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Research Article

An Efficient A-Stable Linear Multistep Hybrid Block Method for Solving Fifth-Order Initial Value Problems in Ordinary Differential Equations

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Abstract

Furthering research on linear multistep hybrid block methods is essential to enhance accuracy, stability, and efficiency in solving ordinary differential equations, enabling advanced modelling of complex systems across science and engineering for better predictive analysis and real-world applications. In this study, an efficient linear multistep hybrid block method with a single-step and seven off-step points for the direct numerical integration of fifth-order initial value problems (IVPs) in ordinary differential equations (ODEs), eliminating the need for reduction to a system of first-order ODEs is proposed. The method is constructed using a collocation approach at both grid and off-grid points, alongside interpolation at five off-grid points, to approximate the solution via a power series polynomial. The resulting system of equations is solved to obtain the necessary discrete and additional formulae that constitute the block approach. A comprehensive theoretical analysis confirms that the method possesses desirable numerical properties, including a well-defined order, zero stability, consistency, convergence, and absolute stability. Comparative numerical experiments against existing methods demonstrate that the proposed approach achieves superior accuracy and efficiency, making it a promising tool for solving both linear and nonlinear fifth-order ODEs.

Keywords

Power Series Polynomials, Grid Points, Off-grid Points, Convergence, Interpolation and Collocation

1. Introduction

The domains of management, engineering, science, technology, and the social sciences all make extensive use of ordinary differential equations (ODEs). It is easier to fully understand these physical phenomena when they are expressed as mathematical equations. Numerous ODEs of different degrees and orders are produced using these mathematical formulae. In this paper, we consider higher-order ODEs of the form:

$$y^{(5)} = f\left(x, y, y', y'', y''', y^{(iv)}\right), \quad y(x_0) = \alpha_0, \quad y'\left(x_0\right) = \alpha_1, \quad y''(x_0) = \alpha_2,\ y'''(x_0) = \alpha_3,\ y^{iv}(x_0) = \alpha_4 \tag{1}$$

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where f is a given real value function that is continuous within the interval of integration. Numerical solutions of higher-order ordinary differential equations (ODEs) are often obtained by reducing them to equivalent systems of first-order ODEs, which can then be solved using well-established numerical methods. This reduction approach has been widely adopted by several researchers Refs. [1-3] to determine general solutions of higher-order ODEs. However, as Awoyemi [4] highlighted, this transformation is not always computationally efficient due to increased computational effort and extended runtime.

To address these challenges, a growing body of research [5-13] has focused on developing direct numerical methods that solve higher-order ODEs without reduction. These methods leverage specialized techniques to maintain computational efficiency while preserving the inherent structure of the equations. Recently, Ref. [14] introduced a Chebyshev-generated block method for directly solving nonlinear and ill-posed fourth-order ODEs, achieving an order of accuracy of seven.

Building on this progress, the present study proposes a novel order-nine linear multistep hybrid block method with one-step formulation and seven off-step points for the direct numerical integration of general fifth-order initial value problems (IVPs). This method aims to enhance accuracy, stability, and computational efficiency, offering a more effective alternative to reduction-based approaches.

2. Derivation of the Method

The derivation presented in this section is significant as it employs the power series as a basis function, ensuring accuracy, flexibility, and analytical strength. This foundation enhances the method's convergence, stability, and applicability in solving complex differential equations across mathematical, physical, and engineering problems. Consider the power series.

$$y(x) = \sum_{j=0}^{(r+s)-1} a_j x^j \tag{2}$$

as basis function. The fifth derivative of Eq. (2) gives

$$y^{(5)}(x) = \sum_{j=0}^{(r+s)-1} j(j-1)(j-2)(j-3)(j-4) a_j x^{j-5} \tag{3}$$

Equating (3) and Eq. (1) yields the differential system.

$$\sum_{j=0}^{(r+s)-1} j(j-1)(j-2)(j-3)(j-4) a_j x^{j-5} = f(x, y(x), y'(x), y''(x), y'''(x), y^{iv}(x)) \tag{4}$$

where $a_j's$ are the parameters to be determined, r and s denotes the number of collocation and interpolation points respectively. By collocating Eq. (4) at the mesh points $x = x_{n+\frac{j}{8}}, j = 0(1)8$ and interpolating Eq. (2) at $x = x_{n+\frac{j}{8}}, j = 3(1)7$ yields a system of equations for interpolation equation,

$$\sum_{j=0}^{(r+s)-1} a_j x^j = y_{n+\frac{j}{8}}, \quad j = 3(1)7 \tag{5}$$

and for collocation,

$$\sum_{j=0}^{(r+s)-1} j(j-1)(j-2)(j-3)(j-4) a_j x^{j-5} = f_{n+\frac{j}{8}}, \quad j = 0(1)8 \tag{6}$$

By putting these equation systems into matrix form and solving them to determine the parameter values $a_j's$, $j = 0,1,\ldots,13$, which when substituted in Eq. (3) yields, after a some simplification, thus provides a continuous hybrid linear multistep formula of the form:

$$y(t) = \sum_{j=3}^{7} \alpha_{\frac{j}{8}} y_{n+\frac{j}{8}}(t) + h^5 \sum_{j=0}^{8} \beta_{\frac{j}{8}} f_{n+\frac{j}{8}}(t) \tag{7}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are the coefficients that defined the method and are obtained as:

$$\alpha_{\frac{3}{8}}(t)=\frac{1}{3}\left(512t^4-1408t^3+1432t^2-638t+105\right)\quad \alpha_{\frac{1}{2}}(t)=-\frac{1}{3}\left(2048t^4-5376t^3+5152t^2-2124t+315\right)$$

$$\alpha_{\frac{5}{8}}(t)=1024t^4-2560t^3+2320t^2-900t+126\quad \alpha_{\frac{3}{4}}(t)=-\frac{2}{3}\left(1024t^4-2432t^3+20960t^2-778t+105\right)$$

$$\alpha_{\frac{7}{8}}(t)=\frac{1}{3}\left(512t^4-1152t^3+592t^2-342t+45\right)$$

$$\beta_0(t)=\frac{16384}{6081075}t^{13}-\frac{1024}{51975}t^{12}+\frac{3328}{51975}t^{11}-\frac{64}{525}t^{10}+\frac{2138}{14175}t^{9}-\frac{89}{700}t^{8}+\frac{29531}{396900}t^{7}-\frac{761}{25200}t^{6}$$
$$+\frac{1}{120}t^{5}-\frac{1041479}{696729600}t^{4}+\frac{9774049}{61312204800}t^{3}-\frac{4116389}{490497638400}t^{2}+\frac{3548693}{19837904486400}t+\frac{107}{7927234560}$$

$$\beta_{\frac{1}{8}}(t)=-\frac{131072}{6081075}t^{13}+\frac{2048}{13365}t^{12}-\frac{74752}{155925}t^{11}+\frac{1472}{1701}t^{10}-\frac{6016}{6075}t^{9}+\frac{698}{945}t^{8}-\frac{3848}{11025}t^{7}+\frac{4}{45}t^{6}$$
$$-\frac{576521}{69672960}t^{4}+\frac{10967303}{3832012800}t^{3}-\frac{11514809}{24524881920}t^{2}+\frac{187049749}{4959476121600}t-\frac{2131}{1981808640}$$

$$\beta_{\frac{1}{4}}(t)=\frac{65536}{868725}t^{13}-\frac{34816}{66825}t^{12}+\frac{244736}{155925}t^{11}-\frac{22912}{8505}t^{10}+\frac{122312}{66825}t^{9}-\frac{18353}{9450}t^{8}+\frac{138}{175}t^{7}-\frac{7}{45}t^{6}$$
$$-\frac{33797}{174182400}t^{4}+\frac{71385989}{15328051200}t^{3}-\frac{179923}{90832896}t^{2}+\frac{29649931}{88562073600}t-\frac{167}{7864320}$$

$$\beta_{\frac{3}{8}}(t)=-\frac{131072}{868725}t^{13}+\frac{2048}{2025}t^{12}-\frac{15256}{51975}t^{11}+\frac{4544}{945}t^{10}-\frac{68608}{14175}t^{9}+\frac{1594}{525}t^{8}-\frac{16024}{14175}t^{7}+\frac{28}{135}t^{6}$$
$$-\frac{4060249}{348364800}t^{4}+\frac{8201143}{958003200}t^{3}-\frac{1940227}{445906944}t^{2}+\frac{765568079}{708496588800}t-\frac{28601}{283115520}$$

$$\beta_{\frac{1}{2}}(t)=\frac{32768}{173745}t^{13}-\frac{16384}{13365}t^{12}+\frac{9728}{2835}t^{11}-\frac{45824}{8505}t^{10}+\frac{43972}{8505}t^{9}-\frac{2914}{4725}t^{8}+\frac{691}{630}t^{7}-\frac{7}{36}t^{6}$$
$$+\frac{64543}{69672960}t^{4}+\frac{11191711}{1226244096}t^{3}-\frac{355902073}{49049763840}t^{2}+\frac{60684329}{25763512320}t-\frac{32101}{113246208}$$

