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Stratification and Related Ideas for Understanding the Smoothness of Black Hole Horizons

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Abstract: In recent work by the author it was found that there is an absence of diffeomorphism symmetry for black hole horizons constrained by the causal structure of spacetime. This absence puts stringent constraints on the black hole spacetime manifold, topology, and smoothness. In particular, it suggests that such manifolds may have only continuity of data but no differentiability. In mathematics, this is described as a spacetime manifold with a C^0 structure, and the assumption of smoothness as a C^∞ structure might not hold. Previously, we supposed that the spacetime manifold may have a Finsler structure and might not be a homogeneous manifold, thus requiring new insights to study black hole spacetime. Investigating the spacetime manifold picture, we presume a mathematical concept called stratification of spacetime with smooth gluing of manifold data as an essential criterion for understanding the topology of black hole spacetime. In basic terms, stratification is a way of gluing spacetime into a collection of disjoint regions called strata such that the strata themselves are smooth, but the whole manifold may or may not have a differentiable structure. The topology of the spacetime manifold then depends on the criteria used for the process called stratification of spacetime. We investigate these and relatable ideas further in this paper. For smooth readability, we have referenced relevant literature grouped by topics or subtopics throughout the text.

Keywords: Black Holes, Stratified Geometry, Fiber Bundles, Null Hypersurfaces, Horizon Smoothness, Causal Structure

1. Introduction

The smoothness of black hole horizons is a major topic of investigation in the black hole physics, quantum field theory, and quantum gravity communities. Almost one and a half decades ago, due to work by Professor Sameer Mathur [1] and subsequently by the authors AMPS [2] and AMPSS [3], it was found that assuming semiclassical gravity for black holes, unitarity and locality imply a contradiction. These questions have been investigated further in the entanglement wedges and island paradigm for black holes. Interestingly, it was found in [4] that black hole spacetime manifolds do not contain diffeomorphism symmetry when considering diffeomorphisms between timelike and null regions.

Since diffeomorphisms of spacetime are assumed to imply that the given manifold is smooth and differentiable, in the absence of this symmetry one needs to rely on more primitive mathematical structures to study the given black hole spacetime manifold [81–84]. Though general relativity has many successful predictions, the author thinks that the mathematical structure of the theory lacks a first-principles understanding [5–8, 120, 124]. Some of these questions are:

1. What is the topology and differential structure of black hole spacetime? Relying only on the line element ds^2 for investigating geometry seems pathological when considering a freely falling observer from timelike to null regions. In the author's viewpoint, the pathological singularity of black hole spacetime at the null hypersurface itself suggests a breakdown of GR [9–11]. Also, unlike in Riemannian geometry, here the line element changes its nature while going from timelike to null regions. Since the line element is zero at null hypersurfaces, we cannot rely on it for investigating the geometry of null hypersurfaces and

hence require additional principles to address it [12–14]. The absence of diffeomorphism symmetry implies that the smoothness of black hole spacetime can at best be of C^0 nature, unlike the prescribed assumption of C^∞ structure [15, 16].

2. Another question is why the Einstein equation is written as a rank-2 tensor equation. A rank-two tensor equation assumes that black hole spacetime is a homogeneous Riemannian manifold; then Einstein equations are mapped from the product of two tangent bundles $TM(P) \times TM(P) \to \mathbb{R}$ (assuming the tangent bundle for black hole spacetime admits a vector space structure) [34–37]. Thus, unlike electrodynamics where Maxwell's equations are mappings from vector space to itself, Einstein's equations of general relativity rely not on the tangent bundle but on a tensor bundle of rank 2. One plausible reason the author presumes is that since Einstein equations are prescribed on a manifold with curvature, and since curvature at a point depends on the holonomy captured at that point, if one tries to study holonomy using vectors, then we may not be able to do so because capturing the holonomy of a manifold requires prescribing it via Lie derivatives $\mathcal{L}_v$, which are commutators of two vector fields; hence the simplest space carrying such commutators is a map given from $V \times V \to V$, where V is the vector space of the associated manifold [38–40]. The question then arises as to why curvature is described using Lie derivatives. This is because curvature measures non-commutativity of orientations along the flow of vector fields; a single vector cannot capture such non-commutativity because the commutator of a single field with itself is zero.
3. Because the mathematical framework of Einstein gravity is not firmly established, it hinders capturing geometric properties of such spaces, such as topological invariants like homology, cohomology, and fundamental group. Since the manifold has curvature, one needs additional geometrical data from topology and differential geometry to study such manifolds [121–124].
4. The geometric intuition of the interplay of spacetime and matter in Einstein gravity is a good physically motivated guiding principle for studying Einstein's equations, but we do not know more basic geometrical or fundamental principles through which one can evidently infer Einstein's equations.[1] A reader does not get the intuition as to why and from where Einstein's equations emerge as solutions to equations of the spacetime manifold [17, 18, 115]. This also hinders a well-motivated understanding to pursue a framework that is a suitable generalization of Einstein gravity in scenarios where Einstein's equations break down. Thus, a first-principles pathway to understanding the emergence of Einstein's equations from a deeper principle leads to a lack of a good framework to address modified gravity. Unfortunately, rather than relying on geometric ideas, modified gravity seems to be a territory of experimentation rather than principles.

