American Journal of Applied Mathematics
2025, Vol. 13, No. 2, pp. 125-142
https://doi.org/10.11648/j.ajam.20251302.13

Research Article

New Symmetry Index Based on Gini Mean Difference

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Abstract

The Gini index is a widely used tool for measuring inequality, but it has several limitations that can lead to misinterpretation or incorrect conclusions, as highlighted in various studies. A significant drawback of the Gini index is that it fails to account for crucial aspects of inequality, such as the heterogeneity within a population, and the asymmetry of the data, meaning how skewed or unbalanced the distribution may be. In response to these shortcomings, a new index has been developed that more accurately captures both inequality and the symmetry of data. This new index builds on Auda's symmetry test and leverages a mathematical relationship between the Gini mean difference and the Gini index, providing a more refined measure. Through a Monte Carlo simulation, the new index demonstrated its superiority over existing ones, as it effectively reveals the distribution of asymmetrical data (whether positively or negatively skewed). Unlike the Gini index, this new index can differentiate between datasets with identical Gini values but different levels of symmetry. Additionally, it is more versatile, able to be applied to datasets of any size, including those that contain negative values. The index's effectiveness is demonstrated with examples, including a scenario where two populations have the same total income and an educational study using data from Helwan University's Faculty of Social Work.

Keywords

Gini Mean Difference, Symmetry Distribution, Symmetry Index

1. Introduction

The Gini index (GI), developed by Italian statistician Corrado Gini [14], is widely used to measure socioeconomic inequality, especially in income and wealth distribution. Despite the existence of over 60 inequality indices [8], the GI remains the most popular. Its application is not limited to socioeconomics where; according to [12], the GI can also be used in fields like sociology, health science, ecology, engineering, and agriculture. In general, for statistical distributions of non-negative values with positive means, the GI serves as a measure of statistical heterogeneity [10]. One of its main advantages is that it summarizes the inequality of an entire data set with a single statistic, as it is ranged from 0 to 1.

Despite the simplicity of the GI, it can sometimes cause confusion when comparing different datasets [26]. For example, [2] pointed out that a lower GI doesn't always mean a distribution is more symmetrical than one with a higher GI. This is because the Lorenz curves of the two datasets may intersect, indicating different distributions. Also, the intersection of Lorenz curves from two datasets with different distributions can result in the same GI value [6].

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Received: 13 February 2025; **Accepted:** 25 February 2025; **Published:** 18 March 2025

Hence, to create a complete ranking of data, such as income data, and measure the difference in income inequality across countries, [1] introduces a social welfare-based inequality index. This index ranges from 0 to 1. The main idea behind the Atkinson index is the concept of an equally distributed equivalent level (EDE), which depends on a parameter called inequality aversion (ε) (see (4)). This parameter (ε) can range from 0 to infinity. As ε increases, the index gives more importance to income transfers at the lower end of the distribution and less importance to transfers at the top. While the Atkinson index has benefits, like providing a full ranking of data distributions and clearly showing the social welfare function behind the income inequality measure (which is helpful for policy decisions), [9] and [17] pointed out that the ranking of income distributions can change significantly depending on the choice of social welfare functions and how much a country dislikes inequality (ε). This dislike for inequality may differ between countries.

To avoid the need for a social welfare judgment, a set of generalized entropy (GE (α)) indices can be used as an alternative way to rank inequality when the Lorenz curves of two or more data sets intersect. The theoretical values of the GE (α) index range from 0 to infinity, with 0 indicating equal distribution and higher values indicating more inequality. The GE (α) index is a group of indices because its form changes depending on the value of the parameter α (see (15)). According to [6], α can be any real number from -∞ to ∞, but in practice, α is usually positive. GE (0) is called the mean logarithmic deviation, and GE (1) is known as the Theil inequality index, named after the creator in 1967 (see (15) and (17)). While the GE (α) indices can overcome the limitations of the Gini and Atkinson indices when Lorenz curves cross, it's important to note that the exact form of the GE (α) index depends on the value of α, which can vary across data sets, making it challenging to compare inequality across different data.

Another limitation of using the GI is that it can show the same value for two different datasets, even when the inequality between them is actually quite different when you consider more details about the data. For example, in 2015, Greece and Thailand both had a GI of 0.360 according to World Bank data. However, the income gap between the richest 10% and the poorest 10% was much larger in Greece (13.8) compared to Thailand (8.9). Additionally, [1] pointed out that the GI is more sensitive to changes in the middle of the income distribution but less sensitive to changes at the top and bottom. Furthermore, the GI cannot be used to measure inequality in distributions that include negative values. For instance, wealth distribution may include people with negative wealth, where their debts are larger than their assets. While the Gini coefficient can be calculated for such distributions, this could result in values greater than 1, making the result hard to interpret.

Additionally, there are several alternative measures with useful features, such as the Zanardi index, which improves upon the GI by capturing asymmetries in income distributions [28]. Also, [22] suggested combining the GI with the income share held by the top and bottom 10% of the population as a joint measure of inequality. The idea behind using multiple measures is to highlight different aspects of the income distribution. However, this approach lacks a systematic analysis to ensure it truly captures all the information in an income distribution. Pleace pointed out that there is no clear agreement on which alternative measure to use because no clear criteria have been established to determine which measure is the best [13].

To overcome these limitations of inequality indices in general and GI for more specific (which are: GI is a relative measure, countries with different income distributions may have the same index, it does not give information about the asymmetry data, the GI has a downward-bias for small sample size and it may give different results when applied to individuals instead of groups for the same data), we devise a new symmetry index that complements the traditional indices of inequality. Our method is quite simple. It utilizes the test of symmetry that proposed by [3] and the mathematical relationship between Gini mean difference (GMD) and GI.

Accordingly, the rest of this paper is organized as follows: Section 2 provides the definition of symmetry distribution. The proposed symmetry index and its properties are provided in Section 3. In Section 4, the simulation study is presented and a summary of the performance of the proposed index as compared with several indices is provided. Section 5 contains real examples for illustrative purpose and finally the conclusions of this study are summarized in Section 6.

2. The Definition of Symmetry Distribution

Symmetry and homogeneity have proven particularly useful from a methodological point of view. They are also the basis for moments and transformation methods, and a more symmetric approach has undoubtedly improved our thinking about a degree of controversial economic issues [18]. In statistics, symmetry is an important assumption, as many statistical procedures rely on the symmetry of the underlying distributions. For instance, errors in regression models are often assumed to be symmetrically distributed [5]. A distribution is considered symmetric when the values of a variable occur at regular frequencies, with the mean, median, and mode typically aligning at the same point. It can be defined as follow: consider a random variable x with cdf F, then the distribution is called symmetric about its median θ when:

$$F(x-\theta) = 1 - F\big(-(x-\theta)\big) \tag{1}$$

which equal to:

$$F(x-\theta) + F\big(-(x-\theta)\big) = 1,\ -\infty < x < \infty \tag{2}$$

hence if $\theta = 0$, the variable x is said to be symmetric around (0) and the above equation reduces to:

$$F(x) + F(-x) = 1 \quad (3)$$

Also, if the median (θ) is known and $\theta \neq 0$, we can define another variable y, where $y = x - \theta$, is said to be symmetric around (0).

3. New Symmetry Index Based on Gini Mean Difference

The main objective here is to propose new index which based on: the test of symmetry proposed by [3] and the mathematical relationship between Gini mean difference (GMD) and Gini index (GI). Therefore, we divide this section into three parts: the first part covers the test of symmetry proposed by [3], the second part introduces the new symmetry index and the properties of the proposed index are presented in third part.

3.1. Test of Symmetry

We propose our new index based on [3] who proposed two statistics based on the actual values of the data. These statistics are S_θ, which used to test the symmetry of the data when median is known, and $S_{\hat{\theta}}$, which used to test the symmetry of the data when median is unknown [4]. In order to define these statistics, let $x_1, x_2, \ldots, x_n$ be random sample of size n from unknown continuous distribution. Hence, y_i^θ can be defined as:

$$y_i^\theta = |x_i - \theta|, i = 1,2, \ldots, n \quad (4)$$

where the sample is partitioned into two sub-samples as follow

$$y_i^\theta = \begin{cases} x_i - \theta \rightarrow x_i \geq \theta \\ \theta - x_i \rightarrow x_i \leq \theta \end{cases}$$

$$i = 1,2,.,n, (all\ transformed\ values\ are\ positive) \quad (5)$$

Since the population assumed to be continuous, then the probability of $x_i = \theta$ is zero and such values $(x_i = \theta)$ should be excluded from the analysis, and sample size adjusted accordingly. Now we can define the test statistic S_θ as follow:

$$S_\theta = \frac{GMD_+^\theta}{GMD_-^\theta} \quad (6)$$

where GMD_+^θ represents the GMD for subset 1 when $x_i \geq \theta$, GMD_-^θ represents the GMD for subset 2 when $x_i \leq \theta$ and the basic mathematical definitions of GMD is:

$$GMD = \frac{\sum_{i<j}^{n} |y_i - y_j|}{n(n-1)/2} \quad (7)$$

Additionally she proposed to use $S_{\hat{\theta}}$ instead of S_θ when the median of the population is unknown, where;

$$S_{\hat{\theta}} = \frac{GMD_+^{\hat{\theta}}}{GMD_-^{\hat{\theta}}} \quad (8)$$

and $\hat{\theta}$ represents an estimate of θ using the sample data.