$$\beta_{\frac{5}{8}}(t)=-\frac{131072}{868725}t^{13}+\frac{63488}{66825}t^{12}-\frac{400384}{155925}t^{11}+\frac{165184}{42525}t^{10}-\frac{152704}{42525}t^{9}+\frac{9782}{4725}t^{8}-\frac{376}{525}t^{7}+\frac{28}{225}t^{6}$$
$$-\frac{2208823}{348364800}t^{4}+\frac{23714711}{3832012800}t^{3}-\frac{189784853}{40874803200}t^{2}+\frac{166432727}{101213798400}t-\frac{20237}{94371840}$$

$$\beta_{\frac{3}{4}}(t)=\frac{65536}{868725}t^{13}-\frac{2048}{4455}t^{12}+\frac{62464}{51975}t^{11}-\frac{1664}{945}t^{10}+\frac{22424}{14175}t^{9}-\frac{187}{210}t^{8}+\frac{4286}{14175}t^{7}-\frac{7}{135}t^{6}$$
$$+\frac{46199}{34836480}t^{4}+\frac{4685113}{15328051200}t^{3}-\frac{640177}{1532805120}t^{2}+\frac{53480839}{354248294400}t-\frac{2791}{141557760}$$

$$\beta_{\frac{7}{8}}(t)=-\frac{131072}{6081075}t^{13}+\frac{59392}{467775}t^{12}-\frac{7168}{22275}t^{11}+\frac{3904}{8505}t^{10}-\frac{17152}{42525}t^{9}+\frac{1054}{4725}t^{8}-\frac{824}{11025}t^{7}+\frac{4}{315}t^{6}$$
$$-\frac{131011}{348364800}t^{4}+\frac{88267}{1916006400}t^{3}+\frac{7591}{3503554560}t^{2}-\frac{4904741}{4959476121600}t+\frac{179}{1981808640}$$

$$\beta_1(t) = \frac{16384}{6081075}t^{13} - \frac{1024}{66825}t^{12} + \frac{5888}{155925}t^{11} - \frac{64}{1215}t^{10} + \frac{1934}{42525}t^{9} - \frac{67}{2700}t^{8} + \frac{121}{14700}t^{7} - \frac{1}{720}t^{6} + \frac{28633}{696729600}t^{4} - \frac{330727}{61312204800}t^{3} - \frac{2089}{32699842560}t^{2} + \frac{26993}{19837904486400}t + \frac{11}{2642411520}$$

where $t = \frac{x - x_n}{h}$. Evaluate Eq. (8) at $t = 0, \frac{1}{8}, \frac{1}{4}, 1$ to obtain the discrete one-step formulas.

$$y_n - 35y_{n+\frac{3}{8}} + 105y_{n+\frac{1}{2}} - 126y_{n+\frac{5}{8}} + 70y_{n+\frac{3}{4}} - 15y_{n+\frac{7}{8}} = \frac{-h^5}{7927234560}\begin{pmatrix} 107f_n + 8524f_{n+\frac{1}{8}} + 168336f_{n+\frac{1}{4}} \\ +800828f_{n+\frac{3}{8}} + 224707f_{n+\frac{1}{2}} + \\ 1699908f_{n+\frac{5}{8}} + 156296f_{n+\frac{3}{4}} \\ -716f_{n+\frac{7}{8}} - 33f_{n+1} \end{pmatrix} \quad (8)$$

$$y_{n+\frac{1}{8}} - 15y_{n+\frac{3}{8}} + 40y_{n+\frac{1}{2}} - 45y_{n+\frac{5}{8}} + 24y_{n+\frac{3}{4}} - 5y_{n+\frac{7}{8}} = \frac{-h^5}{3963617280}\begin{pmatrix} 31f_n - 328f_{n+\frac{1}{8}} + 6348f_{n+\frac{1}{4}} + \\ 77944f_{n+\frac{3}{8}} + 335810f_{n+\frac{1}{2}} + \\ 280104f_{n+\frac{5}{8}} + 2594f_{n+\frac{3}{4}} - \\ 88f_{n+\frac{7}{8}} - 9f_{n+1} \end{pmatrix} \quad (9)$$

$$y_{n+\frac{1}{4}} - 5y_{n+\frac{3}{8}} + 10y_{n+\frac{1}{2}} - 10y_{n+\frac{5}{8}} + 5y_{n+\frac{3}{4}} - y_{n+\frac{7}{8}} = \frac{-h^5}{23781703680}\begin{pmatrix} 41f_n - 388f_{n+\frac{1}{8}} + 1448f_{n+\frac{1}{4}} + \\ 27404f_{n+\frac{3}{8}} + 335810f_{n+\frac{1}{2}} + \\ 330664f_{n+\frac{5}{8}} + 30848f_{n+\frac{3}{4}} - \\ 28f_{n+\frac{7}{8}} - 19f_{n+1} \end{pmatrix} \quad (10)$$

$$y_{n+1} - 5y_{n+\frac{7}{8}} + 10y_{n+\frac{3}{4}} - 10y_{n+\frac{5}{8}} + 5y_{n+\frac{1}{2}} - y_{n+\frac{3}{8}} = \frac{-h^5}{23781703680}\begin{pmatrix} 19f_n - 212f_{n+\frac{1}{8}} + 1072f_{n+\frac{1}{4}} - \\ 3044f_{n+\frac{3}{8}} - 25010f_{n+\frac{1}{2}} - 338204f_{n+\frac{5}{8}} \\ -329048f_{n+\frac{3}{4}} - 31532f_{n+\frac{7}{8}} + 199f_{n+1} \end{pmatrix} \quad (11)$$

In order to implement the derived method in block mode, we adopted the block formula proposed in Awoyemi *et al*. [15]. The formula in its normalized form is given as

$$A^{(0)}Y_m = ey_n + h^{\mu-\lambda}df(y_m) + h^{\mu-\lambda}bF(y_m) \quad (12)$$

By evaluating Eq. (7) at $t = 1$; the first, second, third and fourth derivatives at $x = x_{n+i}$, $i = 0\left(\frac{1}{8}\right)1$ and substituting into Eq. (12) gives the coefficients matrices as:

$$y_{n+\frac{1}{8}} = y_n + \frac{1}{8}hy'_n + \frac{1}{128}h^2y''_n + \frac{1}{3072}h^3y'''_n + \frac{1}{98304}h^4y^{iv} + \frac{h^5}{2856658\,246041600}\begin{bmatrix} 480013145f_n + 600057294f_{n+\frac{1}{8}} - 897106000f_{n+\frac{1}{4}} \\ +1146150950f_{n+\frac{3}{8}} - 1053111990f_{n+\frac{1}{2}} + 666135050f_{n+\frac{5}{8}} \\ -275432584f_{n+\frac{3}{4}} + 67088130f_{n+\frac{7}{8}} - 7308235f_{n+1} \end{bmatrix} \quad (13)$$

$$y_{n+\frac{1}{4}} = y_n + \frac{1}{4}hy'_n + \frac{1}{32}h^2y''_n + \frac{1}{384}h^3y'''_n + \frac{1}{6144}h^4y^{iv} + \frac{h^5}{11158\,821273600}\begin{bmatrix} 42847915f_n + 96091480f_{n+\frac{1}{8}} - 119779674f_{n+\frac{1}{4}} \\ +150074680f_{n+\frac{3}{8}} - 136659740f_{n+\frac{1}{2}} + 86003400f_{n+\frac{5}{8}} \\ -35444150f_{n+\frac{3}{4}} + 8613544f_{n+\frac{7}{8}} - 936735f_{n+1} \end{bmatrix} \quad (14)$$

$$y_{n+\frac{3}{8}} = y_n + \frac{3}{8}hy'_n + \frac{9}{128}h^2y''_n + \frac{9}{1024}h^3y'''_n + \frac{27}{32768}h^4y^{iv} + \frac{27h^5}{11755\,795251200}\begin{bmatrix} 9700895f_n + 28590750f_{n+\frac{1}{8}} - 29397480f_{n+\frac{1}{4}} \\ +37769046f_{n+\frac{3}{8}} - 34430130f_{n+\frac{1}{2}} + 21669690f_{n+\frac{5}{8}} \\ -8929760f_{n+\frac{3}{4}} + 2169810f_{n+\frac{7}{8}} - 235941f_{n+1} \end{bmatrix} \quad (15)$$