Fortunately, mathematicians have developed deep geometric frameworks, and the author believes these territories are of prime necessity in understanding the nature of black holes and gravity. The author believes that tools and techniques of algebraic topology, differential geometry, and differential topology are of utmost necessity in investigating deeper and more fundamental truths of nature and in establishing general relativity on firm mathematical grounds. Our prime motivation has been investigating the problem of what happens to an infalling observer along the horizon: does the observer experience fuzzballs, firewalls, islands, or a Cauchy horizon, addressing it by relying on more basic mathematical rules. The author motivates the introduction of sheaves and stratified spaces for understanding the problem of describing an infalling observer along the null horizon [23–27]. The motivation has been to study the problem from the most basic mathematical tools, allowing for generality and making the problem mathematically tractable. The author is also interested in knowing if black hole spacetime admits the structure of homogeneous spaces so that the underlying manifold can be investigated using isometries of Lie groups [72–75].

2. A Thought Experiment for Minkowski Spacetime

The thought experiment presented here may have relevance for black hole spacetimes. Suppose we consider a light cone in Minkowski spacetime and diffeomorphisms between surfaces with values of the line element $ds_A^2 > 0$ (A is timelike) and $ds_B^2 = 0$ (B is null). We label a point in A as χ and a point in B as η. Then such a diffeomorphism labeled as D is of the form:

$$\eta = D\chi. \tag{1}$$

Also, because the inner product of the element satisfies

$$ds^2|_B = n^T g|_B n = 0, \tag{2}$$

we have from (1) and (2) that [2]

$$D^T g D = 0. \tag{3}$$

Here g is the induced metric along the null hypersurface. Assuming the metric g is non-zero and taking the determinant, we find

$$\det D^T \det D = 0 \Rightarrow \det D = 0. \tag{4}$$

[1] In a previous paper by the author [4], it was found that the Ricci scalar for null hypersurfaces computed using Gauss-Codazzi equations was non-zero, hinting towards a more generic framework for understanding classical gravity.

[2] In previous paper there was a mistake by author in not considering metric for inner product this has been remedied here. The notion of Superspace arrived previously is therefore also incorrect.

Hence the matrix D is non-invertible. Also, suppose we consider the line element along timelike region A with metric g as

$$\chi^T g \chi = c, \tag{5}$$

where[3] c is some numerical constant. Using a new diffeomorphism from region $B \to A$ via G, we have:

$$\chi = G\eta. \tag{6}$$

Substituting (6) into (5), we find:

$$\eta^T G^T g G \eta = c. \tag{7}$$

Now using (1) in (7), we get:

$$\chi^T D^T G^T g G D \chi = c = \chi^T g \chi. \tag{8}$$

Matching terms, we find:

$$D^T G^T g G D = g. \tag{9}$$

By supposing $GD = B$, we get:

$$B^T g B = g. \tag{10}$$

Using (10) and applying the determinant on both sides, we find:

$$\det B = \pm 1. \tag{11}$$

Thus B is a nice group with determinant ± 1; in fact, using the equations listed earlier. For simplicity, we take $\det B = +1$. Then we find:

$$\det(GD) = +1. \tag{12}$$

This shows that combining different phenomena from region $A \to B$ along with some other diffeomorphism from $B \to A$ is a smooth action with determinant ± 1. Thus, the combined operation of mapping points from $A \to B$ and a diffeomorphism from $B \to A$ can be assumed as a smooth Lie group action.[4] This suggests that given the symmetry action along $A \to B$, one can find a group action from $B \to A$. However, we can notice that:

$$\det G = \frac{1}{\det D} = \infty.$$

Hence, the diffeomorphism group from $B \to A$ has infinity as the product of eigenvalues. There can be many interpretations of this:

1. Recovering information from B about A has infinite complexity.
2. The mapping group G is an infinite-dimensional group since D is found to be a smooth group with zero determinant; hence G is a smooth group in infinite dimensions.
3. The two combined morphism act as B if manifold admits a homogenous space structure so that techniques from Lie Theory can be applied to it.

