elzaki Transform

Theorem 5.3. If at the point $(\rho,\sigma,\alpha,\beta)$, the integral

$$F_1(\rho,\sigma,\alpha,\beta) = \frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\rho x-\frac{t}{\sigma}-\alpha y-\frac{z}{\beta}}f_1(x,t,y,z)dxdtdydz$$

converges and if

$$F_2(\rho,\sigma,\alpha,\beta) = \frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\rho x-\frac{t}{\sigma}-\alpha y-\frac{z}{\beta}}f_2(x,t,y,z)dxdtdydz$$

is absolutely convergent, then

$$F(\rho,\sigma,\alpha,\beta) = \frac{\beta}{\sigma\alpha}F_1(\rho,\sigma,\alpha,\beta)F_2(\rho,\sigma,\alpha,\beta)$$

is a quadruple Laplace-Sumudu-Aboodh-Elzaki transform of the function

$$f(\rho,\sigma,\alpha,\beta) = \int_0^z\int_0^y\int_0^t\int_0^x f_1(x-x_1,t-t_1,y-y_1,z-z_1)$$

$$g_2(x_1, t_1, y_1, z_1)dx_1dt_1dy_1dz_1$$

and the integral

$$F(\rho, \sigma, \alpha, \beta) = \frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\rho x - \frac{t}{\sigma} - \alpha y - \frac{z}{\beta}} f(x, t, y, z)dxdtdydz$$

which converge at the point $(\rho, \sigma, \alpha, \beta)$.

Proof:

$$F(\rho, \sigma, \alpha, \beta) = \frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\rho x - \frac{t}{\sigma} - \alpha y - \frac{z}{\beta}} f(x, t, y, z)dxdtdydz$$

$$= \frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\rho x - \frac{t}{\sigma} - \alpha y - \frac{z}{\beta}} \left[\int_0^z\int_0^y\int_0^t\int_0^x f_1(x - x_1, t - t_1, y - y_1, z - z_1)\right.$$

$$\left. f_2(x_1, t_1, y_1, z_1)dx_1dt_1dy_1dz_1\right] dxdtdydz$$

Using Heaviside unit set function, we have

$$F(\rho, \sigma, \alpha, \beta) = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty f_2(x_1, t_1, y_1, z_1)dx_1dt_1dy_1dz_1\times$$

$$\left[\frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty f_1(x - x_1, t - t_1, y - y_1, z - z_1)\ \boldsymbol{H}(x - x_1, t - t_1, y - y_1, z - z_1)dxdtdydz\right]$$

Hence, it follows from Theorem (5.1) that

$$F(\rho, \sigma, \alpha, \beta) = \frac{\beta}{\sigma\alpha}F_1(\rho, \sigma, \alpha, \beta)F_2(\rho, \sigma, \alpha, \beta)$$

5.3. Change of Scale

Theorem 5.4. If $f(x, t, y, z)$ be a function such that

$$L_xS_tA_yE_z[h(x, t, y, z)] = H(\rho, \sigma, \alpha, \beta)$$

then for some positive constants a, b, c and d; we have

$$L_xS_tA_yE_z[h(ax, bt, cy, dz)] = \frac{1}{abcd}H(\frac{\rho}{a}, b\sigma, \frac{\alpha}{c}, d\beta)$$

Proof: We can deduce using quadruple Laplace-Sumudu-Aboodh-Elzaki transform that

$$L_xS_tA_yE_z[h(ax, bt, cy, dz)] = \frac{\beta}{\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\rho x + \frac{t}{\sigma} + \alpha y + \frac{z}{\beta}\right)}h(ax, bt, cy, dz)dxdtdydz$$

If $\mu = ax, \xi = bt, \phi = cy$ and $\tau = dz$, then we have

$$L_xS_tA_yE_z[h(\mu, \xi, \phi, \tau)] = \frac{\beta}{abcd\sigma\alpha}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\rho}{a}\mu + \frac{\xi}{b\sigma} + \alpha y + \frac{z}{\beta}\right)}$$

$$h(\mu, \xi, \phi, \tau)d\mu d\xi d\phi d\tau = \frac{1}{abcd}H(\frac{\rho}{a}, b\sigma, \frac{\alpha}{c}, d\beta)$$

5.4. Derivatives Properties

The derivatives properties of quadruple Laplace-Sumudu-Aboodh-Elzaki transform are given below without proof. If $L_xS_tA_yE_z[h(x, t, y, z)] = H(\rho, \sigma, \alpha, \beta)$ then,