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}hy'_n + \frac{1}{8}h^2y''_n + \frac{1}{48}h^3y'''_n + \frac{1}{384}h^4y^{iv} + \frac{27h^5}{21\,794572800}\begin{bmatrix} 1645948f_n + 5645028f_{n+\frac{1}{8}} - 4816217f_{n+\frac{1}{4}} \\ +6688948f_{n+\frac{3}{8}} - 6081180f_{n+\frac{1}{2}} + 3830092f_{n+\frac{5}{8}} \\ -1578983f_{n+\frac{3}{4}} + 383772f_{n+\frac{7}{8}} - 41738f_{n+1} \end{bmatrix} \quad (16)$$

$$y_{n+\frac{5}{8}} = y_n + \frac{5}{8}hy'_n + \frac{25}{128}h^2y''_n + \frac{125}{3072}h^3y'''_n + \frac{625}{98304}h^4y^{iv} + \frac{625h^5}{114266\,329841664}\begin{bmatrix} 35166685f_n + 132509950f_{n+\frac{1}{8}} \\ -95373600f_{n+\frac{1}{4}} + 148782550f_{n+\frac{3}{8}} \\ -132350750f_{n+\frac{1}{2}} + 83554842f_{n+\frac{5}{8}} \\ -34458200f_{n+\frac{3}{4}} + 8376850f_{n+\frac{7}{8}} \\ -911175f_{n+1} \end{bmatrix} \quad (17)$$

$$y_{n+\frac{3}{4}} = y_n + \frac{3}{4}hy'_n + \frac{9}{32}h^2y''_n + \frac{9}{128}h^3y'''_n + \frac{27}{2048}h^4y^{iv} + \frac{27h^5}{45\,921075200}\begin{bmatrix} 697855f_n + 2802264f_{n+\frac{1}{8}} - 1733130f_{n+\frac{1}{4}} \\ +3085880f_{n+\frac{3}{8}} - 12627940f_{n+\frac{1}{2}} + 1680840f_{n+\frac{5}{8}} \\ -692454f_{n+\frac{3}{4}} + 168360f_{n+\frac{7}{8}} - 18315f_{n+1} \end{bmatrix} \quad (18)$$

$$y_{n+\frac{7}{8}} = y_n + \frac{7}{8}hy'_n + \frac{49}{128}h^2y''_n + \frac{343}{3072}h^3y'''_n + \frac{2401}{98304}h^4y^{iv} + \frac{16807h^5}{58299\,147878400}\begin{bmatrix} 2690765f_n + 11310450f_{n+\frac{1}{8}} - 6109768f_{n+\frac{1}{4}} \\ +12431930f_{n+\frac{3}{8}} - 9988230f_{n+\frac{1}{2}} + 6595190f_{n+\frac{5}{8}} \\ -2687440f_{n+\frac{3}{4}} + 654558f_{n+\frac{7}{8}} - 71215f_{n+1} \end{bmatrix} \quad (19)$$

$$y_{n+1}=y_n+hy'_n+\frac{1}{2}h^2y''_n+\frac{1}{6}h^3y'''_n+\frac{1}{24}h^4y^{iv}+\frac{h^5}{681080400}\begin{bmatrix}915115f_n+3981280f_{n+\frac{1}{8}}-1904280f_{n+\frac{1}{4}}\\+4404064f_{n+\frac{3}{8}}-3315620f_{n+\frac{1}{2}}+2301600f_{n+\frac{5}{8}}\\-906920f_{n+\frac{3}{4}}+224800f_{n+\frac{7}{8}}-24369f_{n+1}\end{bmatrix} \quad (20)$$

$$y'_{n+\frac{1}{8}}=hy'_n+\frac{1}{8}h^2y''_n+\frac{1}{128}h^3y'''_n+\frac{1}{3072}h^4y^{iv}+\frac{h^4}{3923\,981107200}\begin{bmatrix}24396497f_n+36501816f_{n+\frac{1}{8}}\\-52883276f_{n+\frac{1}{4}}+67126376f_{n+\frac{3}{8}}\\-61500210f_{n+\frac{1}{2}}+38838088f_{n+\frac{5}{8}}\\-160419160f_{n+\frac{3}{4}}\\+3904536f_{n+\frac{7}{8}}-425111f_{n+1}\end{bmatrix} \quad (21)$$

$$y'_{n+\frac{1}{4}}=hy'_n+\frac{1}{4}h^2y''_n+\frac{1}{32}h^3y'''_n+\frac{1}{384}h^4y^{iv}+\frac{h^4}{15328051200}\begin{bmatrix}1035731f_n+2719504f_{n+\frac{1}{8}}-3139836f_{n+\frac{1}{4}}\\+3933392f_{n+\frac{3}{8}}-3577790f_{n+\frac{1}{2}}+2254704f_{n+\frac{5}{8}}\\-926764f_{n+\frac{3}{4}}+225136f_{n+\frac{7}{8}}-24477f_{n+1}\end{bmatrix} \quad (22)$$

$$y'_{n+\frac{3}{8}}=hy'_n+\frac{3}{8}h^2y''_n+\frac{9}{128}h^3y'''_n+\frac{9}{1024}h^4y^{iv}+\frac{9h^4}{16148070400}\begin{bmatrix}456059f_n+1529352f_{n+\frac{1}{8}}-1386324f_{n+\frac{1}{4}}\\+1846040f_{n+\frac{3}{8}}-1685430f_{n+\frac{1}{2}}+1061496f_{n+\frac{5}{8}}\\-437572f_{n+\frac{3}{4}}+106344f_{n+\frac{7}{8}}-11565f_{n+1}\end{bmatrix} \quad (23)$$

$$y'_{n+\frac{1}{2}}=hy'_n+\frac{1}{2}h^2y''_n+\frac{1}{8}h^3y'''_n+\frac{1}{48}h^4y^{iv}+\frac{h^4}{59875200}\begin{bmatrix}38084f_n+145056f_{n+\frac{1}{8}}-104780f_{n+\frac{1}{4}}\\+160352f_{n+\frac{3}{8}}-144375f_{n+\frac{1}{2}}+90976f_{n+\frac{5}{8}}\\-37516f_{n+\frac{3}{4}}+9120f_{n+\frac{7}{8}}-992f_{n+1}\end{bmatrix} \quad (24)$$

$$y'_{n+\frac{5}{8}}=hy'_n+\frac{5}{8}h^2y''_n+\frac{25}{128}h^3y'''_n+\frac{125}{3072}h^4y^{iv}+\frac{125h^4}{156959244288}\begin{bmatrix}1611373f_n+6624920f_{n+\frac{1}{8}}-3926460f_{n+\frac{1}{4}}\\+7167880f_{n+\frac{3}{8}}-6133450f_{n+\frac{1}{2}}+3899112f_{n+\frac{5}{8}}\\-1608620f_{n+\frac{3}{4}}+391160f_{n+\frac{7}{8}}-42555f_{n+1}\end{bmatrix} \quad (25)$$

$$y'_{n+\frac{3}{4}} = hy'_n + \frac{3}{4}h^2 y''_n + \frac{9}{32}h^3 y'''_n + \frac{9}{128}h^4 y^{iv} + \frac{9h^4}{63078400}\begin{bmatrix} 15881f_n + 68688f_{n+\frac{1}{8}} - 34308f_{n+\frac{1}{4}} \\ +74768f_{n+\frac{3}{8}} - 59130f_{n+\frac{1}{2}} + 39024f_{n+\frac{5}{8}} \\ -15988f_{n+\frac{3}{4}} + 3888f_{n+\frac{7}{8}} - 423f_{n+1} \end{bmatrix} \quad (26)$$

$$y'_{n+\frac{7}{8}} = hy'_n + \frac{7}{8}h^2 y''_n + \frac{49}{128}h^3 y'''_n + \frac{343}{3072}h^4 y^{iv} + \frac{2401h^4}{560568729600}\begin{bmatrix} 853103f_n + 3825192f_{n+\frac{1}{8}} - 1645028f_{n+\frac{1}{4}} \\ +4226936f_{n+\frac{3}{8}} - 3059070f_{n+\frac{1}{2}} + 2168152f_{n+\frac{5}{8}} \\ -853972f_{n+\frac{3}{4}} + 209928f_{n+\frac{7}{8}} - 22841f_{n+1} \end{bmatrix} \quad (27)$$

$$y'_{n+1} = hy'_n + h^2 y''_n + \frac{1}{2}h^3 y'''_n + \frac{1}{6}h^4 y^{iv} + \frac{h^4}{3742200}\begin{bmatrix} 20648f_n + 95104f_{n+\frac{1}{8}} - 35808f_{n+\frac{1}{4}} \\ +106880df_{n+\frac{3}{8}} - 70760f_{n+\frac{1}{2}} + 54912f_{n+\frac{5}{8}} \\ -19744f_{n+\frac{3}{4}} + 5248f_{n+\frac{7}{8}} - 555f_{n+1} \end{bmatrix} \quad (28)$$

$$y''_{n+\frac{1}{8}} = h^2 y''_n + \frac{1}{8}h^3 y'''_n + \frac{1}{128}h^4 y^{iv} + \frac{h^3}{20437401600}\begin{bmatrix} 3619903f_n + 6779886f_{n+\frac{1}{8}} - 9359135f_{n+\frac{1}{4}} \\ +11774146f_{n+\frac{3}{8}} - 10745445f_{n+\frac{1}{2}} + 6771082f_{n+\frac{5}{8}} \\ -2792861f_{n+\frac{3}{4}} + 679110f_{n+\frac{7}{8}} - 73886f_{n+1} \end{bmatrix} \quad (29)$$