3.2. The Proposed Symmetry Index

As previous subsection, the new symmetry indices are (SI_θ) which used when median is known and $(SI_{\hat{\theta}})$ that is used when median is unknown. First SI_θ can be defined as follow:

$$SI_\theta = \frac{GI_+^\theta}{GI_-^\theta} = \frac{\frac{GMD_+^\theta}{2\bar{y}_1}}{\frac{GMD_-^\theta}{2\bar{y}_2}} = \frac{GMD_+^\theta}{GMD_-^\theta} \times \frac{\bar{y}_2}{\bar{y}_1} \quad (9)$$

$$\therefore SI_\theta = S_\theta \times \frac{\bar{y}_2}{\bar{y}_1} \quad (10)$$

where $\bar{y}_1$ presents the sample mean for subset 1 when $x_i \geq \theta$, $\bar{y}_2$ presents the sample mean for subset 2 when $x_i \leq \theta$, that:

$$\bar{y}_1 = \frac{\sum_{k=1}^{n_1} y_{1k}}{n_1},\ \bar{y}_2 = \frac{\sum_{j=1}^{n_2} y_{2j}}{n_2} \quad (11)$$

For unknown median, we suggest to use $SI_{\hat{\theta}}$ instead of SI_θ where;

$$SI_{\hat{\theta}} = \frac{GI_+^{\hat{\theta}}}{GI_-^{\hat{\theta}}} = \frac{\frac{GMD_+^{\hat{\theta}}}{2\bar{y}_1}}{\frac{GMD_-^{\hat{\theta}}}{2\bar{y}_2}} = \frac{GMD_+^{\hat{\theta}}}{GMD_-^{\hat{\theta}}} \times \frac{\bar{y}_2}{\bar{y}_1} \quad (12)$$

$$\therefore SI_{\hat{\theta}} = S_{\hat{\theta}} \times \frac{\bar{y}_2}{\bar{y}_1} \quad (13)$$

3.3. The Properties of Proposed Index

1) The proposed index (SI_θ) bounded between 0 and ∞. That as the value of SI_θ tends to 1, the distribution of the data tends to symmetry.
2) SI_θ gives information about the distribution of asymmetry data, that when SI_θ is greater than 1, the data is said to have positive skewed distribution, and when SI_θ is less than 1 (tends to 0), the data is said to have negative skewed distribution.
3) In contrast of GI (which defined as an equality index), SI_θ can be defined as a symmetry index since it combines between three main concepts:
   a. Asymmetry: which measured by the statistic S_θ (see (6) and (10)).
   b. Concentration: which introduced by GI that calcu-

lated for each subset of the data separately (see (9)).

c. Heterogeneity: This is introduced through the discriminant point, which is determined by measuring the dispersion between pairs of observations (using GMD), and by the process of transforming the data using (4).

4) Also in contrast of GI, we can use SI_θ to measure the symmetry for distributions that may include negative values. The reason behind that is; SI_θ does not depend on the original values of the data $(x_1, x_2, \ldots, x_n)$, it, instead, depends on its transformed values (see (4)).

5) SI_θ does not change when all observations of the data change proportionally. That the change of location do alter (SI_θ) where the mean for subset 1 when $x_i \geq \theta$ does not equal the mean for subset 2 when $x_i \leq \theta$ and $GMD_+^\theta \neq GMD_-^\theta$, then $GI_+^\theta \neq GI_-^\theta$.

6) As GI, SI_θ can be used to assess the inequality in several fields such as: income data, education, social stud-etc…

4. Simulation Study

Simulation study was conducted to investigate the performance of the proposed index (SI_θ). Accordingly, we select several well-known indices, some of which have been recognized as effective indicators for a variety of distributions. These indices are listed below.

1) *Lorenz curve, Gini's and Pietra's indices*

a) The Lorenz curve defines as an increasing and convex curve that lies in the first quadrant of the Cartesian plane the values of the variable are positive [6]. In the case of complete equality, it coincides with the 45-degree line. However, under inequality cases the curve lies below the 45-degree line (see Figure 1). Intuitively, the closer the Lorenz curve is to the 45-degree line, the more equal the distribution [16]. The Lorenz curve gives an intuitive basis to understand both the Gini index and the Pietra index.

b) Gini index [14, 15]: it is defined as the normalized area between the Lorenz curve of the distribution and the 45-degree line and it can be represented in several equivalent mathematical forms [27]. The index ranges between (0) and (1), where it equals to zero in the case of complete equality and approaches one in the case of complete inequality.

c) Pietra index (PI) [19]: this index is considered a fundamental measure of statistical heterogeneity, especially valuable when dealing with asymmetric and skewed probability distributions. Similar to the Gini index, the Pietra index equals zero for complete equality and increases towards one in cases of complete inequality. The mathematical formulation of PI can be expressed as follow:

$$PI = 0.5E\left[\left|\frac{X}{\mu} - 1\right|\right] \tag{14}$$

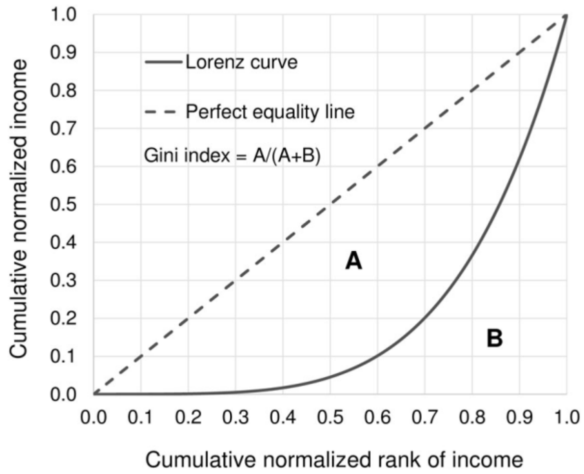


Figure 1. *Illustrates the relation between Lorenz curve and GI [7].*

It is important to note that both the Gini and Pietra indices can be calculated for variables with both positive and negative values. However, in cases where negative values are included, these indices do not have an upper bound. This means that as the magnitude of inequality increases, the indices can grow larger without limit, especially in distributions with substantial negative values. A higher value of these indices indicates a greater degree of inequality within the distribution, reflecting a larger disparity between the variable's values. This characteristic makes the indices useful for measuring inequality across diverse datasets, including those where negative values might be present.

2) *Generalized Entropy and the Theil's indices*

a) Generalized entropy (GE) [20, 21]: The GE indices are based on a parameter, α, which determines how sensitive the indicator is to different parts of the distribution. By adjusting α, the index can focus more on either the lower or upper ends of the distribution, allowing for a flexible measure of inequality depending on which part of the distribution is of most interest. The mathematical formulation of this index can be expressed as follow:

$$GE(\alpha) = \frac{1}{n(\alpha^2-\alpha)}\sum_{i=1}^{n}\left[\left(\frac{x_i}{\bar{x}}\right)^\alpha - 1\right] \tag{15}$$

in our simulation study we assume $\alpha = 0.5$.

b) Our simulation also involved Theil indices (T_T and T_L) [24, 25]: that present specific cases of GE when α=1 and α=0, respectively, and they can be defined using the following forms:

$$T_T = \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{\bar{x}}\ ln\left(\frac{x_i}{\bar{x}}\right) \tag{16}$$

$$T_L = \frac{-1}{n}\sum_{i=1}^{n}\ ln\left(\frac{x_i}{\bar{x}}\right) \tag{17}$$

the GE indicators are equal to 0 in the case of complete equality. A larger value of each of them indicates larger inequality in the distribution. The GE is unbounded when considered for a theoretical distribution but is bounded in the case of a finite independent and identically distributed sample (for example, in the case of the Theil index, the bound is $\ln(n)$, where n is the sample size, and it corresponds to the case of complete inequality). Besides, The GE indices can only be computed for non-negative values of the variable in certain cases, depending on the parameter α chosen.