1.

$$L_xS_tA_yE_z\left[\frac{\partial h(x, t, y, z)}{\partial x}\right] = \rho H(\rho, \sigma, \alpha, \beta) - S_tA_yE_z[h(0, t, y, z)]$$

2.

$$L_xS_tA_yE_z\left[\frac{\partial h(x, t, y, z)}{\partial t}\right] = \frac{1}{\sigma}H(\rho, \sigma, \alpha, \beta) - \frac{1}{\sigma}L_xA_yE_z[h(x, 0, y, z)]$$

3.
$$L_xS_tA_yE_z\left[\frac{\partial h(x,t,y,z)}{\partial y}\right]=\alpha H(\rho,\sigma,\alpha,\beta)-L_xS_tE_z[h(x,t,0,z)]$$

4.
$$L_xS_tA_yE_z\left[\frac{\partial h(x,t,y,z)}{\partial z}\right]=\frac{1}{\beta}H(\rho,\sigma,\alpha,\beta)-\beta L_xS_tA_y[h(x,t,y,0)]$$

5.
$$L_xS_tA_yE_z\left[\frac{\partial^2 h(x,t,y,z)}{\partial x^2}\right]=\rho^2H(\rho,\sigma,\alpha,\beta)-\rho S_tA_yE_z[h(0,t,y,z)]-S_tA_yE_z[h_x(0,t,y,z)]$$

6.
$$L_xS_tA_yE_z\left[\frac{\partial^2 h(x,t,y,z)}{\partial t^2}\right]=\frac{1}{\sigma^2}H(\rho,\sigma,\alpha,\beta)-\frac{1}{\sigma^2}L_xA_yE_z[h(x,0,y,z)]-\frac{1}{\sigma}L_xA_yE_z[h_t(x,0,y,z)]$$

7.
$$L_xS_tA_yE_z\left[\frac{\partial^2 h(x,t,y,z)}{\partial y^2}\right]=\alpha^2H(\rho,\sigma,\alpha,\beta)-L_xS_tE_z[h(x,t,0,z)]-\frac{1}{\alpha}L_xS_tE_z[h_y(x,t,0,z)]$$

8.
$$L_xS_tA_yE_z\left[\frac{\partial^2 h(x,t,y,z)}{\partial z^2}\right]=\frac{1}{\beta^2}H(\rho,\sigma,\alpha,\beta)-L_xS_tA_y[h(x,t,y,0)]-\beta L_xS_tA_y[h_z(x,t,y,0)]$$

9.
$$L_xS_tA_yE_z\left[\frac{\partial^2 h(x,t,y,z)}{\partial y\partial z}\right]=\frac{\alpha}{\beta}H(\rho,\sigma,\alpha,\beta)-\alpha\beta L_xS_tA_y[h(x,t,y,0)]-\frac{1}{\alpha}L_xS_tE_z[h_z(x,t,0,z)]$$

10.
$$L_xS_tA_yE_z\left[\frac{\partial^2 h(x,t,y,z)}{\partial x\partial t}\right]=\frac{\rho}{\sigma}H(\rho,\sigma,\alpha,\beta)-\frac{\rho}{\sigma}L_xA_yE_z[h(x,0,y,z)]-S_tA_yE_z[h_t(0,t,y,z)]$$

11.
$$L_xS_tA_yE_z\left[\frac{\partial^2 h(x,t,y,z)}{\partial x\partial y}\right]=\alpha\rho H(\rho,\sigma,\alpha,\beta)-\frac{\rho}{\alpha}L_xS_tE_z[h(x,t,0,z)]-L_xS_tE_z[h_y(0,t,y,z)]$$

6. Numerical Applications

In this section, we will illustrate the effectiveness of quadruple Laplace-Sumudu-Aboodh-Elzaki transform by applying it solve different examples.

6.1. Example 1

Consider the following integral equation:

$$a^2=\int_0^x\int_0^t\int_0^y\int_0^z h(x-\mu,t-\xi,y-\phi,z-\tau)h(\mu,\xi,\phi,\tau)d\mu d\xi d\phi d\tau$$

where a represents constant.