Given that different maps from $A \to B$ and $B \to A$ are not inverses of each other, i.e.,

$$G = BD^{-1} \to \infty, \tag{13}$$

this suggests that though D as a mapping is non-invertible, hence it is not possible to go from $B \to A$, information about A can be recovered at B using infinite-dimensional groups as resources for information recovery.

2.1. Proceedings & Initiation

From here we see that studying diffeomorphisms from region $B \to A$ requires studying infinite-dimensional groups[5] Since the action of $GD = B$ with $\det B = +1$ is smooth, we presume that the null hypersurface itself is a smooth manifold.[6]

To justify the argument, suppose we consider Minkowski spacetime with coordinates x_0, x_1, x_2, x_3. We find:

$$x_0^2 - \sum_{i=1}^{3} x_i^2 = c. \tag{14}$$

Similarly, for the null hypersurface:

$$x_0^2 - \sum_{i=1}^{3} x_i^2 = 0. \tag{15}$$

Thus, in Minkowski spacetime, the region labeled A is smooth, as is region B. However, transitioning from $A \to B$ requires infinite acceleration; hence information propagating from $A \to B$ forms a black hole, thereby restricting smoothness along traversability from $A \to B$. Suppose we label the region in the black hole interior as C. Then we presume that as submanifolds, A, B, and C are smooth manifolds and are joined via the process of stratification [28–31].

2.2. The Smoothness Across the Null Horizon

In Minkowski spacetime, regions $ds^2 > 0$, $ds^2 = 0$, and $ds^2 < 0$ have different topologies[7]. Hence, we expect the same in black hole spacetime: the three strata contain different topologies [32, 33]. This is an indication that the process of finding a foliation, tubular or smooth horizon requires investigating the stratification process for the three regions. However, there is a word of caution: since the BH horizon forms a trapped surface with singular extrinsic curvature, it suggests a breakdown of the null region satisfying Einstein's equations. This also follows from the Gauss-

[3] c is not the speed of light.
[4] We need to assume that the manifold is a Homogenous space to proceed with it.
[5] since the map is infinite dimensional, we do not know if we can assume an invertible group structure in this scenario.
[6] We give additional arguments and suggestions for this later in the paper.
[7] Please refer to Section 5.

Codazzi equation[8] linking null and timelike regions. We assume the 3-Ricci scalar along the null hypersurface as ${}^3R$. Then:

$$ {}^3R + k_{ab}k^{ab} - k^2 = {}^4R,^9 \tag{16}$$

using $k_{ab}k^{ab} = |\nabla n|^2$ and $k^2 = (\operatorname{div} n)^2$ as per [4], we find:

$$ {}^3R = -|\nabla n|^2 + (\operatorname{div} n)^2. \tag{17}$$

Here n is normal to the upper null hypersurface. Thus,

$$ n_\alpha = \partial_\alpha(n^T gn). \tag{18}$$

Using (15-17), we find:

$$ {}^3R = -\partial^\alpha(n^T gn)\partial_\alpha(n^T gn) + (\partial_\alpha\partial^\alpha(n^T gn))^2. $$

In general, it is non-zero, suggesting a breakdown of vacuum Einstein equations. Using this, we notice that R of the null hypersurface is smooth, suggesting that the null hypersurface can be assumed to be a smooth manifold.[10] Also, because the maps $D_{A\to B}$ and $G_{B\to A}$ are different, this suggests that under regularization of singularities, the geometry encountered by an infalling observer is a Finsler geometry [19–22].

1. As ${}^3R$ is non-zero but continuous, it suggests that diffeomorphisms of infinite-dimensional Lie groups, groupoids, or topological semigroups are the arena for studying topology and geometry encountered by infalling observers [76–78].
2. The transition from $D_{A\to B}$ is degenerate, and $G_{B\to A}$ is infinite-dimensional; thus, this is an obstruction[11] to realizing a bundle structure for the two different mappings, with $D_{A\to B}$ being the projection to the base manifold, which here is the null hypersurface. Similarly, the map from the null hypersurface to timelike region A can be defined as sections along A, and the sections $G_{A\to B}$ here are infinite-dimensional groups, groupoids, or just topological spaces.