$$y''_{n+\frac{1}{4}} = h^2 y''_n + \frac{1}{4}h^3 y'''_n + \frac{1}{32}h^4 y^{iv} + \frac{h^3}{319334400}\begin{bmatrix} 286967f_n + 911204f_{n+\frac{1}{8}} - 926646f_{n+\frac{1}{4}} \\ +1173140f_{n+\frac{3}{8}} - 1067950f_{n+\frac{1}{2}} + 671628f_{n+\frac{5}{8}} \\ -276634f_{n+\frac{3}{4}} + 67196f_{n+\frac{7}{8}} - 7305f_{n+1} \end{bmatrix} \quad (30)$$

$$y''_{n+\frac{3}{8}} = h^2 y''_n + \frac{3}{8}h^3 y'''_n + \frac{9}{128}h^4 y^{iv} + \frac{9h^3}{252313600}\begin{bmatrix} 61128f_n + 237306f_{n+\frac{1}{8}} - 173739f_{n+\frac{1}{4}} \\ +255430f_{n+\frac{3}{8}} - 233295f_{n+\frac{1}{2}} + 147102f_{n+\frac{5}{8}} \\ -60681f_{n+\frac{3}{4}} + 14754f_{n+\frac{7}{8}} - 1605f_{n+1} \end{bmatrix} \quad (31)$$

$$y''_{n+\frac{1}{2}} = h^2 y''_n + \frac{1}{2}h^3 y'''_n + \frac{1}{8}h^4 y^{iv} + \frac{9h^3}{19958400}\begin{bmatrix} 80293f_n + 342816f_{n+\frac{1}{8}} - 188120f_{n+\frac{1}{4}} \\ +358816f_{n+\frac{3}{8}} - 310800f_{n+\frac{1}{2}} + 196192f_{n+\frac{5}{8}} \\ -80936f_{n+\frac{3}{4}} + 19680f_{n+\frac{7}{8}} - 2141f_{n+1} \end{bmatrix} \quad (32)$$

$$y''_{n+\frac{5}{8}} = h^2 y''_n + \frac{5}{8}h^3 y'''_n + \frac{25}{128}h^4 y^{iv} + \frac{125h^3}{817496064}\begin{bmatrix} 42025f_n + 189622f_{n+\frac{1}{8}} - 82371f_{n+\frac{1}{4}} \\ +20298f_{n+\frac{3}{8}} - 157445f_{n+\frac{1}{2}} + 103362f_{n+\frac{5}{8}} \\ -42617f_{n+\frac{3}{4}} + 10366f_{n+\frac{7}{8}} - 1128f_{n+1} \end{bmatrix} \quad (33)$$

$$y''_{n+\frac{3}{4}} = h^2 y''_n + \frac{3}{4}h^3 y'''_n + \frac{9}{32}h^4 y^{iv} + \frac{9h^3}{3942400}\begin{bmatrix} 4113f_n + 19236f_{n+\frac{1}{8}} - 6870f_{n+\frac{1}{4}} \\ +21396f_{n+\frac{3}{8}} - 14370f_{n+\frac{1}{2}} + 10572f_{n+\frac{5}{8}} \\ -4186f_{n+\frac{3}{4}} + 1020f_{n+\frac{7}{8}} - 111f_{n+1} \end{bmatrix} \quad (34)$$

$$y''_{n+\frac{7}{8}} = h^2 y''_n + \frac{7}{8}h^3 y'''_n + \frac{49}{128}h^4 y^{iv} + \frac{343h^3}{2919628800}\begin{bmatrix} 109918f_n + 527226f_{n+\frac{1}{8}} - 59299f_{n+\frac{1}{4}} \\ +603190f_{n+\frac{3}{8}} - 356475f_{n+\frac{1}{2}} + 305662f_{n+\frac{5}{8}} \\ -104321f_{n+\frac{3}{4}} + 27474f_{n+\frac{7}{8}} - 2975f_{n+1} \end{bmatrix} \quad (35)$$

$$y''_{n+1} = h^2 y''_n + h^3 y'''_n + \frac{1}{2}h^4 y^{iv} + \frac{343h^3}{1247400}\begin{bmatrix} 21203f_n + 103616f_{n+\frac{1}{8}} - 27024f_{n+\frac{1}{4}} \\ +121280f_{n+\frac{3}{8}} - 63820f_{n+\frac{1}{2}} + 63552f_{n+\frac{5}{8}} \\ -16816f_{n+\frac{3}{4}} + 6464f_{n+\frac{7}{8}} - 555f_{n+1} \end{bmatrix} \quad (36)$$

$$y'''_{n+\frac{1}{8}} = h^3 y'''_n + \frac{1}{8}h^4 y^{iv} + \frac{h^2}{464486400}\begin{bmatrix} 1624505f_n + 4124232f_{n+\frac{1}{8}} - 5225624f_{n+\frac{1}{4}} \\ +648832f_{n+\frac{3}{8}} - 5888310f_{n+\frac{1}{2}} + 3698920f_{n+\frac{5}{8}} \\ -1522672f_{n+\frac{3}{4}} + 369744f_{n+\frac{7}{8}} - 40187f_{n+1} \end{bmatrix} \quad (37)$$

$$y'''_{n+\frac{1}{4}} = h^3 y'''_n + \frac{1}{4}h^4 y^{iv} + \frac{h^2}{7257600}\begin{bmatrix} 58193f_n + 235072f_{n+\frac{1}{8}} - 183708f_{n+\frac{1}{4}} \\ +247328f_{n+\frac{3}{8}} - 227030f_{n+\frac{1}{2}} + 143232f_{n+\frac{5}{8}} \\ -59092f_{n+\frac{3}{4}} + 14368f_{n+\frac{7}{8}} \text{-}1563f_{n+1} \end{bmatrix} \tag{38}$$

$$y'''_{n+\frac{3}{8}} = h^3 y'''_n + \frac{3}{8}h^4 y^{iv} + \frac{3h^2}{5734400}\begin{bmatrix} 23887f_n + 109536f_{n+\frac{1}{8}} - 50208f_{n+\frac{1}{4}} + 105000f_{n+\frac{3}{8}} - 93810f_{n+\frac{1}{2}} \\ +59088f_{n+\frac{5}{8}} - 24376f_{n+\frac{3}{4}} + 5928f_{n+\frac{7}{8}} \text{-}645f_{n+1} \end{bmatrix} \tag{39}$$

$$y'''_{n+\frac{1}{2}} = h^3 y'''_n + \frac{1}{2}h^4 y^{iv} + \frac{h^2}{453600}\begin{bmatrix} 7703f_n + 37248f_{n+\frac{1}{8}} - 11600f_{n+\frac{1}{4}} + 40064f_{n+\frac{3}{8}} \\ -29610f_{n+\frac{1}{2}} + 19072f_{n+\frac{5}{8}} - 7888f_{n+\frac{3}{4}} + 1920f_{n+\frac{7}{8}} \text{-}209f_{n+1} \end{bmatrix} \tag{40}$$

$$y'''_{n+\frac{5}{8}} = h^3 y'''_n + \frac{5}{8}h^4 y^{iv} + \frac{25h^2}{18579456}\begin{bmatrix} 15953f_n + 79480f_{n+\frac{1}{8}} - 18600f_{n+\frac{1}{4}} + 91760f_{n+\frac{3}{8}} \\ -51350f_{n+\frac{1}{2}} + 40824f_{n+\frac{5}{8}} - 16480f_{n+\frac{3}{4}} + 4000f_{n+\frac{7}{8}} \text{-}435f_{n+1} \end{bmatrix} \tag{41}$$

$$y'''_{n+\frac{3}{4}} = h^3 y'''_n + \frac{3}{4}h^4 y^{iv} + \frac{3h^2}{89600}\begin{bmatrix} 775f_n + 3936f_{n+\frac{1}{8}} - 732f_{n+\frac{1}{4}} + 4736f_{n+\frac{3}{8}} \\ -2130f_{n+\frac{1}{2}} + 2400f_{n+\frac{5}{8}} - 756f_{n+\frac{3}{4}} + 192f_{n+\frac{7}{8}} \text{-}21f_{n+1} \end{bmatrix} \tag{42}$$

$$y'''_{n+\frac{7}{8}} = h^3 y'''_n + \frac{7}{8}h^4 y^{iv} + \frac{49h^2}{66355200}\begin{bmatrix} 41219f_n + 212016f_{n+\frac{1}{8}} - 32144f_{n+\frac{1}{4}} + 261464f_{n+\frac{3}{8}} \\ -98490f_{n+\frac{1}{2}} + 144256f_{n+\frac{5}{8}} - 20776f_{n+\frac{3}{4}} + 12024f_{n+\frac{7}{8}} \text{-}1169f_{n+1} \end{bmatrix} \tag{43}$$