3) *Atkinson's index*

Atkinson index (AI) [1]: it is defined as the normalized ratio of the equally distributed equivalent level of observed data to the mean of the actual data distribution; it can be defined using the following form:

$$AI(\varepsilon) = \begin{cases} 1 - \left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i}{\bar{x}}\right)^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}, \varepsilon \neq 1 \\ 1 - \frac{\prod_{i=1}^{n}\left((x_i)^{\frac{1}{n}}\right)}{\bar{x}}, \varepsilon = 1 \end{cases} \quad (18)$$

where the parameter $\varepsilon > 0$ is interpreted as the level of aversion of inequality. The index is bounded between 0 and 1 and increases with inequality, and it only can be computed for positive values of the variable.

4) *Zanardi index*

Zanardi index (ZI) [28, 29]: which also known as symmetry index, can be defined as follow:

$$ZI = 2K_d\,\frac{GI^r - GI^p}{GI} \quad (19)$$

where d is defined as discriminant point $D(p_d, q_d)$ on Lorenz curve (see Figure 2 that provided by [7]) and $q\%$ is the percentage of the total values of the data that hold by $p\%$ of them. The heterogeneity introduced by $K_d = (p_d q_d)/2$ and the concentration introduced with GI [23]. The index ranges from -1 to $+1$, and it approaches 0 in the case of complete equality.

5) *Eliazar index*

In our study we depend on Vertical-diameter inequality index (I_V) that introduced by [10, 11]. The index can be defined as:

$$I_V = \frac{1}{\mu}\,E\left[\left|X - \theta\right|\right] \quad (20)$$

where θ presents the median of the data. The index is bounded between 0 and 1, which approaches zero under complete equality and one for complete inequality.

The simulation covers multi scenarios included: symmetric and non-symmetric (positive and negative skewed) distributions, for each scenario we generate 10000 samples with sample sizes 20, 50 and 100. The detailed setup under each of these scenarios and its results are explained next.

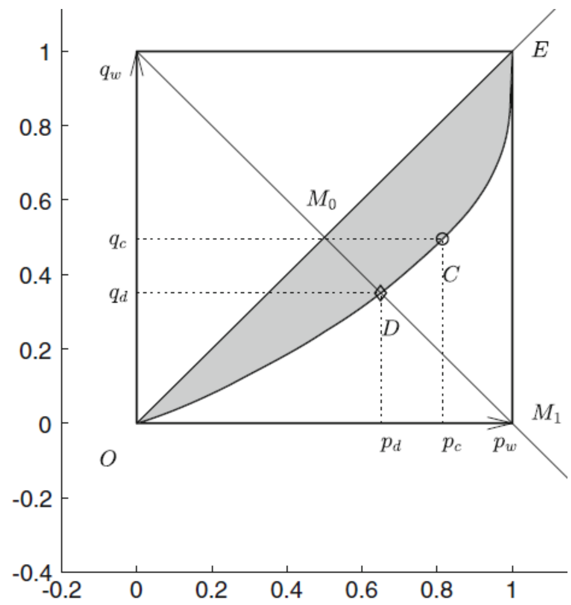


***Figure 2.** Lorenz curve and its characteristic points [7].*

4.1. Symmetric Distributions

4.1.1. The Characteristics of Symmetric Distributions

Under this scenario, several distributions with different characteristics have been generated such as: standard normal distribution that involves negative and non-negative values, standard uniform, from Beta family we generate two distributions (B (3,3) and B (5,5)) and from t-distribution we generate two distributions (t(3) and t(5)). The details of these distributions with their corresponding density and Lorenz curves are explained next.

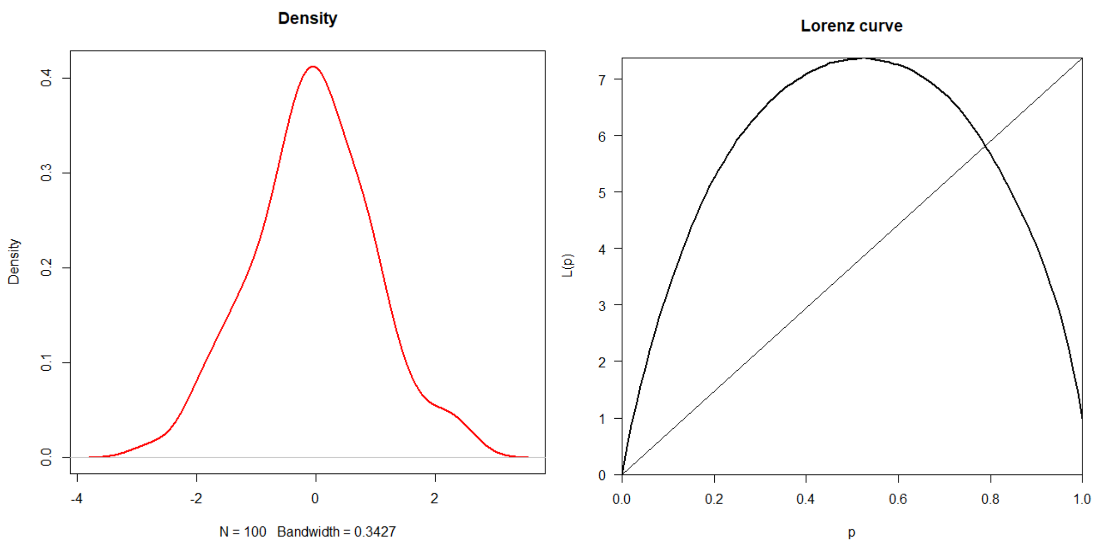
***Table 1.** The summary statistics of symmetric distributions.*

Distributions	Min	Q1	Median	Mean	Q3	Max
Normal (0, 1)	-2.79622	-0.65045	-0.05757	-0.05734	0.63126	2.51546
Uniform (0, 1)	0.002111	0.201003	0.454077	0.461732	0.719649	0.983808
Beta (3, 3)	0.1050	0.3455	0.5113	0.5040	0.6470	0.8804
Beta (5, 5)	0.1467	0.3933	0.5239	0.5256	0.6485	0.7966

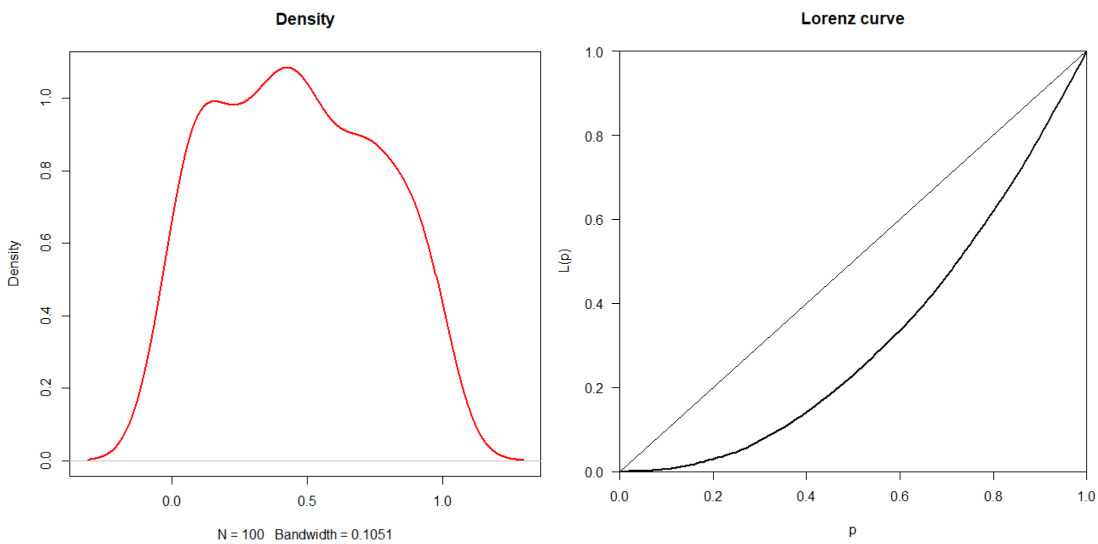
Distributions	Min	Q1	Median	Mean	Q3	Max
t (3)	-3.2013	-0.6126	0.2061	0.4186	0.9407	15.1049
t (5)	-2.5522	-0.9673	-0.1791	-0.1325	0.5438	2.9057

Based on Table 1 we can observe that, although all distributions are generated from symmetric distributions, both standard normal and t-distributions involve negative values on their data which may affect negatively on Lorenz curve. The corresponding density curves and Lorenz curves of these distributions are explained next.