Considering the strata A, B, and C, we can hypothesise that the three spaces form a principal G-bundle.[12] Here G is supposed to be the action of a group or groupoid with an invertible structure. Assuming d is number of spatial dimensions of manifold we see, Since $B \in SL(D, \mathbb{R}^c)$, we can study sections as left action of infinite-dimensional groups or semigroups [79, 80]. We need to note that the group B is not necessarily commutative or Abelian because:

$$ G \circ D = B \neq D \circ G. $$

Outlining these arguments, we propose that strata A, B, and C preserve the bundle structure of spacetime manifold M, and the orbit action of the groups is not necessarily abelian. Whether or not B and C are homogeneous spaces is a question for future investigations. If true, the problem can be crafted as maps between homogeneous spaces might be useful in preserving smoothness across the strata. Whether or not the three spaces have the structure of a homogeneous space is under investigation. Please refer to Prof. Terence Tao's book referenced at the end [76].

2.3. Assumptions

1. We assume that the maps linking $A \to B$ preserve the presheaf structures assigned along the strata, and they also abide by uniqueness and glueability criteria for being sheaves [41–45].
2. The natural arena for studying sheaf-theoretic strata is Finsler geometry.
3. Sections or diffeomorphisms from $B \to A$ are infinite-dimensional groups, semigroups, or groupoids.
4. We do not know with certainty if maps from $B \to A$ are homogeneous spaces; however, we conjecture that in the strata there exist nice foliations correctly subdividing submanifolds A, B, and C.

Whether the transverse foliations linking the strata are smooth or not is a question of investigation because:

Giving a sheaf structure is a mathematically rich but limiting criterion, requiring additional structure to give the surfaces topology and geometry [46–49].

These minimal assumptions are our primary criteria for proceeding further.

3. Maps Between Three Strata and Some Interpretation

As previously, we consider maps between three strata A, B, and C. For notational simplicity, we label the metric along B as g, along A as g', and along C as g''. We label vectors along A as χ, along B as η, and along C as α. We label the action of T between D from $A \to B$, and G from B to A and map $T : \chi \to n$. Thus we have the following equations:

$$ \eta = D\chi, \tag{19}$$
$$ \alpha = T\eta, \tag{20}$$
$$ \chi^T g'\chi = C > 0, \tag{21}$$
$$ \alpha^T g''\alpha = -\beta \quad (\beta > 0), \tag{22}$$
$$ \eta^T g\eta = 0. \tag{23}$$

Using (18) and (19), we can write:

$$ \alpha = TD\chi. \tag{24}$$

[8] One can refer to *Relativistic Toolkit* by Prof. Eric Poisson.
[9] Please refer to [4].
[10] We cant say it with certainty.
[11] Please refer to literature on Charactersitic classes and obstructions to bundle structures.
[12] Its justification is a question of investigation for us for now.

Implementing (23) into (21), we find:

$$(TD\chi)^T g'' TD\chi = -\beta < 0. \tag{25}$$

Thus,

$$\chi^T D^T T^T g'' TD\chi = -\beta. \tag{26}$$

Using (18) and (22), we find:

$$\chi^T D^T g D\chi = 0. \tag{27}$$

We have (20), (21), and (22) as equations for the three strata. Taking the determinant in (20), we find:

$$\det(\chi^T g' \chi) > 0. \tag{28}$$

Since $\det(AB) = \det A \det B$, we get:

$$\left|(\det \chi)^2 \det g'\right| > 0. \tag{29}$$

Similarly, for (24) we find:

$$(\det(DT))^2(\det g'') < 0, \tag{30}$$

and from (26) we find:

$$(\det D)^2 \det g = 0. \tag{31}$$

Assuming $\det(g) \neq 0$, we find:

$$\det D = 0. \tag{32}$$

From (29) we find:

$$\det(DT)^2 \det g'' < 0 \quad \Rightarrow \quad \det g'' < 0. \tag{33}$$

Thus we have the following scenario: From (31) and (32), we find $\det T \to \infty$. Thus, the pushforward map G from $B \to A$ is analogous to the pullback map T from B to C. Equation (32) implies $\det g'' < 0$. Thus, we can say there is a reversal in orientation as we traverse from $A \to C$. We see that information cannot propagate from $B \to A$ because $\det G \to \infty$, and also information cannot propagate from $B \to C$ because $\det T \to \infty$. Hence, strata A and C are identical with reference to the causal structure of spacetime but with orientation reversed. Since information cannot propagate from B to A or from B to C, the information that falls across the horizon does not seem to propagate to states A and C-perhaps localized at the horizon. This is a version of holography of information localized along the null hypersurface. Eventually, by relaxing this setup using Finsler geometry, surgery theory, and stratification, we hope to convey a better picture.