$$y'''_{n+1} = h^3 y'''_n + h^4 y^{iv} + \frac{h^2}{28350}\begin{bmatrix} 989f_n + 5152f_{n+\frac{1}{8}} - 696f_{n+\frac{1}{4}} + 6560f_{n+\frac{3}{8}} \\ -2270f_{n+\frac{1}{2}} + 3936f_{n+\frac{5}{8}} - 232f_{n+\frac{3}{4}} + 736f_{n+\frac{7}{8}} \end{bmatrix} \tag{44}$$

$$y^{iv}_{n+\frac{1}{8}} = h^4 y^{iv} + \frac{h}{29030400}\begin{bmatrix} 1070017f_n + 4467094f_{n+\frac{1}{8}} \text{-}4604594f_{n+\frac{1}{4}} + 5595358f_{n+\frac{3}{8}} \\ -5033120f_{n+\frac{1}{2}} + 3146338f_{n+\frac{5}{8}} \text{-}1291214f_{n+\frac{3}{4}} + 312874f_{n+\frac{3}{4}} - 33953f_{n+1} \end{bmatrix} \tag{45}$$

$$y^{iv}_{n+\frac{1}{4}} = h^4 y^{iv} + \frac{h}{907200}\begin{bmatrix} 32377f_n + 182584f_{n+\frac{1}{8}} \text{-}42494f_{n+\frac{1}{4}} + 120088f_{n+\frac{3}{8}} \\ -116120f_{n+\frac{1}{2}} + 74728f_{n+\frac{5}{8}} \text{-}31154f_{n+\frac{3}{4}} + 7624f_{n+\frac{7}{8}} - 833f_{n+1} \end{bmatrix} \tag{46}$$

$$y^{iv}_{n+\frac{3}{8}}=h^4y^{iv}+\frac{h}{358400}\begin{bmatrix}12881f_n+70902f_{n+\frac{1}{8}}+3438f_{n+\frac{1}{4}}+79934f_{n+\frac{3}{8}}\\ -56160f_{n+\frac{1}{2}}+34434f_{n+\frac{5}{8}}-14062f_{n+\frac{3}{4}}+3402f_{n+\frac{7}{8}}-369f_{n+1}\end{bmatrix} \quad (47)$$

$$y^{iv}_{n+\frac{1}{2}}=h^4y^{iv}+\frac{h}{113400}\begin{bmatrix}4063f_n+22576f_{n+\frac{1}{8}}+244f_{n+\frac{1}{4}}+32752f_{n+\frac{3}{8}}\\ -9080f_{n+\frac{1}{2}}+9232f_{n+\frac{5}{8}}-3956f_{n+\frac{3}{4}}+976f_{n+\frac{7}{8}}-107f_{n+1}\end{bmatrix} \quad (48)$$

$$y^{iv}_{n+\frac{5}{8}}=h^4y^{iv}+\frac{5h}{1161216}\begin{bmatrix}8341f_n+46030f_{n+\frac{1}{8}}+1510f_{n+\frac{1}{4}}+63670f_{n+\frac{3}{8}}\\ -800f_{n+\frac{1}{2}}+34186f_{n+\frac{5}{8}}-9830f_{n+\frac{3}{4}}+2290f_{n+\frac{7}{8}}-245f_{n+1}\end{bmatrix} \quad (49)$$

$$y^{iv}_{n+\frac{3}{4}}=h^4y^{iv}+\frac{h}{11200}\begin{bmatrix}401f_n+2232f_{n+\frac{1}{8}}+18f_{n+\frac{1}{4}}+3224f_{n+\frac{3}{8}}\\ -360f_{n+\frac{1}{2}}+2664f_{n+\frac{5}{8}}+158f_{n+\frac{3}{4}}+72f_{n+\frac{7}{8}}-9f_{n+1}\end{bmatrix} \quad (50)$$

$$y^{iv}_{n+\frac{7}{8}}=h^4y^{iv}+\frac{7h}{4147200}\begin{bmatrix}21361f_n+116662f_{n+\frac{1}{8}}+6958f_{n+\frac{1}{4}}+155134f_{n+\frac{3}{8}}\\ +7840f_{n+\frac{1}{2}}+105154f_{n+\frac{5}{8}}+74578f_{n+\frac{3}{4}}+31882f_{n+\frac{7}{8}}-1169f_{n+1}\end{bmatrix} \quad (51)$$

$$y^{iv}_{n+1}=h^4y^{iv}+\frac{h}{28350}\begin{bmatrix}989f_n+5888f_{n+\frac{1}{8}}-928f_{n+\frac{1}{4}}+10496f_{n+\frac{3}{8}}\\ -4540f_{n+\frac{1}{2}}+10496f_{n+\frac{5}{8}}-928f_{n+\frac{3}{4}}+5888f_{n+\frac{7}{8}}+989f_{n+1}\end{bmatrix} \quad (52)$$

Above are the discrete one step with seven off-steps hybrid block method which are used to solve some sample fifth-order ordinary differential equations (IVPs) directly without requiring predictor to start the corrector.

3. Analysis of the Method

In this section, the analysis of the basic properties of the method was carried out as follows.

3.1. Order and Error Constant of the Method

The formula in Eq. (11) in a conventional linear multistep method can be express as

$$\sum_{j=3}^{7}\alpha_{\frac{j}{8}}y_{n+\frac{j}{8}}=h^5\sum_{j=0}^{8}\beta_{\frac{j}{8}}y^{(5)}_{n+\frac{j}{8}} \quad (53)$$

According to Lambert [1], the local truncation error associated with Eq. (53) was defined by the difference operator.

$$L_{\frac{j}{8}}\{y(x):h\}=\sum_{j=3}^{7}\left\{\alpha_{\frac{j}{8}}y\left(x_n+\frac{j}{8}h\right)\right\}-h^5\sum_{j=0}^{8}\left\{\beta_{\frac{j}{8}}y^{(5)}\left(x_n+\frac{j}{8}h\right)\right\} \quad (54)$$

$y(x)$ is assumed to have continuous derivative of a sufficiently high order. Therefore expanding (11) in Taylor series about the point x to obtain the expression.

$$L_{\frac{j}{8}}\{y(x):h\} = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + ... + c_{p+4} h^{p+4} y^{(p+4)}(x) + c_{p+5} h^{p+5} y^{(p+5)}(x) \quad (55)$$

The term c_{p+5} is called the error constant and implies that the local truncation error is given by:

$$t_{n+k} = c_{p+5} h^{(p+5)} y^{(p+5)}(x_n) + 0h^{(p+6)} \quad (56)$$

Since $c_0 = c_1 = ... = c_{p+4} = 0, c_{p+5} \neq 0$. See Ref. [16]; the method has order $p = 9$ with error constant $c_{p+5} = \frac{374071}{910608676208400295552640}$

3.2. Zero Stability of the Method

According to Fatula [17], the block method represented in (13) – (52) is zero stable provided the roots $z_j, j = 1(1)k$ of the first characteristic polynomial $\rho(r)$ specified as

$$\rho(z) = \det\left[\sum_{j=0}^{k} A^{(j)} Z^{k-j}\right] = 0,\ A^{(0)} = -1 \quad (57)$$

Satisfies $|z_j| \leq 1$, and for those roots with $|z_j| = 1$, the multiplicity must not exceed 2. By definition (3.2), the block is zero stable since the roots of the characteristic polynomial satisfy $|z| \leq 1$ and the root $|z| = 1$ has multiplicity not exceeding the order of the differential equation. Moreover, as $h^{\mu} \to 0, \rho(z) = z^{r-\mu}(\lambda - 1)^{\mu}$, where μ is the order of the differential equation, for the block method, $r = 32$, and $\mu = 5$

$$\rho(z) = \lambda^{27}(\lambda - 1)^5 = 0$$

$$\Rightarrow \lambda = 0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1$$

Hence, worth concluding that the method is Zero Stable.

3.3. Consistency of the Method

From Eq. (11), the first and second characteristics polynomials of the method are given by

$$\rho(r) = r - 5r^{\frac{7}{8}} + 10r^{\frac{3}{4}} - 10r^{\frac{5}{8}} + 5r^{\frac{1}{2}} - r^{\frac{3}{8}}$$

$$\sigma(r) = \frac{-19}{23781703680} + \frac{212}{23781703680} r^{\frac{1}{8}} - \frac{1072}{23781703680} r^{\frac{1}{4}} + \frac{3044}{23781703680} r^{\frac{3}{8}} + \frac{25010}{23781703680} r^{\frac{1}{2}} + \frac{338204}{23781703680} r^{\frac{5}{8}} + \frac{329048}{23781703680} r^{\frac{3}{4}} + \frac{31532}{23781703680} r^{\frac{7}{8}} - \frac{199}{23781703680} r$$

This implies that the method presented in this report is consistent since it satisfies the following conditions:

i. The order of the method is $p = 9 > 1$ which is obvious.

ii. For the method, $\alpha_1 = 1, \alpha_2 = -5, \alpha_3 = 10, \alpha_4 = -10, \alpha_5 = 5$ *and* $\alpha_6 = -1$, thus $\sum_{j=1}^{6} \alpha_j = 1 - 5 + 10 - 10 + 5 - 1 = 0$, show the condition (ii) is satisfied.

iii. If $\rho(r) = r - 5r^{\frac{7}{8}} + 10r^{\frac{3}{4}} - 10r^{\frac{5}{8}} + 5r^{\frac{1}{2}} - r^{\frac{3}{8}}$ and $\rho'(r) = 1 - \frac{35}{8}r^{-\frac{1}{8}} + \frac{15}{2}r^{-\frac{1}{4}} - \frac{25}{4}r^{-\frac{3}{8}} + \frac{5}{2}r^{-\frac{1}{2}} - \frac{3}{8}r^{-\frac{5}{8}}$

It follows from here that $\rho(1) = 0 = \rho'(1)$

Show that the condition (iii) is satisfied as well.