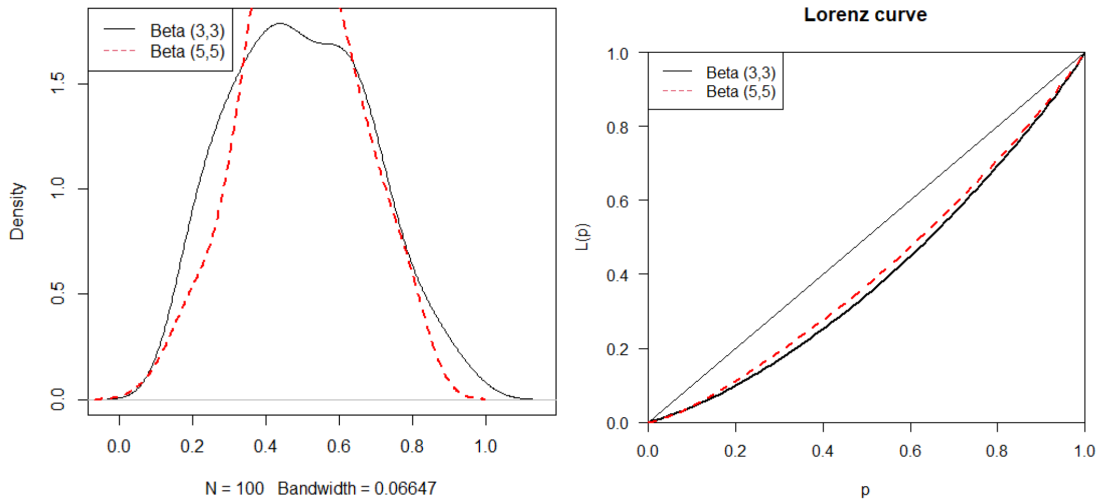
From the following table (Figure 3), un-acceptable Lorenz curve can be observed under the cases of standard normal distribution and t-distributions since data include negative values. However, according to uniform and Beta distributions we observe an acceptable Lorenz curves that is because all values of their generated data are non-negative. Also, their distributions appear as not perfectly equally (symmetry) distributed where there are gap between its equality line and their Lorenz curves.



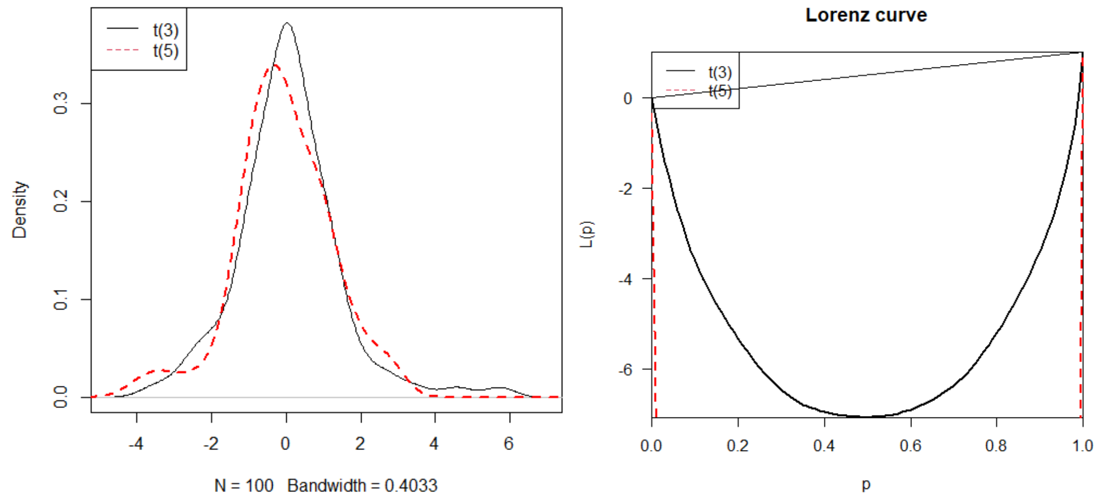
a) Density curve and Lorenz curve of standard normal distribution



b) Density curve and Lorenz curve of standard uniform distribution



c) Density curves and Lorenz curves of Beta distributions



d) Density curves and Lorenz curves of t-distributions

***Figure 3**. The density curves and Lorenz curves of symmetric distributions.*

4.1.2. The Results of Symmetric Distributions

Table 2 shows the results of the study when data are generated from symmetric distributions with different characteristics.

***Table 2**. The results of symmetric distributions.*

N	Distribution	GI	SI_θ	PI	AI	T_T	T_L	GE	ZI	I_V
20	Normal (0, 1)	1.583	1.018	1.220	-	-	-	-	-0.217	6.2195
	Uniform (0, 1)	0.321	1.030	0.2465	0.109	0.1912	0.300	0.225	-0.353	0.0779
	Beta (3, 3)	0.207	1.029	0.1528	0.039	0.074	0.087	0.079	-0.224	0.035
	Beta (5, 5)	0.164	1.028	0.120	0.0235	0.045	0.050	0.047	-0.208	0.022
	t (3)	1.218	1.037	1.008	-	-	-	-	-0.1353	1.139
	t (5)	-0.085	1.030	2.354	-	-	-	-	-0.1736	1.284
50	Normal (0, 1)	-1.432	1.020	-1.0175	-	-	-	-	-2.5278	0.001

N	Distribution	GI	SI_θ	PI	AI	T_T	T_L	GE	ZI	I_V
	Uniform (0, 1)	0.329	1.018	0.248	0.1102	0.192	0.304	0.227	-0.358	0.0773
	Beta (3, 3)	0.213	1.010	0.155	0.039	0.076	0.089	0.081	-0.219	0.035
	Beta (5, 5)	0.168	1.014	0.1220	0.024	0.047	0.052	0.049	-0.172	0.022
	t (3)	-1.688	1.023	0.325	-	-	-	-	-0.102	1.045
	t (5)	11.751	1.015	7.954	-	-	-	-	-0.159	1.107
	Normal (0, 1)	-3.909	1.008	-2.826	-	-	-	-	-2.070	<0.001
100	Uniform (0, 1)	0.331	1.012	0.249	0.1106	0.1927	0.305	0.228	-0.358	0.0771
	Beta (3, 3)	0.215	1.010	0.156	0.040	0.076	0.090	0.081	-0.222	0.035
	Beta (5, 5)	0.170	1.009	0.123	0.024	0.047	0.052	0.049	-0.174	0.022
	t (3)	-5.755	1.011	-3.931	-	-	-	-	-0.090	1.0197
	t (5)	-7.357	1.009	-5.066	-	-	-	-	-0.151	1.033

The results shows that all values of SI_θ are close to 1 indicating symmetric distributions. Also it is obvious that under normal and t distributions, except the proposed index, all other indices give either misleading values or no values (empty cells), since they cannot be used when data contain negative values. This problem is not encountered when dealing with the proposed index SI_θ since it does not depend on the original values of the data but it depends on its transformed form as provided in (5).

Additionally, the values of the proposed index (SI_θ) approach one as the sample size increases. See for example results of uniform distribution where, as the sample size increases from 20 to 50 to 100, the corresponding results of SI_θ decreases from 1.030 to 1.018 to 1.012, this result cannot be obtained under competitive indicators. Also it is observed that, the values of all indicators are lower when the distributions are more symmetric.

4.2. Non-symmetric Positive Skewed Distributions

4.2.1. The Characteristics of Non-symmetric Positive Skewed Distributions

In this scenario, we generate four distributions from gamma family and four distributions from Beta family, both of them were generated with skewness coefficients: 0.5, 1, 2, and 4, respectively. Besides, two distributions from log-normal distribution have been generated with parameters 0.6 and 0.8. The skewness coefficients of these distributions and their corresponding kurtosis values can be shown in the following table.

Table 3. *Generate data from positive skewed distributions.*

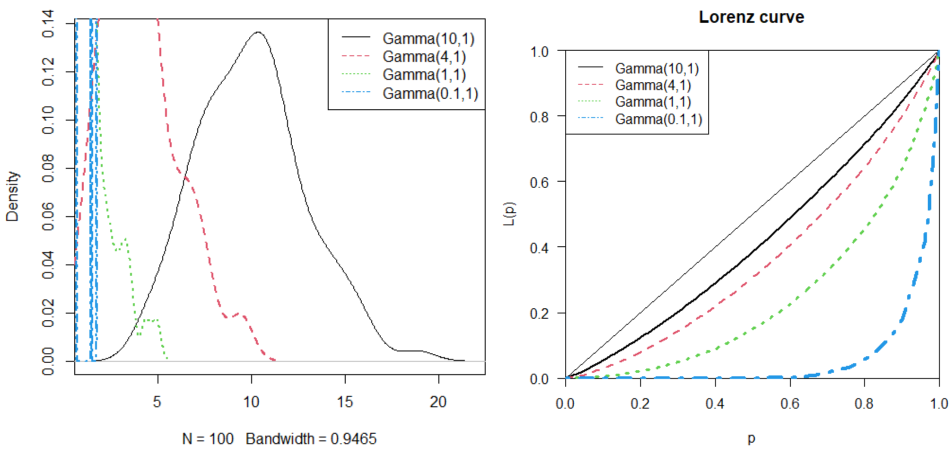
Distribution	Skew	Kurtosis
G1: G (10, 1)	0.5	0.386
G2: G (4, 1)	1	1.819
G3: G (1, 1)	2	4.058
G4: G (0.10, 1)	4	16.676
BP1: B (1, 1.5)	0.5	-0.873
BP2: B (1, 3.698)	1	1.555
BP3: B (0.5, 5.552)	2	4.784
BP4: B (0.25, 6)	4	22.925
LN (0.6)	1.242	1.229
LN (0.8)	2.306	6.725

For each family of the above distributions (Gamma, Beta and Log-normal), we provide a brief descriptions using the summary statistics in the following table.