4. The Bundle Structure of Spacetime

[13]We have three strata: A along timelike, B along null direction, and C being spacelike. With associated maps between the strata, assuming the null hypersurface as the base manifold and A and C as different spaces, they can be represented pictorially as a principal G-bundle [55]. We label these maps in Figure 1.

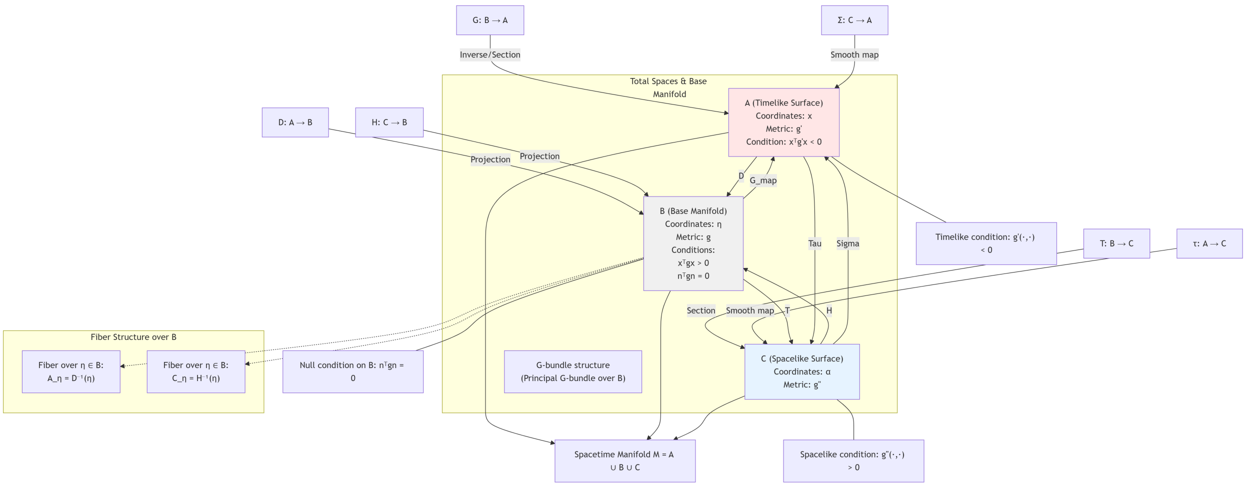

Figure 1. *Fibre bundle for the three strata.*

Please note that in the diagram, the spacelike interval is positive and the timelike interval has a negative line element; this is opposite to the convention used elsewhere.

[13]One can refer to Wikipedia articles on Fibre Bundle and Principal G-bundle [34, 55, 56] or Prof. Dale Husemoller's book on Fibre Bundles [54] for reviewing bundle theory. It is used very mildly here.

4.1. Fibre Bundle Structure with Three Strata: Timelike, Spacelike, and Null Surfaces

The following maps are defined between the strata:

1. $D : A \to B$ (projection from timelike to base)
2. $G : B \to A$ (section/inverse map)
3. $H : C \to B$ (projection from spacelike to base)
4. $T : B \to C$ (section/inverse map)
5. $\tau : A \to C$ (map from timelike to spacelike)
6. $\sigma : C \to A$ (map from spacelike to timelike)

4.1.1. Composition Relations

$$G \circ D = I_\chi = \sigma \circ \tau, \tag{34}$$
$$T \circ H = I_\alpha = \tau \circ \sigma, \tag{35}$$
$$D \circ G = I_\eta = H \circ T. \tag{36}$$

4.1.2. Mapping Relations

$$H = D \circ \sigma, \tag{37}$$
$$G = \sigma \circ T, \tag{38}$$
$$\tau = T \circ D, \tag{39}$$
$$\sigma = G \circ H, \tag{40}$$
$$D = H \circ \tau, \tag{41}$$
$$T = \tau \circ G. \tag{42}$$