Note that

$$\rho^{(5)}(r) = -\frac{133875}{32768}r^{-\frac{33}{8}} + \frac{8775}{512}r^{-\frac{17}{4}} - \frac{423225}{16384}r^{-\frac{35}{8}} + \frac{525}{32}r^{-\frac{9}{2}} - \frac{118755}{32768}r^{-\frac{37}{8}}$$

$$\Rightarrow \rho^{(5)}(1) = \frac{15}{4096} = 5!\sigma(1)$$

Thus, the condition (iv) is satisfied. Hence the method is consistent.

3.4. Convergence of the Method

According to Henrici [18], the necessary and sufficient condition for a numerical method to be convergent is to be Zero Stable and Consistent. Thus, since it has been successful shown in (3.2) and (3.3) above respectively. Hence, the method is said to convergent.

3.5. Region of Absolute Stability of the Method

Worth considering the stability polynomial in the general form:

$$\pi(r, \bar{h}) = \rho(r) - \bar{h}\sigma(r) = 0 \tag{58}$$

Where $\bar{h} = h^2\lambda$ and $\lambda = \frac{\partial f}{\partial y}$ is assumed constant. The first and second characteristics polynomials of Eq. (11) are given by

$$\rho(r) = r - 5r^{\frac{7}{8}} + 10r^{\frac{3}{4}} - 10r^{\frac{5}{8}} + 5r^{\frac{1}{2}} - r^{\frac{3}{8}}$$

$$\sigma(r) = \frac{-19}{23781703680} + \frac{212}{23781703680}r^{\frac{1}{8}} - \frac{1072}{23781703680}r^{\frac{1}{4}} + \frac{3044}{23781703680}r^{\frac{3}{8}} + \frac{25010}{23781703680}r^{\frac{1}{2}}$$
$$+ \frac{338204}{23781703680}r^{\frac{5}{8}} + \frac{329048}{23781703680}r^{\frac{3}{4}} + \frac{31532}{23781703680}r^{\frac{7}{8}} - \frac{199}{23781703680}r$$

The boundary of the region of the absolute stability is

$$\bar{h} = \frac{\rho(r)}{\sigma(r)} = \frac{-23781703680\left(r - 5r^{\frac{7}{8}} + 10r^{\frac{3}{4}} - 10r^{\frac{5}{8}} + 5r^{\frac{1}{2}} - r^{\frac{3}{8}}\right)}{19 - 212r^{\frac{1}{8}} + 1072r^{\frac{1}{4}} - 3044r^{\frac{3}{8}} - 25010r^{\frac{1}{2}} - 338204r^{\frac{5}{8}} - 329048r^{\frac{3}{4}} - 31532r^{\frac{3}{4}} + 199r} \tag{59}$$

By setting $r = e^{i\theta}$, then Eq. (59) becomes

$$\bar{h}(\theta)9\frac{-23781703680\left(e^{i\theta}-5e^{\frac{7}{8}i\theta}+10e^{\frac{3}{4}i\theta}-10e^{\frac{5}{8}i\theta}+5e^{\frac{1}{2}i\theta}-e^{\frac{3}{8}i\theta}\right)}{19-212e^{\frac{1}{8}i\theta}+1072e^{\frac{1}{4}i\theta}-3044e^{\frac{3}{8}i\theta}-25010e^{\frac{1}{2}i\theta}-338204e^{\frac{5}{8}i\theta}-329048e^{\frac{3}{4}i\theta}-31532e^{\frac{7}{8}i\theta}+199e^{i\theta}} \tag{60}$$

Evaluate Eq. (60), and equate the imaginary part to zero gives

$$\bar{h}(\theta)=\frac{-23781703680\left(\begin{array}{l}19\cos\theta-226369\cos\frac{1}{8}\theta+485216\cos\frac{1}{4}\theta-85995\cos\frac{3}{8}\\-291386\cos\frac{1}{2}\theta-10913\cos\frac{5}{8}\theta+2322\cos\frac{3}{4}\theta-307\cos\frac{6}{7}\theta+127413\end{array}\right)}{\left(\begin{array}{l}7562\cos\theta+260219207780\cos\frac{1}{8}\theta+39711472112\cos\frac{1}{4}\theta+2731248864\frac{3}{8}\theta-381017960\cos\frac{1}{2}\theta\\+57848480\cos\frac{5}{8}\theta+1292400\cos\frac{3}{4}\theta-1282592\cos\frac{7}{8}\theta+116120485352\end{array}\right)} \tag{61}$$

Evaluating Eq. (61) at the interval of 30^0 gives the following results of the boundaries for the region of absolute stability of the method as tabulated below.

Table 1. *Boundaries for region of absolute stability.*

θ	0^0	30^0	60^0	90^0	120^0	150^0	18 180^0
$\bar{h}(\theta)$	0	−475.1	−1872.5	−4109.6	−7050.8	−10499.0	−14261.0

From table 1 above, it could be deduced that the region of absolute stability of the method is given by $x(\theta)=(-14261,0)$ which satisfies the condition for A-stability.

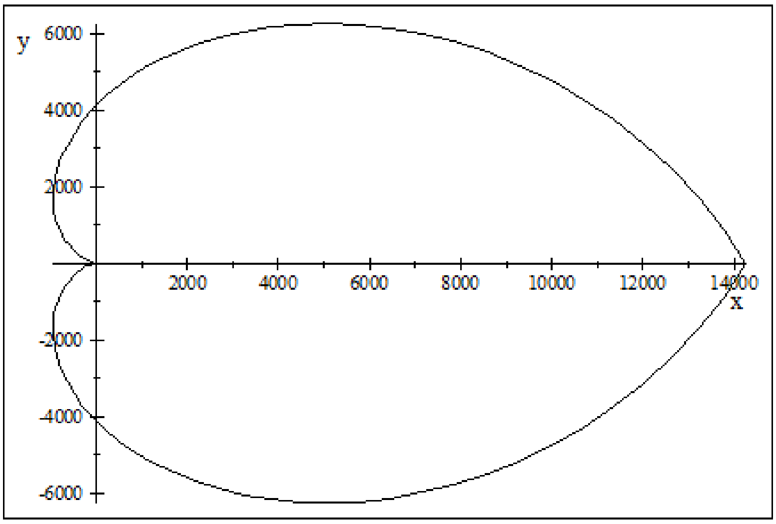

Figure 1. *Domain of stability (i.e. absolute) of the proposed method.*

4. Numerical Results

To verify the method's accuracy and feasibility, some specific initial value problems involving fifth-order ordinary differential equations were solved in this section. The table show that $E_{rc} = |y_{ex} - y_c|$ where y_{ex} is the exact solution, y_c is the computed result and absolute error defined as E_{rc}.

4.1. Problem 1: The General Fifth-order IVP of Ordinary Differential Equation

$$y^{(5)} = 2y^{(1)}y^{(2)} - yy^{(4)} - y^{(1)}y^{(3)} - 8x + \left(x^2 - 2x - 3\right)e^x. \text{ for } 0 \le x \le 1,$$

$$y(0)=1,\ y^{(1)}(0)=1,\ y^{(2)}(0)=3,\ y^{(3)}(0)=1,\ y^{(4)}(0)=1,\ \mathrm{h}=0.1$$

whose exact solution is reported as $y(x) = e^x + x^2$. is considered as the first test problem. The solutions were obtained within $[0,1]$ over 10 iterations following Ref. [8]. Table 2 presents a comparison between the exact solution (ERC) and the numerical result (NRC) of a newly developed method for solving initial value problems in ordinary differential equations. Across all values of the independent variable (XVC), the numerical results precisely match the exact results up to many significant digits, indicating high accuracy. The computed error (ERR) of the proposed method remains extremely low, consistently in the range of 10^{-27} to 10^{-23}, highlighting the method's strong precision and numerical stability. In contrast, the error values from method [8] are significantly higher, increasing progressively from 10^{-12} at XVC = 0.1 to 10^{-7} at XVC = 1.0. As seen in Figure 2, this performance comparison demonstrates that the new method offers several orders of magnitude improvement in accuracy over method Ref. [8]. Overall, the results confirm the effectiveness of the proposed method as a reliable and superior tool for solving higher-order initial value problems in ordinary differential equations.