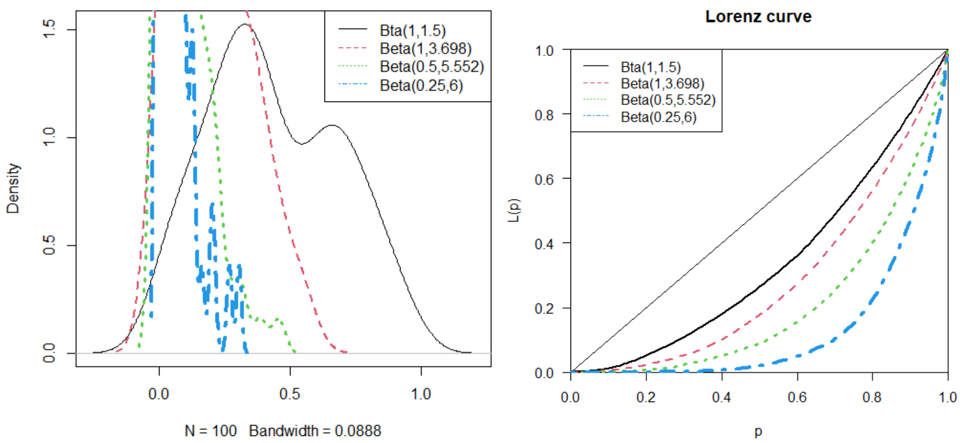
***Table 4**. The summary statistics of positive skewed distributions.*

Distribution	Min	Q1	Median	Mean	Q3	Max
G1: G (10, 1)	4.088	8.261	10.225	10.243	12.173	19.874
G2: G (4, 1)	1.139	2.720	3.681	4.025	5.051	11.351
G3: G (1, 1)	0.01418	0.34435	0.77641	1.24477	1.55669	6.51136
G4: G (0.10, 1)	<0.0001	0.0000052	0.0006252	0.1262594	0.0321569	2.3488801
BP1: B (1, 1.5)	0.01063	0.19292	0.33211	0.40671	0.62232	0.98263
BP2: B (1, 3.698)	0.002196	0.080997	0.182057	0.212075	0.298862	0.801788
BP3: B (0.5, 5.552)	<0.0001	0.0048936	0.0308123	0.0834487	0.1261889	0.5862884
BP4: B (0.25, 6)	<0.0001	0.0006983	0.0079445	0.0353128	0.0338933	0.5188197
LN (0.6)	0.1533	1.2318	2.1321	3.1098	4.3363	12.216
LN (0.8)	0.253	1.205	2.175	3.313	3.774	20.625

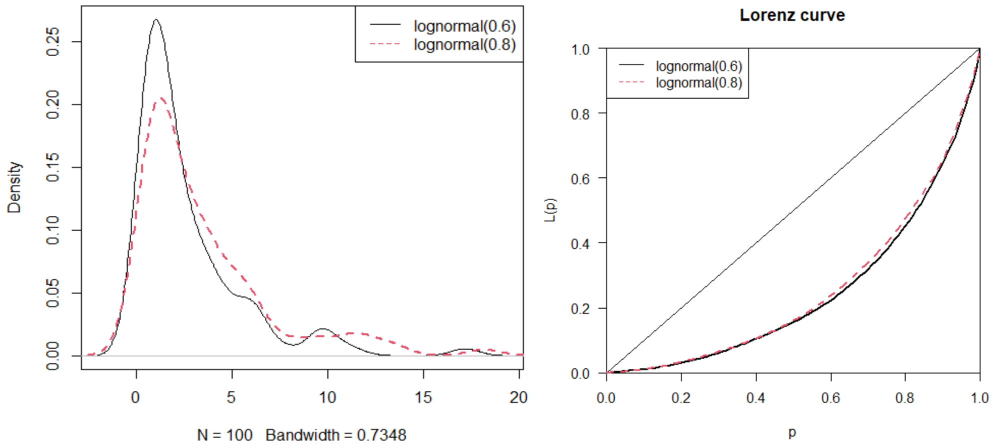
From Table 4 it is observed that under positive skewed scenario, all values of the generated distributions are non-negative which may results in obtain an acceptable Lorenz curves and accepted indicator values. In the following figure (Figure 4) the corresponding density curves and Lorenz curves of these distributions are explained.



a) Density curve and Lorenz curve of Gamma distribution



b) Density curve and Lorenz curve of Beta distribution



c) Density curves and Lorenz curves of Log-normal distributions

Figure 4*. The density curves and Lorenz curves of positive skewed distributions.*

As observed from Figure 4, all distributions (Gamma, Beta and Log-normal distributions) give an acceptable Lorenz curves since all values are non-negative. Also we found that, as skewness parameter increases, the gap between the equality line and Lorenz curve increase which indicates large degree of asymmetry. Besides, Log-normal (0.6) appear to be more skewed than Log-normal (0.8), however the intersection between them may lead to obtain the same values from equality indices.

4.2.2. The Results of Non-symmetric Positive Skewed Distributions

The results when data are generated from positive skewed distributions can be seen in the next table (Table 5).

Table 5*. The results of positive skewed distributions.*

N	Distribution	GI	SI_θ	PI	AI	T_T	T_L	GE	ZI	I_V
	G1	0.167	1.169	0.122	0.023	0.046	0.048	0.047	-0.107	0.324
	G2	0.260	1.263	0.191	0.058	0.113	0.124	0.117	-0.091	0.322
	G3	0.474	1.651	0.358	0.204	0.398	0.551	0.433	-0.171	0.309
	G4	0.839	25.969	0.736	0.719	1.657	7.885	1.898	-	0.222
20	BP1	0.361	1.191	0.273	0.129	0.231	0.353	0.267	-0.294	0.079
	BP2	0.422	1.432	0.318	0.165	0.309	0.450	0.347	-0.225	0.063
	BP3	0.582	2.090	0.453	0.323	0.6195	1.145	0.714	-0.219	0.058
	BP4	0.714	3.620	0.581	0.503	1.0168	2.672	1.190	-0.215	0.054
	LN (0.6)	0.480	1.806	0.365	0.198	0.428	0.460	0.421	-0.066	0.589
	LN (0.8)	0.480	1.817	0.365	0.198	0.428	0.460	0.420	-0.065	0.632
	G1	0.172	1.160	0.123	0.024	0.048	0.0497	0.0485	-0.0619	0.328
	G2	0.268	1.267	0.1932	0.059	0.117	0.127	0.120	-0.094	0.328
	G3	0.490	1.677	0.364	0.210	0.413	0.567	0.446	-0.151	0.323
	G4	0.865	9.341	0.748	0.740	1.785	8.018	1.972	-0.139	0.273
50	BP1	0.370	1.182	0.277	0.131	0.234	0.361	0.272	-0.297	0.079
	BP2	0.433	1.444	0.323	0.169	0.316	0.462	0.355	-0.212	0.064
	BP3	0.598	2.107	0.459	0.331	0.638	1.170	0.730	-0.197	0.060
	BP4	0.734	3.498	0.590	0.5167	1.058	2.730	1.223	-0.181	0.059

N	Distribution	GI	SI_θ	PI	AI	T_T	T_L	GE	ZI	I_V
	LN (0.6)	0.504	1.877	0.377	0.212	0.469	0.485	0.451	-0.028	0.646
	LN (0.8)	0.504	1.872	0.377	0.212	0.469	0.485	0.451	-0.028	0.688
	G1	0.174	1.161	0.124	0.024	0.049	0.050	0.049	-0.0596	0.331
	G2	0.271	1.263	0.194	0.059	0.1185	0.129	0.121	-0.090	0.331
	G3	0.495	1.686	0.366	0.212	0.417	0.572	0.450	-0.145	0.327
	G4	0.875	9.105	0.751	0.748	1.831	8.079	1.997	-0.120	0.299
	BP1	0.373	1.177	0.278	0.132	0.235	0.363	0.274	-0.296	0.079
100	BP2	0.437	1.444	0.323	0.171	0.318	0.465	0.358	-0.210	0.064
	BP3	0.603	2.116	0.461	0.334	0.643	1.178	0.736	-0.190	0.060
	BP4	0.740	3.524	0.592	0.521	1.071	2.748	1.233	-0.171	0.061
	LN (0.6)	0.480	1.906	0.379	0.217	0.485	0.493	0.461	-0.013	0.673
	LN (0.8)	0.512	1.805	0.379	0.217	0.485	0.493	0.461	-0.013	0.714

It is observed that, under positive skewed distributions, all values of SI_θ are greater than 1 that as the skewness coefficient increases, the corresponding value of SI_θ tends to ∞. The same conclusion can be obtained for the other indices, thst as the skewness coefficient increases, the values of all indices increase indicating high level of inequality/ asymmetry.