4.1.3. Coordinate Relations

$$\eta = D\chi = H\alpha, \tag{43}$$
$$\chi = \sigma\alpha = G\eta, \tag{44}$$
$$\alpha = \tau\chi = T\eta. \tag{45}$$

4.1.4. Mapping Conditions

1. H is degenerate: $\det(H) = 0$
2. D is also degenerate: $\det(D) = 0$
3. T and G are singular maps
4. τ and σ are regular and smooth maps

4.1.5. Metric Conditions on Strata

Timelike Region (A)
Coordinates: χ, Metric: g'

$$\chi^T g' \chi > 0. \tag{46}$$

Base/Null Region (B)
Coordinates: η, Metric: g

$$\eta^T g \eta = 0. \tag{47}$$

Spacelike Region (C)
Coordinates: α, Metric: g''

$$\alpha^T g'' \alpha < 0. \tag{48}$$

4.1.6. Induced Metric Conditions and Algebraic Varieties

To proceed further, we use the metric conditions on strata and rewrite the three metric conditions in terms of the timelike vector χ using coordinate relations. Because maps T and G are singular as per mapping conditions and maps τ and σ are regular and smooth, we rewrite relations in terms of D, τ, and σ, where τ and σ are inverses of each other. One notices that the parameter for maintaining bundle structure implies we can use the triple D, σ, and τ or H, σ, and τ, where $H = D \circ \sigma = D \circ \tau^{-1}$. To avoid complications, we use the triple D, τ, and χ. Thus, we use the bundle structure of the three strata to rewrite metric conditions in terms of the triples. We find the following relations:

$$A_1 = \chi^T g' \chi > 0, \tag{49}$$
$$A_2 = \chi^T D^T g D \chi = 0, \tag{50}$$
$$A_3 = \chi^T \tau^T g'' \tau \chi < 0. \tag{51}$$

Here A_1, A_2, and A_3 are algebraic varieties[14] or quadratic forms described on the spacetime manifold with the three strata [92, 93]. The motivation for calling them algebraic varieties is that we do not know the metrics along the three strata[15] but expect them to be some form of equations labeled as varieties. The three varieties A_1, A_2, and A_3 represent the three strata of our spacetime. We assume that the varieties are smooth themselves[16] but do not know if the transition between varieties is smooth. We can also foliate spacetime into subvarieties, usually called Cauchy slicing of spacetime or codimension-2 slices. We expect topology changes while moving from one variety to another[17] but assume maps in subvarieties are smooth foliations with continuous and differentiable structure. We observe that for the same vector χ, A_1, A_2, and A_3 are different. We can alternatively say that given a vector χ, the group of transformation

$$(g \to g'), \quad (b : g' \to D^T g D \quad \text{or} \quad \tau^T g'' \tau) \tag{52}$$

maps degrees of freedom from one stratum to another using the same coordinate χ. Hence, given a metric along some hypersurface, the mapping D or τ and the metric along the strata determine the induced structure of the strata for any pair of 4-vector χ. Thus, we can say that the degrees of freedom in the strata are the mappings D or τ and the induced metrics along the strata. Thus, we hope to say that:

1. Metric g' is timelike.
2. Induced metric $D^T g D$ is null.
3. $\tau^T g'' \tau$ is the induced metric along the spacelike region.

Thus, given the induced metric (g, g', or g'') and mapping

[14] Please refer to literature on Algebraic geomtry and Algebraic varieties and Topology of Algebraic varieties.
[15] For generality, we label the metric for the timelike vector as g'.
[16] Which we need to prove; it implies the three strata are smooth themselves
[17] Why this is so is a question of investigation for us

across the strata (D or τ), any arbitrary vector along the three strata can be mapped using this. To refine this further, we need to establish some relation between the metrics along the three strata. We assume that the transition of coordinates from timelike to null regions is continuous; that is, as we move along a neighboring region from the timelike surface to the null hypersurface, this can be stated as the condition that the timelike 4-vector χ motivated in equation (1), when displaced along a neighboring region along the null hypersurface Σ, is the same as η. Thus,

$$\chi|_B = \eta. \tag{53}$$

This means as the radius r[18] of the topological semigroup [78][19] D shrinks to zero, it becomes identical to the identity, but during the transition from timelike to null, coordinates are mapped to a codimension-1 surface; hence one of the four elements of matrix D becomes 0. For simplicity, we assume D is equivalent to the identity group, but the identity has only three unit elements as one identity component shrinks to zero. For notational simplicity, we write hypersurfaces as A (timelike), B (null), C (spacelike). Thus, along the null hypersurface B, we find:

$$\lim_{r\to 0} D_{A\to B}(r) \cong \begin{bmatrix} 1_{3\times 3} & 0 \\ 0 & 0 \end{bmatrix}. \tag{54}$$