Solution: Applying quadruple Laplace-Sumudu-Aboodh-Elzaki transform and the convolution theorem on this integral equation, we have:

$$\frac{\beta^2a^2}{\rho\alpha^2}=\frac{\alpha\sigma}{\beta}[L_xS_tA_yE_z(h(x,t,y,z))]^2$$

So that

$$L_xS_tA_yE_z(h(x,t,y,z))=\sqrt{\frac{\beta^3a^2}{\rho\sigma\alpha^3}}$$

Making use of Laplace-Sumudu-Aboodh-Elzaki transform inverse, we get:

$$h(x,t,y,z)=L_x^{-1}S_t^{-1}A_y^{-1}E_z^{-1}\left[\sqrt{\frac{\beta^3a^2}{\rho\sigma\alpha^3}}\right]=\frac{a\pi^2}{\sqrt{xtyz}}$$

6.2. Example 2

Consider the following integral equation:

$$xtyz = \int_0^x \int_0^t \int_0^y \int_0^z e^{-((x-\mu)+(t-\xi)+(y-\phi)+(z-\tau))}$$
$$h(\mu,\xi,\phi,\tau) d\mu d\xi d\phi d\tau$$

Solution: Applying quadruple Laplace-Sumudu-Aboodh-Elzaki transform, it follows that

$$L_x S_t A_y E_z(h(x,t,y,z)) = \left(\frac{1}{\rho} + \frac{1}{\rho^2}\right)(1+\sigma)\left(\frac{1}{\alpha^2} + \frac{1}{\alpha^3}\right)(\beta^2 + \beta^3)$$

Taking the inverse, we have

$$h(x,t,y,z) = (1+x)(1+t)(1+y)(1+z)$$

6.3. Example 3

Consider the following homogeneous wave equation:

$$U_{tt}(x,t,y,z) = U_{xx}(x,t,y,z) \tag{1}$$

Subject to the conditions:

$$U(x,0,y,z) = \sin(x+y+z), U(0,t,y,z) = 2tyz$$
$$U_t(x,0,y,z) = 2, U_x(0,t,y,z) = \cos(t+y+z)$$

Solution: Applying the quadruple Laplace-Sumudu-Aboodh-Elzaki transform to (4), we have

$$L_x S_t A_y E_z[U_{tt}(x,t,y,z)] = L_x S_t A_y E_z[U_{xx}(x,t,y,z)]$$

Making use of the derivative properties, we get

$$\frac{1}{\sigma^2}H(\rho,\sigma,\alpha,\beta) - \frac{1}{\sigma^2}L_x A_y E_z[h(x,0,y,z)] - \frac{1}{\sigma}L_x A_y E_z[h_t(x,0,y,z)]$$
$$= \rho^2 H(\rho,\sigma,\alpha,\beta) - \rho S_t A_y E_z[h(0,t,y,z)] - S_t A_y E_z[h_x(0,t,y,z)]$$

Then

$$H(\rho,\sigma,\alpha,\beta) = \left(\frac{\sigma^2}{1-\rho^2}\right)\left(\frac{1}{\sigma^2}L_x A_y E_z[h(x,0,y,z)]\right.$$
$$\left.+\frac{1}{\sigma}L_x A_y E_z[h_t(x,0,y,z)] - \rho S_t A_y E_z[h(0,t,y,z)] - S_t A_y E_z[h_x(0,t,y,z)]\right)$$

Making use of the conditions, we have

$$H(\rho,\sigma,\alpha,\beta) = \left(\frac{\sigma^2}{1-\rho^2}\right)\left(\frac{1}{\sigma^2}L_x A_y E_z[\sin(x+y+z)]\right.$$
$$\left.+\frac{1}{\sigma}L_x A_y E_z[2] - \rho S_t A_y E_z[2tyz] - S_t A_y E_z[\cos(t+y+z)]\right)$$

Evaluating and simplifying, we get

$$H(\rho,\sigma,\alpha,\beta) = \left(\frac{\sigma^2}{1-\rho^2}\right)\left(\frac{\beta^2(\alpha\beta\rho + \alpha - \beta + \rho)}{\alpha\sigma^2(\alpha^2+1)(\beta^2+1)(\rho^2+1)}\right.$$
$$\left.+\frac{2\beta^2}{\alpha^2\rho\sigma} - \frac{2\beta^3\rho\sigma}{\alpha^3} + \frac{\beta^2(\alpha(\beta\sigma-1)+\beta+\sigma)}{\alpha(\alpha^2+1)(\beta^2+1)(\rho^2+1)}\right)$$