Besides it should also be noted that, except the proposed index, other indices did not give information about the direction of non-symmetric data if it is positive or negative skewed, while all values of the proposed index SI_θ exceeded 1 reflecting positive skewed distribution.

Also under log normal distributions, except the proposed index, all other indices give the same values for both LN (0.6) and LN (0.8) that they could not recognize the difference between them especially under small sample sizes.

4.3. Non-symmetric Negative Skewed Distributions

4.3.1. The Characteristics of Non-symmetric Negative Skewed Distributions

A negatively skewed distribution is characterized by more values clustering towards the right side of the graph. Negatively skewed data can have a value of zero or a negative value, indicating the extent of the negative skewness in the distribution. The human life cycle presents one of the most popular examples of negatively skewed distribution as many live the average life, some living very less, and some live a very high life in age. Accordingly, three distributions from both normal (with shape parameters -2, -4 and -6) and Beta family have been generated with different negative skewness coefficients. The skewness coefficients of these distributions and their corresponding kurtosis values can be shown in the following table:

Table 6. *Generate data from negative skewed distributions.*

Distribution	Skew	Kurtosis
NN1 (-2)	-0.09127	-0.3640731
NN2 (-4)	-0.1415391	0.3806735
NN3 (-6)	-1.296241	2.479258
BN1: B (5, 2)	-0.6578	-0.2036
BN2: B (5, 0.7)	-1.5377	2.151608
BN3: B (5, 0.1)	-2.006	4.488146

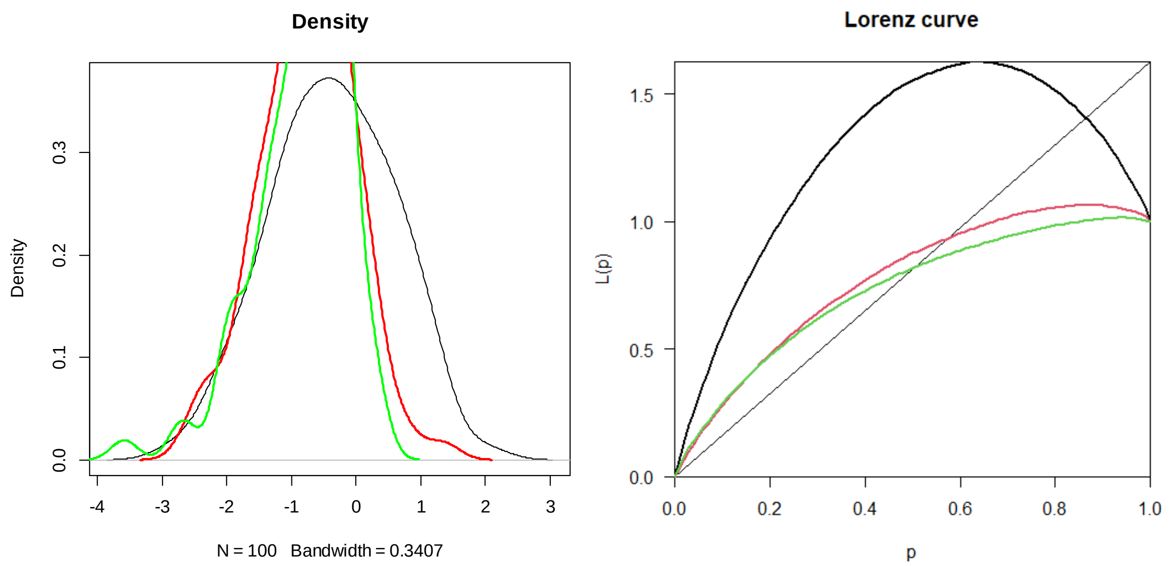
For each family of the above distributions, we provide brief descriptions using the summary statistics in the following table.

Table 7. *The summary statistics of negative skewed distributions.*

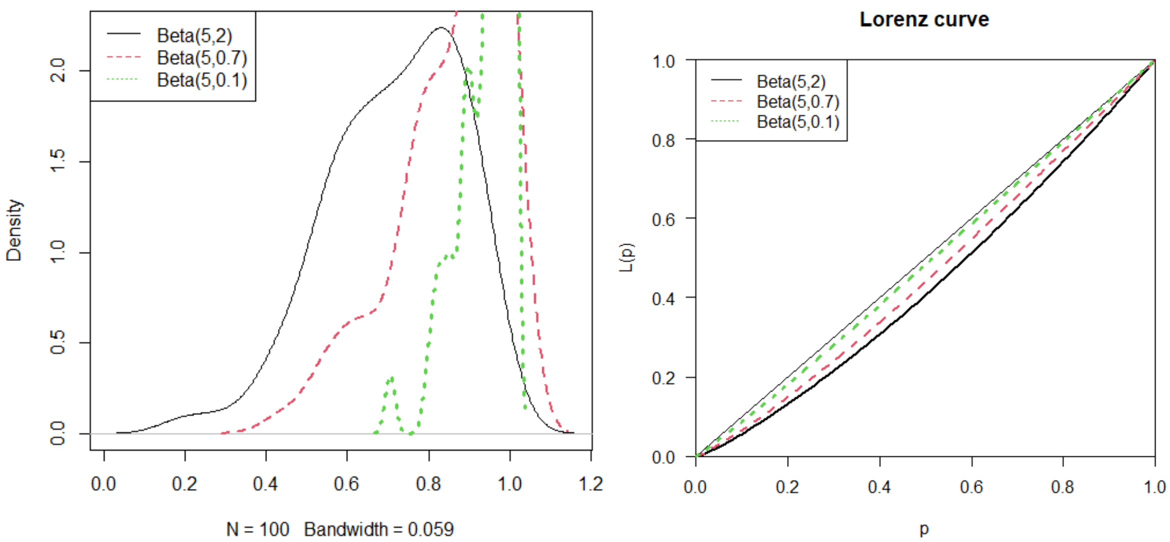
Distribution	Min	Q1	Median	Mean	Q3	Max
NN1 (-2)	-2.8180	-1.0398	-0.2851	-0.3662	0.3724	2.0154

Distribution	Min	Q1	Median	Mean	Q3	Max
NN2 (-4)	-2.5792	-1.1782	-0.6282	-0.7375	-0.2458	1.3572
NN3 (-6)	-3.5821	-1.130	-0.6185	-0.7874	-0.3515	0.3435
BN1: B (5, 2)	0.3665	0.6562	0.7786	0.7507	0.8545	0.9674
BN2: B (5, 0.7)	0.3507	0.8028	0.9082	0.8594	0.9691	0.9998
BN3: B (5, 0.1)	0.3941	0.8907	0.9618	0.9135	0.9930	1.0000

From Table 7 it is observed that under the cases of negative skewed normal distributions the data contain negative values (which indicate unacceptable Lorenz curves), however under Beta cases, although the distributions are negatively distributed, their data don't include negative values which indicate acceptable Lorenz curves. The corresponding density curves and Lorenz curves of these distributions are explained in the following figure (Figure 5).



a) Density curve and Lorenz curve of negative skewed normal distribution



b) Density curve and Lorenz curve of negative skewed Beta distribution

Figure 5. *The density curves and Lorenz curves of negative skewed distributions.*

From Figure 5, as expected, unacceptable Lorenz curves when data are generated from negative skewed normal distributions can be observed, however under Beta cases, although the distributions are negatively distributed, their data have an accepted Lorenz curves. Also under normal cases it is obvious that, the normal with shape parameter (-6) is the most skewed distribution of them, however under Beta cases we found that, as skewness parameter increases, the gap between the equality line and Lorenz curve decrease which giving misleading indications.

4.3.2. The Results of Non-symmetric Negative Skewed Distributions

The results when data are generated from negative skewed distributions can be seen in the next table (Table 8). As mentioned in symmetric case under normality distributions, except the proposed index, the other indices give misleading values since they are not applicable under negative values. This problem could not be faced when we use SI_θ since it does not depend on the original values of the data but it depends on its transformed form (see (5)). Also it is observed that, under negative skewed distributions, all values of SI_θ are less than 1, that as the skewness coefficient increases, the corresponding value of SI_θ tends to 0. This note did not observe for other indices where their values are closer to zero (reflecting symmetry) as the negative skewness parameter increases, indicating misleading information.