We have similar relations for coordinates α and η and therefore for mapping groups H, τ, and σ. Thus, we have:

$$\alpha|_B = \eta, \tag{55}$$

which implies

$$\lim_{r'\to 0} H|_{C\to B}(r') \cong \begin{bmatrix} 1_{3\times 3} & 0 \\ 0 & 0 \end{bmatrix}. \tag{56}$$

Finally, for coordinates α and η, we have:

$$\alpha|_A = \chi, \tag{57}$$

which implies

$$\lim_{p'\to 0} \sigma|_{C\to A}(p') \cong 1_{4\times 4}. \tag{58}$$

Similarly,

$$\chi|_C = \alpha, \tag{59}$$

using the coordinate relations, we find:

$$\lim_{p''\to 0} \tau|_{A\to C}(p'') \cong 1_{4\times 4}. \tag{60}$$

Conditions from (34-45) and (53-60) are very much like cocycle conditions on restriction maps imposed on presheaves.[20] with mappings D, H, τ, and σ forming two kinds of isomorphism classes[21] on some indefinite metric space[22] structure between the strata (because the length interval in this scenario can also take 0 and negative values for null and spacelike regions[23]). Such indefinite metric spaces are studied as Krein spaces and Pontryagin spaces [94–100]. The structure of topological semigroups and groupoids also plays a key role here [110, 111].

5. Lessons from Minkowski Spacetime

Just as we consider strata in black hole spacetime, we can consider strata in Minkowski spacetime. Since the metric of Minkowski spacetime is known, we can write equations for the three strata: $ds^2 > 0$ for timelike region A, $ds^2 = 0$ for null region B, and $ds^2 < 0$ for spacelike region C. Suppose we label coordinate charts via parameters x_0, x_1, x_2, x_3. Then the equation for region A is:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 > 0. \tag{61}$$

Similarly, for region B:

$$x_0^2 = x_1^2 + x_2^2 + x_3^2, \tag{62}$$

and for region C:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 < 0. \tag{63}$$

5.1. Topology of Three Regions: A, B, and C

For region A, we see x_1, x_2, and x_3 are free parameters and $x_0^2 > x_1^2 + x_2^2 + x_3^2$; thus, for any parameter $\epsilon > 0$, region A is defined as:

$$x_0^2 = x_1^2 + x_2^2 + x_3^2 + \epsilon. \tag{64}$$

We see from above that x_0 is determined uniquely. Hence, the range for $x_0 \in (0, \infty)$. *Thus timelike slices has topology* $\mathbb{R}^3 \times \mathbb{R}$.

Similarly, for null region B, we find that given x_0, parameters x_1, x_2, x_3 are constrained by:

$$x_1^2 + x_2^2 + x_3^2 = x_0^2. \tag{65}$$

Thus, x_0 is the free parameter and $x_1, x_2, x_3 \in S^2$. *Thus, the topology of B is* $S^2 \times \mathbb{R}$.

Similarly, for spacelike region C, we have for some $\epsilon' > 0$:

$$x_1^2 + x_2^2 + x_3^2 = x_0^2 + \epsilon'. \tag{66}$$

[18] One can refer to references on metric spaces and topology for further understanding.

[19] Please note that semigroup D is not a diffeomorphism group as the inverse map is singular [79].

[20] One can infer the basic idea of sheaves and presheaves from Wikipedia articles or references mentioned at the end. Also note that, unlike in presheaves where the restriction map is identity, the constraint that $\det(D)$ or $\det(H)$ is zero collapses one value to zero; thus, unlike identity, the map shrinks one coordinate to 0.

[21] Isomorphism class in Equations (54, 56) and (60).

[22] Please go through references at the end for these topics.

[23] In our convention taken elsewhere, the length of the spacelike region is assumed negative to suggest this is not a usual metric space.