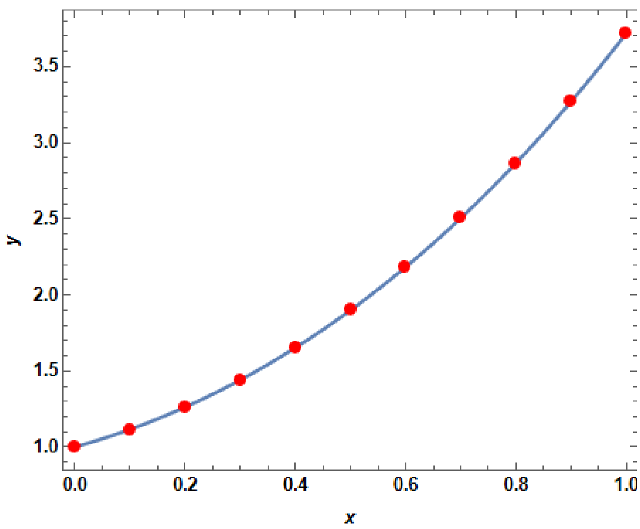

Figure 2. *Numerical and exact solution of problem 1.*

Table 2. *Numerical result for problem 1.*

XVC	ERC	NRC	ERR	ERR in [8]
0.10	1.1151709180756477	1.1151709180756477	2.331630031933065E-27	1.459721E-12
0.20	1.26140275816017	1.26140275816017	3.873466667706724E-26	4.187584E-11
0.3	1.43985807576003	1.43985807576003	1.959270647939141E-25	3.221776E-10
0.4	1.651824697641270	1.651824697641270	6.154612158754346E-25	1.365175E-09
0.5	1.8987212707001282	1.8987212707001282	1.490545433327853E-24	4.166737E-09
0.6	2.18211880039050	2.18211880039050	3.062255121730985E-24	1.033164E-08
0.7	2.50375270747047	2.50375270747047	5.615494577751005E-24	2.218906E-08
0.8	2.8655409284924676	2.8655409284924676	9.475191543275425E-24	4.288554E-08
0.9	3.2696031111569495	3.2696031111569495	1.500325406008122E23	7.645583E-08
1.0	3.71828182845904	3.71828182845904	2.259677853297352E-23	1.278750E-07

4.2. Problem 2: Consider a Linear Fifth Order Problem

$$y^{v} = 32y + \cos x - 32\sin x$$

$$y(0)=1,\ y'(0)=3,\ y''(0)=4,\ y'''(0)=7,\ y^{iv}(0)=16,\ \mathrm{h}=0.1$$

whose exact solution is $y(x) = \sin x + e^{2x}$ presented by Ref. [19]. Table 3 presents a detailed comparison between the exact results (ERC) and the numerical results (NRC) obtained using the newly developed method for solving initial value problems in ordinary differential equations. The numerical results align remarkably well with the exact solutions across all values of the independent variable (XVC), with negligible differences. The absolute errors (ERR) associated with the proposed method are extremely small, ranging from 10^{-23} to 10^{-19}, reflecting exceptional numerical precision. Worth observing in Figure 3 that when compared to the results from Ogunrinde et al. [19], the new method consistently shows significantly lower errors, especially as XVC increases. For instance, at XVC = 1.0, the proposed method yields an error of only 6.17×10^{-19}, while the error in Ref. [19] is 8.17×10^{-10}, illustrating an improvement by nearly nine orders of magnitude. These findings confirm that the newly developed method offers superior accuracy and stability, making it a robust alternative to existing numerical methods for solving high-order differential equations.

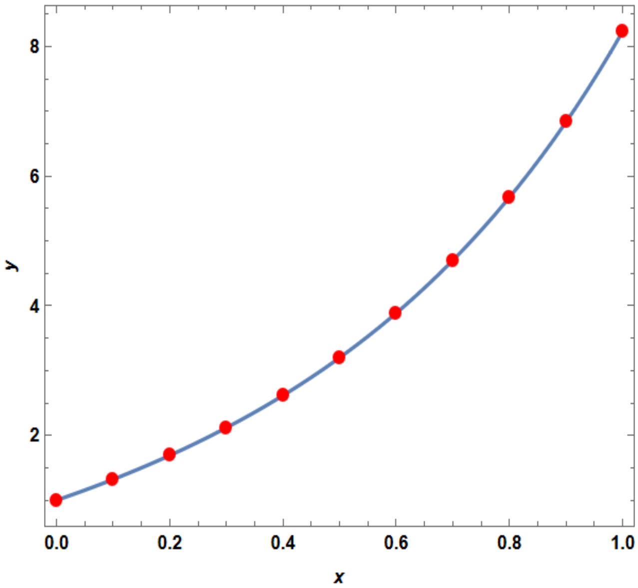

Figure 3. Numerical and exact solution of problem 2.

Table 3. Numerical result for problem 2.

XVC	ERC	NRC	ERR	ERR in [19]
0.10	1.321236174806998	1.321236174806998	4.097816747016119E-23	0.000000E+000
0.20	1.6904940284363315	1.6904940284363317	7.006902596219831E-22	4.440892E-016
0.3	2.1176390070518485	2.1176390070518485	3.689091584562187E-21	6.705747E-014
0.4	2.6149592708011182	2.6149592708011182	1.211955799815713E-20	9.467982E-013
0.5	3.197707367063248	3.197707367063248	3.079902542407762E-20	5.706546E-012
0.6	3.8847593961131583	3.88475939611315825	6.658412253296306E-20	2.265743E-011
0.7	4.699417654082366	4.699417654082366	1.288200402557018E-19	6.985079E-011
0.8	5.670388515294637	5.670388515294638	2.298838987614749E-19	1.814815E-010
0.9	6.83297437404043	6.83297437404043	3.858608133497839E-19	4.166596E-010
1.0	8.230527083738547	8.230527083738547	6.173890371406142E-19	8.710561E-010

4.3. Problem 3: Consider a Linear Fifth Order Problem

$$y^{(5)} = y^{(4)} + y' - y$$

$$y(0)=y'(0)=y'''(0)=y^{(4)}(0)=0,\ y''(0)=1,\quad x\in[0,1].$$

with theoretical solution $y(x)=\frac{1}{4}e^{-x}+\frac{1}{4}e^{x}-\frac{1}{2}\cos(x)$ presented by Ref. [20].

Table 4. *Numerical result for problem 3.*

XVC	ERC	NRC	ERR	ERR in [20]
0.10	0.005000001388888916	0.005000001388888842	2.33923079068786E-27	1.64909E-26
0.20	0.0200000888881711	0.02000008888817134	4.039547421786097E-26	4.12382E-25
0.30	0.04500101250162723	0.04500101250162725	2.111892891265468E-25	4.55707E-24
0.40	0.08000568891778487	0.08000568891778492	6.851722415640375E-25	9.81575E-24
0.50	0.12502170165800403	0.12502170165800397	1.714606688146714E-25	2.83674E-23
0.60	0.1800648016662947	0.18006480166629463	3.642883347702843E-24	6.80914E-23
0.70	0.2451634091732273	0.24516340917322726	6.914917222246244E-24	1.43194E-22
0.80	0.3203641184788396	0.32036411847883967	1.208878950600296E-23	2.72799E-22
0.90	0.4057382085890549	0.40573820858905507	1.984883320032055E-23	4.81322E -22
1.00	0.5013891644735521	0.501389164473552	3.102039357132054E-23	3.02101E-22

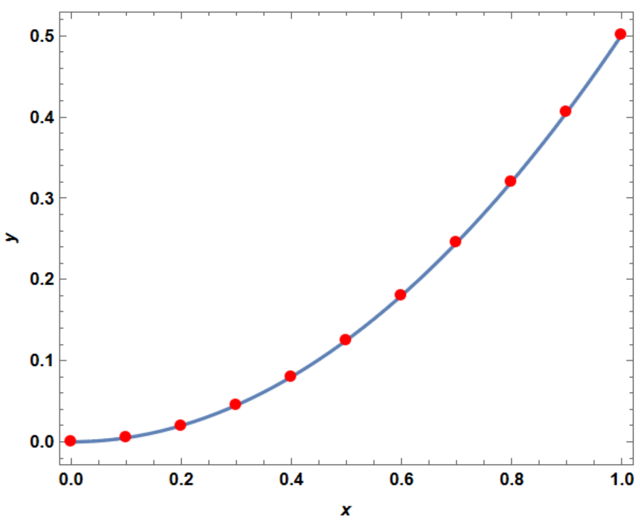

Figure 4. *Numerical and exact solution of problem 3.*

Table 4 and Figure 4 compare the exact results (ERC) and numerical results (NRC) of the newly developed method for solving initial value problems in ordinary differential equations across various values of the independent variable (XVC). The numerical outcomes closely match the exact solutions, with errors (ERR) consistently remaining in the extremely low range of 10^{-27} to 10^{-23}, demonstrating the method's high precision. When compared with the error values reported in Ramos and Momoh [20], the proposed method consistently yields smaller errors by at least one order of magnitude at every XVC point. For example, at XVC = 0.30, the error in the proposed method is approximately 2.11×10^{-25}, whereas the corresponding error in Ref. [20] is 4.56×10^{-24}, showing a clear improvement. The errors in Ref. [20] generally increase with XVC, while the new method maintains extremely stable and minimal error growth, even at XVC = 1.00. Worth deducing from Table 4 and Figure 4 that this analysis confirms that the proposed method significantly outperforms the method in Ref. [20] in terms of numerical accuracy and is highly effective for solving initial value problems in higher-order differential equations.