Additionally it is observed that as the sample size increases, the efficiency of the proposed index increases. See for example the results under BN1, when the sample size increases from 20 to 50 to 100, the corresponding results of SI_θ decreases from 0.876 to 0.852 to 0.840 (tends to zero) indicating negative skewed distribution, this conclusion is also obtained under competitive indicators.

Table 8. The results of negative skewed distributions.

N	Distribution	GI	SI_θ	PI	AI	T_T	T_L	GE	ZI	I_V
20	NN1	1.940	1.022	-1.422	-	-	-	-	-0.219	0.0299
	NN2	-0.543	0.942	-0.394	-	-	-	-	-0.219	-0.625
	NN3	-0.438	0.866	-0.322	-	-	-	-	-0.219	-0.372
	BN1	0.120	0.876	0.089	0.0135	0.025	0.0288	0.0271	-0.203	0.0176
	BN2	0.0715	0.627	0.0546	0.0058	0.011	0.0126	0.0118	-0.4193	0.009
	BN3	0.058	0.531	0.0452	0.0043	0.0083	0.0093	0.0087	-0.5188	0.007
50	NN1	-1.505	1.009	-1.075	-	-	-	-	-0.211	-13.93
	NN2	-0.546	0.922	-0.391	-	-	-	-	-0.210	-0.557
	NN3	-0.446	0.844	-0.323	-	-	-	-	-0.210	-0.357
	BN1	0.1236	0.852	.0903	0.0139	0.0265	0.0296	0.0278	-0.2721	0.0176
	BN2	0.0733	0.595	0.0553	0.006	0.0114	0.0128	0.0121	-0.4595	0.0091
	BN3	0.059	0.500	0.0457	0.004	0.008	0.0094	0.009	-0.517	0.0069
100	NN1	-1.595	1.003	-1.134	-	-	-	-	-0.207	6.492
	NN2	-0.547	0.914	-0.389	-	-	-	-	-0.208	-0.539
	NN3	-0.450	0.835	-0.324	-	-	-	-	-0.207	-0.354
	BN1	0.1248	0.840	0.0907	0.014	0.0267	0.0299	0.0281	-0.268	0.0175
	BN2	0.0738	0.582	0.055	0.006	0.012	0.0129	0.0121	-0.451	0.0091
	BN3	0.059	0.4882	0.0458	0.005	0.0084	0.0096	0.0089	-0.527	0.0069

5. Empirical Examples

5.1. Example 1

As we refer before, GI has some mathematical limitations as well. For example, when the total income of two populations is the same, in certain situations two countries with different income distributions can have the same GI especially these cases when income Lorenz Curves cross (see [6]). The following table (Table 9) illustrates one such situation.

Table 9*. Different income distributions with the same Gini index.*

Individuals	**Income distribution of city (A)**	**Income distribution of city (B)**
n.1	2427	4417
n.2	7800	5400
n.3	8489	6500
n.4	10072	10072
n.5	12957	15346
Total	41735	41735
Gini index (GI)	0.2	0.2
Proposed index (SI_θ)	0.5	1.4

As observed from this table, although both countries have a GI of 0.2, the income distributions for household groups are different since the Lorenz curves of the two data are intersected (see Figure 6). However, SI_θ can distinguish this difference between them since city (A) has $SI_\theta = 0.5$, whereas city (B) has $SI_\theta = 1.4$. Thus, by using the proposed index we can say that, city (A) appear to have a bit higher level of inequality than city (B), which cannot be explained by the GI. Moreover, by using SI_θ we can detect the direction of asymmetry for both data as follow: since city (A) has $SI_\theta = 0.5$, which is less than 1, then city (A) is said to have a negative skewed distribution. In contrast of city (B), where $SI_\theta = 1.4$, which is greater than 1, then city (B) is said to have a positive skewed distribution (this conclusion can also be conducted in the following figure (Figure 6)).

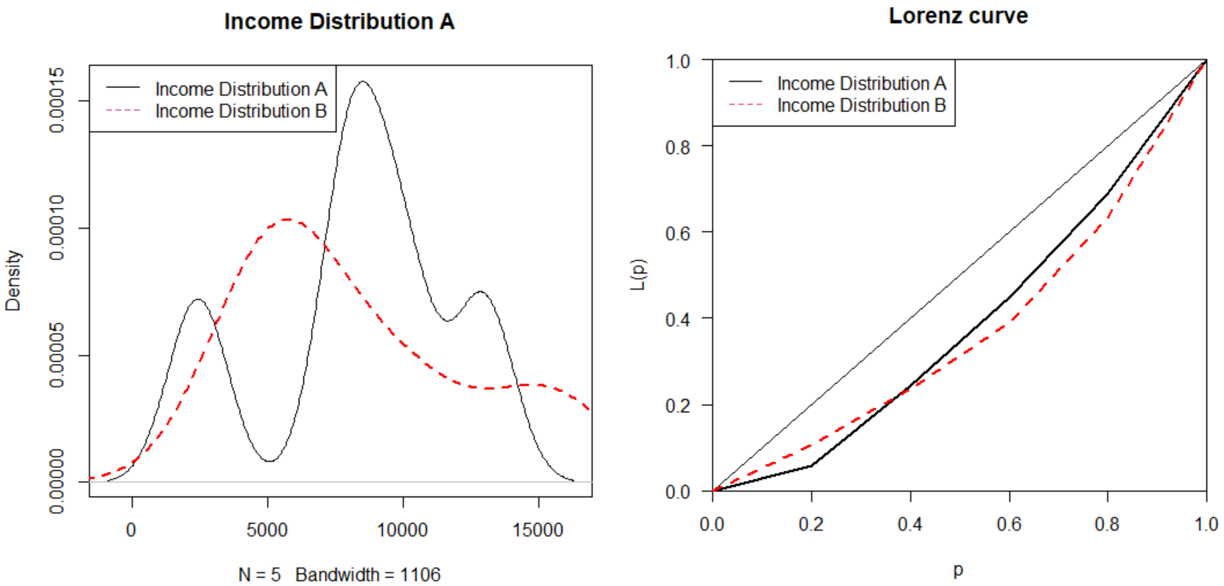


Figure 6*. Density and Lorenz curves of different income distributions with the same GI.*

5.2. Example 2

In education, engineering and management sciences studies, it is important to check if the underlying distribution has a particular form. In this sub-section, we present another example based on real data in education area to show the behavior of the proposed index in real cases. The Gini education index measures the distribution of years of schooling across a population, where a score of zero represents perfect equality and a score of one signifies maximum inequality. This index allows for the comparison of educational inequality across different groups and over time.

In this example, we aim to measure educational inequality across different sections. The dataset consists of 433 students from 14 sections in the Faculty of Social Work at Helwan University. We examine two time points: the first when the students were in their first year (2015) of undergraduate study, and again two years later in their third year (2017). The analysis includes the students' scores from the statistics module at both time points, along with their gender. To demonstrate the advantages of our proposed index, we compare two distributions shown in Figure 7, based on the statistics module scores from section 1 and section 6 in the first year (2015).

We chose these sections because when measured by the Gini coefficient they seem to exhibit the same level of inequality that is a Gini of approximately (0.111) (see Table 10). However, when considering the Lorenz curve of their distributions, it becomes evident that the distribution of scores differs between them.

In addition, we provide a visual description for the distributions of 14 sections in both first and third years using boxplots (see Figure 8).