Applying the inverse, we obtain

$$h(x,t,y,z) = 2tyz\cosh x - \sinh x\left(\sin(x+y+z) + \frac{t^2}{2}\cos(t+y+z) - 2t\right)$$

6.4. Example 4

Consider the following non homogeneous wave equation:

$$U_y(x,t,y,z) = U_{xx}(x,t,y,z) + \quad U_{tt}(x,t,y,z) + 2\cos(x+t) \tag{2}$$

Subject to the conditions:

$$\begin{aligned}U(0,t,y,z) &= e^{-2y}\sin t + \cos t,\\ U(x,0,y,z) &= e^{-2y}\sin x + \cos x,\\ U_x(0,t,y,z) &= e^{-2y}\cos t - \sin t\\ U(x,t,0,z) &= \sin(x+t) + \cos(x+t),\\ U_t(x,0,y,z) &= e^{-2y}\cos x - \sin x\end{aligned}$$

Solution: Applying the quadruple Laplace-Sumudu-Aboodh-Elzaki transform to (5), we have

$$L_xS_tA_yE_z[U_y(x,t,y,z)] = L_xS_tA_yE_z\left[U_{xx}(x,t,y,z) + U_{tt}(x,t,y,z) + 2\cos(x+t)\right]$$

Applying the derivative properties, we have

$$\alpha H(\rho,\sigma,\alpha,\beta) - L_xS_tE_z[h(x,t,o,z)] = \rho^2 H(\rho,\sigma,\alpha,\beta)$$

$$-\rho S_tA_yE_z[h(o,t,y,z)] - S_tA_yE_z[h_x(o,t,y,z)] + \frac{1}{\sigma^2}H(\rho,\sigma,\alpha,\beta) - \frac{1}{\sigma^2}L_xA_yE_z[h(x,o,y,z)]$$

$$-\frac{1}{\sigma}L_xA_yE_z[h_t(x,o,y,z)] + \frac{2\beta^2(\rho-\sigma)}{\alpha^2(\rho^2+1)(\sigma^2+1)} \tag{3}$$

Applying the conditions on equation (3) and simplifying, we have

$$H(\rho,\sigma,\alpha,\beta)\left(\alpha - \rho^2 - \frac{1}{\sigma^2}\right) = L_xS_tA_yE_z[\sin(x+t) + \cos(x+t)] - \rho S_tA_yE_z[e^{-2y}\sin t + \cos t]$$

$$-S_tA_yE_z[e^{-2y}\cos t - \sin t] - \frac{1}{\sigma^2}L_xA_yE_z[e^{-2y}\sin x + \cos x] - \frac{1}{\sigma}L_xA_yE_z[e^{-2y}\cos x + \sin x] + \frac{2\beta^2(\rho-\sigma)}{\alpha^2(\rho^2+1)(\sigma^2+1)}$$

Evaluating, then followed by some algebraic operation, we obtain

$$H(\rho,\sigma,\alpha,\beta) = \left(\frac{\sigma^2}{\alpha\sigma^2 - \rho^2\sigma^2 - 1}\right)$$

$$\left[\frac{(1-\beta\sigma+\beta\rho+\rho\sigma)(\alpha\sigma^2-\rho^2\sigma^2-1)}{\alpha^2\sigma^2(\beta^2+1)(\sigma^2+1)(\rho^2+1)} - \frac{\beta^2(\beta\sigma\rho+\beta+\sigma-\rho)(\alpha\sigma^2-\rho^2\sigma^2-1)}{\alpha\sigma^2(\alpha+2)(\beta^2+1)(\sigma^2+1)(\rho^2+1)}\right]$$

Simplifying and taking Quadruple L-S-A-E inverse on both sides, we get

$$h(x,t,y,z) = \sin(x+t+z) - e^{-2y}\cos(x+t+z)$$

7. Conclusion

In this paper, we combined the Laplace transform, Sumudu transform, Aboodh transform and Elzaki transform to produce a new transform called Quadruple Laplace-Sumudu-Aboodh-Elzaki (L-S-A-E) transform. We then applied this transform to some functions and presented some fundamental properties and theorems about the transform. Moreover, the applicability and effectiveness of the transform were investigated by using the transform to solve some integral and partial differential equations. It is worth mentioning that the L-S-A-E transform can be used as an alternative to other similar techniques in the solution of integral and partial differential equations.

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Conflicts of Interest

The authors declare no conflicts of interest.

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