4.4. Problem 4: Consider a Fifth Order Problem

$$y^{(v)}=6\left(2\left(y'\right)^{3}\right)+6yy'y''+y^{2}y'''$$

$$y(1)=1,\ y'(1)=-1,\ y''(1)=2,\ y'''(1)=-6,\ y^{(iv)}(1)=24.\quad x\in[1,3].$$

with theoretical solution $y(x)=\frac{1}{x}$ is in Ref. [21]. Figure 5 presents a comparison between the exact (ERC) and numerical (NRC) results of a newly developed numerical method for solving an initial value problem in ordinary differential equations. The numerical results closely match the exact values across all tested points, with errors (ERR) in the range of 10^{-16} to 10^{-13}, indicating exceptional accuracy of the proposed method.. Although the error increases with the value of XVC, this growth is extremely gradual and well-controlled, further confirming the method's efficiency over longer intervals. Overall, the analysis highlights the robustness and accuracy of the new numerical method for solving initial value problems.

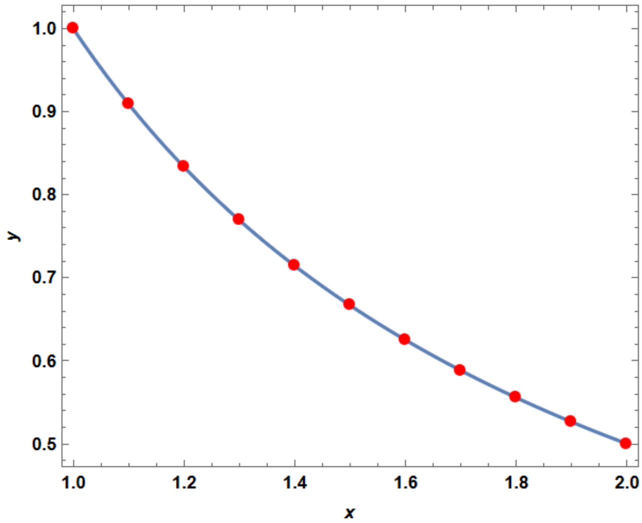

Figure 5. Numerical and exact solution of problem 4.

Table 5. Numerical result for problem 4.

XVC	ERC	NRC	ERR
1.0	1.0	1.0	0
1.10	0.9090909090909090	0.9090909090909091	1.036583182360166E-16
1.20	0.8333333333333316	0.8333333333333334	1.699754792581852E-15
1.30	0.7692307692307613	0.7692307692307693	7.983410155079497E-15
1.40	0.7142857142856911	0.7142857142857143	2.319132433273376E-14
1.50	0.6666666666666614	0.666666666666666	5.236573438706911E-14
1.60	0.6249999999998986	0.625	0.131818260874E-13
1.70	0.5882352941174704	0.5882352941176471	1.766366165336689E-13
1.80	0.5555555555552698	0.5555555555555556	2.856999582544477E-13
1.90	0.5263157894732475	0.5263157894736842	4.366919922205567E-13
2.00	0.4999999999993614	0.5	6.386131136225942E-13

4.5. Problem 5: Consider a Linear Fifth Order Problem

$$y^{v}=5y'''-4y',$$

$$y(0)=3,\ y'(0)=-5,\ y''(0)=11, y'''(0)=-23,\ y^{iv}(0)=47,\quad h=0.1$$

with theoretical solution $y(x)=1-e^{x}+3e^{-2x}$ available in Ref. [19].

Table 6. *Numerical result for problem 5.*

XVC	ERC	NRC	ERR	ERR in [19]
0.10	2.351021341158298	2.351021341158298	1.026908204376053E-22	2.664535E-15
0.20	1.7895573799467481	1.789557379946748	1.728314033805348E-21	1.896261E-13
0.30	1.296576100706076	1.2965761007060763	8.701324028207974E-21	2.172484E-12
0.40	0.8561621947103945	0.8561621947103943	2.717296285010156E-20	1.211808E-11
0.50	0.4549170528141988	0.4549170528141988	6.560622644085899E-20	4.576428E-11
0.60	0.0814638353460973	0.08146383534609747	1.349302530365906E-19	1.352662E-10
0.70	-0.2739618156456571	-0.27396181564565714	2.48857939203751E-19	3.379645E-10
0.80	-0.6198513745085014	-0.6198513745085017	4.2436969228687085E-19	7.473215E-10
0.90	-0.9637064464921901	-0.9637064464921903	6.823750086760096E-19	1.506349E-09
1.00	-1.312275978749207	-1.312275978749207	1.048572435146556E-18	2.824062E-09

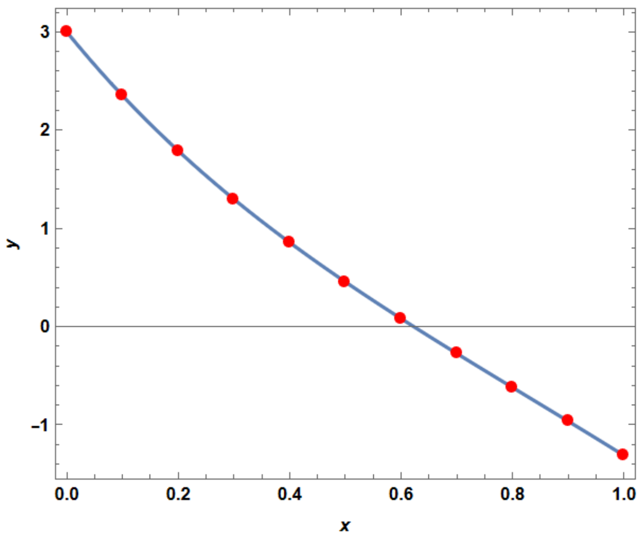

Figure 6. *Numerical and exact solution of problem 5.*

Table 6 and Figure 6 compares exact results (ERC) with numerical results (NRC) generated by a newly developed numerical method for solving initial value problems in ordinary differential equations. The proposed method yields errors (ERR) in the range of 10^{-22} to 10^{-18}, which are remarkably small and demonstrate extremely high precision in approximating the exact solution. Compared to the corresponding errors in reference [19], the new method consistently outperforms it by several orders of magnitude at each step of the independent variable (XVC). For instance, at XVC=0.10, the error of the proposed method is approximately 1.03×10^{-22}, while that of [19] is 2.66×10^{-15}, clearly highlighting the improvement. Although the absolute error increases slightly as XVC grows, the values remain negligible and within acceptable computational bounds, confirming the method's stability over the interval [0.1,1.0]. Overall, the data underscores the accuracy, reliability, and numerical superiority of the newly developed method, making it a promising tool for solving initial value problems in scientific computing.

Discussion of Results

In this study, collocation and interpolation techniques were employed to construct a linear hybrid multistep method for solving initial value problems of fifth-order ordinary differential equations. The developed method demonstrates a high order of accuracy and computational efficiency, making it suitable for complex initial value problems. Numerical results presented in Tables 2, 3, 4 and 6 confirm the superior performance of the new method over existing approaches by Ramos and Momoh [20], Ogunrinde et al. [19], and Kayode [8]. Figures 2–6 illustrate the close agreement between the numerical solutions obtained using the proposed method and the exact solutions. The newly constructed ninth-order method exhibits improved precision and stability, outperforming earlier methods in both error minimization and convergence behaviour. Sequel to Animasaun et al. [22], improved precision and stability in numerical solutions are essential to ensure accurate modeling of complex systems. Outperforming earlier methods in error minimization and convergence behavior enhances reliability, especially for sensitive or long-time simulations. As mentioned by Li et al. [23], and Wang et al. [24], such advancements reduce computational cost, prevent error accumulation, and support more robust, real-world scientific and engineering applications.

5. Conclusion

The linear multistep hybrid block method with a single step and seven off-step points for the direct solution of fifth-order ordinary differential equations. The main and additional formulas that constitute the method were obtained from the same continuous scheme derived via interpolation and collocation procedures. The stability properties and region of the method were discussed. The method is applied in block form. Numerical results obtained using the block method show that they are efficient and adequate for solving general fifth-order problems of ordinary differential equations. In fact, when the results were compared to different articles, this proposed method by the new results obtained was better in terms of accuracy and efficiency.

Abbreviations

XVC	Value of the Independent Variable Where a Numerical Value Is Taken
ERC	Exact Result at XVC
NRC	Numerical Result of XVC
ERR	Error in Proposed Method at XVC

Author Contributions

Duromola Monday Kolawole is the sole author. The author read and approved the final manuscript.

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Conflicts of Interest

The author declares no conflicts of interest.

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