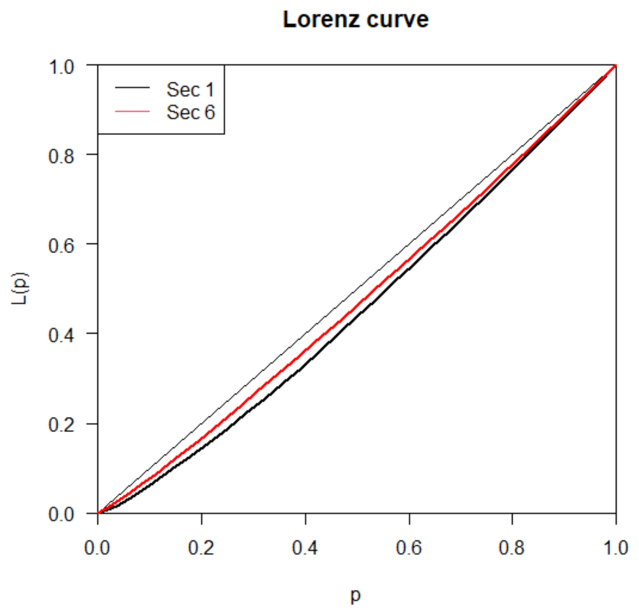


Figure 7. *Plotting the Lorenz curve for distributions of the scores in the first year* (2015) *from section (1) and section (6).*

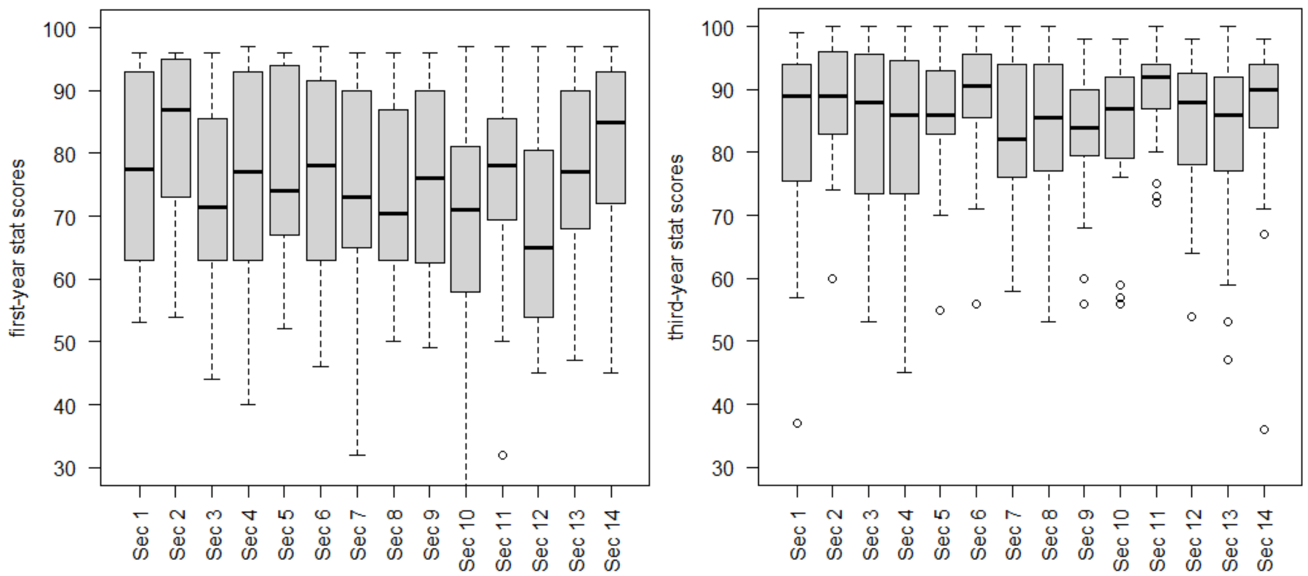


Figure 8. *The boxplot shows the students' test scores from their first and third years.*

It is immediately apparent that, on average, the students' test scores in their third year are higher than in their first year, with scores typically being greater than or equal to eighty. Sections 6, 11, and 14 seem to have the highest average scores. Additionally, it is clear that the score distributions differ across all sections. There is also considerable variation in individual scores, with some students scoring lower than their section’s average, indicating the presence of outliers in the data.

Next, we calculate the values of the proposed index and all other competitive indices under this study for the scores of statistics in both first and third years. The results are obtained and provided in Table 10. The results show that the proposed index (SI_θ) is capable of distinguishing score inequality in cases where two or more sections have the same Gini index but differ in the score gap between the top and bottom of their Lorenz curves. For example, in the first year, sections 1 and 6, as well as sections 4 and 8, share the same level of inequality when measured by the Gini index. Similarly, in the third year, sections 13 and 14 exhibit the same level of inequality according to the Gini index.

However, our symmetry index (SI_θ) can tell this difference since section (1) has $SI_\theta = 0.830$, whereas section (6) has $SI_\theta = 0.944$. Thus, by using the proposed index, we can say that section (1) has a higher level of inequality than section (6), which cannot be explained by the Gini index. This conclusion can also be concluded for section (4) versus section (8) and for section (13) versus section (14).

Table 10. *The results of the proposed index and other competitive indices for the scores of statistics for both first and third years.*

Year	Section	GI	SI_θ	PI	AI	T_T	T_L	GE	ZI	I_V
1st year	1	0.111	0.830	0.088	0.010	0.019	0.019	0.019	-0.248	0.602
	2	0.092	0.586	0.072	0.008	0.015	0.016	0.015	-0.408	0.547
	3	0.134	0.596	0.093	0.021	0.037	0.048	0.041	-0.307	0.699
	4	0.124	0.715	0.096	0.013	0.025	0.026	0.020	-0.177	0.651
	5	0.103	0.705	0.081	0.009	0.017	0.017	0.017	-0.082	0.572
	6	0.111	0.944	0.084	0.010	0.020	0.020	0.020	-0.194	0.598
	7	0.121	0.466	0.089	0.013	0.025	0.028	0.026	-0.131	0.649
	8	0.124	0.625	0.088	0.017	0.031	0.039	0.034	-0.147	0.667
	9	0.113	0.851	0.084	0.010	0.020	0.020	0.020	-0.095	0.599
	10	0.136	1.054	0.098	0.017	0.032	0.035	0.033	-0.076	0.686
	11	0.101	0.589	0.069	0.011	0.020	0.024	0.022	-0.327	0.590
	12	0.141	0.810	0.108	0.015	0.030	0.031	0.030	0.056	0.684
	13	0.107	0.798	0.078	0.010	0.019	0.020	0.019	-0.204	0.584
	14	0.096	0.471	0.074	0.009	0.017	0.018	0.017	-0.475	0.565
3rd year	1	0.089	0.834	0.068	0.009	0.016	0.019	0.017	-0.380	0.559
	2	0.057	0.560	0.041	0.003	0.005	0.006	0.006	-0.341	0.334
	3	0.084	0.689	0.065	0.006	0.012	0.013	0.013	-0.303	0.505
	4	0.087	0.769	0.065	0.006	0.013	0.014	0.013	-0.316	0.509
	5	0.059	0.624	0.041	0.003	0.007	0.007	0.007	-0.286	0.364
	6	0.054	0.671	0.038	0.003	0.006	0.006	0.006	-0.291	0.330
	7	0.099	0.487	0.067	0.020	0.031	0.054	0.039	-0.378	0.652
	8	0.077	0.759	0.054	0.005	0.010	0.011	0.011	-0.183	0.458
	9	0.058	0.679	0.040	0.003	0.006	0.007	0.006	-0.304	0.335
	10	0.079	0.743	0.058	0.006	0.012	0.013	0.012	-0.404	0.487
	11	0.047	0.933	0.035	0.002	0.004	0.004	0.004	-0.253	0.261
	12	0.074	1.145	0.056	0.005	0.010	0.010	0.010	-0.361	0.439
	13	0.077	0.893	0.056	0.006	0.012	0.013	0.012	-0.258	0.480
	14	0.077	0.437	0.052	0.006	0.013	0.015	0.014	-0.435	0.495

In addition, the proposed index could be able to distinguish scores inequality of two or more sections that have the same PI, AI, T_T, T_L and GE indices but have different values of the Gini index. See for example, section (6) versus section (9) in the first year, that demonstrate identical levels of inequality if measured by the PI (0.084), AI (0.010), T_T (0.020), T_L (0.020) and GE (0.020), however SI_θ can tell this difference since section (6) has $SI_\theta = 0.944$, whereas section (9) has $SI_\theta = 0.851$ which means that section (9) displays a greater degree of inequality than section (6). Furthermore, the results in the previous table also show that the rankings of scores inequality among sections have been changed when comparing them using our proposed symmetry index in contrast to the Gini index.

6. Conclusion

In this paper, our proposed symmetry index is provided via a new combination between the concept of symmetry test (based on the symmetry test proposed by [3]) and the mathematical definition of Gini index. The proposed index seems to be a natural choice for evaluating both the equality and the symmetry of the underlying data since it can differentiate inequality in cases where two or more groups (populations) have the same Gini index but differ in the gap between the top and bottom of the Lorenz curve. Besides, it gives information about the distribution of asymmetry data (positive or negative skewed) which cannot be explained by the Gini index. Unlike other indices, the use of the proposed index is not limited to positive data only, as it can be used when data contains some negative values (see section 4). As a future work, we aim to use the proposed index to analyze income data from Egypt.

Abbreviations

GI	Gini Index
EDE	An Equivalent Level of Equal Distribution
ε	The Parameter of Inequality Degree
GE (α)	The Class of Generalized Entropy Indices
GMD	Gini Mean Difference
SI_θ	The Proposed Index
PI	Pietra Index
T_T and T_L	Theil Indices
AI	Atkinson Index
ZI	Zanardi Index
I_V	Vertical-diameter Inequality Index

Conflicts of Interest

The authors declare no conflicts of interest.

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