Thus, x_1, x_2, and x_3 are constrained to lie on a 2-sphere with radius $\sqrt{x_0^2+\epsilon'}$. *Thus, the topology for region C is $S^2\times\mathbb{R}\times\mathbb{R}$.*

Thus, we see that regions B and C have topologies $S^2\times\mathbb{R}$ and $S^2\times\mathbb{R}\times\mathbb{R}$, but topology for region A is $\mathbb{R}^3\times\mathbb{R}$ [104–107]. However, the origin in all three scenarios has a conical singularity as for the light cone; thus, we need to mod out the singular point $\{0\}$. Also, since the fundamental group of $\mathbb{R}^3$ is different from $S^2\times\mathbb{R}$, there does not exist a diffeomorphism or homeomorphism from region A to B or from region A to C. However, regions B and C are smoothly connected in Minkowski spacetime. Thus, we learn that though the topology of Minkowski spacetime $\mathbb{M}$ minus the origin $\{O\}$ is a combination of the following causal strata:

$$\mathbb{M}\setminus\{O\}=\bigcup_{\varepsilon>0}\underbrace{\mathscr{H}_\varepsilon^+}_{\mathbb{R}^3}\cup\bigcup_{\varepsilon>0}\underbrace{\mathscr{H}_\varepsilon^-}_{\mathbb{R}^3}\cup\bigcup_{\varepsilon<0}\underbrace{\mathscr{S}_\varepsilon}_{S^2\times\mathbb{R}}\cup\big(\underbrace{\mathscr{N}^+}_{S^2\times\mathbb{R}}\cup\underbrace{\mathscr{N}^-}_{S^2\times\mathbb{R}}\big),$$

where:

1. $\mathscr{H}_\varepsilon^+$: Future timelike hyperboloid, $Q=\varepsilon>0$, $x_0>0$
2. $\mathscr{H}_\varepsilon^-$: Past timelike hyperboloid, $Q=\varepsilon>0$, $x_0<0$
3. $\mathscr{S}_\varepsilon$: Spacelike hyperboloid, $Q=-\varepsilon'<0$
4. $\mathscr{N}^\pm$: Future/past null cone without origin, $Q=0$, $x_0>0$ or <0

with $Q(x)=x_0^2-(x_1^2+x_2^2+x_3^2)$. Thus, we see that going from timelike region A to region B or C undergoes a topology change, and there are no smooth maps from A to B or A to C, but as a whole, Minkowski spacetime is smooth except at the origin.

5.2. Implications

We see that Minkowski spacetime is a combination of 5 strata, and the union of all 5 strata is smooth except with a conical singularity at the origin. *This suggests that even though there might not be homeomorphisms or diffeomorphisms between strata with different topology, the manifold as a whole contains only a finite number of singularities. Thus, modulo a finite number of singular points or a singular region, we expect black hole spacetime to also be smooth.*

5.3. The Null Mapping Group D for Minkowski Spacetime

Using the condition of vanishing norm along the null hypersurface, we can find the semigroup D for morphism from timelike to null region using:

$$D^T\Lambda D=0, \tag{67}$$

where Λ is the uniform Minkowski metric. We leave the calculation to the reader, who can verify that using parameters $d_{00}, d_{01}, d_{02}, d_{03}$ and θ, ϕ, the mapping group is:

$$D=\begin{bmatrix} d_{00} & d_{01} & d_{02} & d_{03}\\ d_{00}\cos\theta\cos\phi & d_{01}\cos\theta\cos\phi & d_{02}\cos\theta\cos\phi & d_{03}\cos\theta\cos\phi\\ d_{00}\cos\theta\sin\phi & d_{01}\cos\theta\sin\phi & d_{02}\cos\theta\sin\phi & d_{03}\cos\theta\sin\phi\\ d_{00}\sin\theta & d_{01}\sin\theta & d_{02}\sin\theta & d_{03}\sin\theta \end{bmatrix}. \tag{68}$$

One can verify that $\det(D)=0$. Thus, we see that despite the mapping group being degenerate and hence non-invertible, Minkowski spacetime as a whole is smooth except at the origin. We expect that except for foliations gluing strata A, B, and C for Schwarzschild spacetime, the three regions A, B, and C, which may have different topologies, but as submanifolds, they contain the same topologies across foliations of submanifolds of A, B, and C [85–91]. *Thus, we conjecture that as submanifolds in black hole spacetime, regions A, B, and C are smooth and differentiable, with all submanifolds foliating A, B, or C smooth as submanifolds of A, B, or C.* The only pathologies occur across strata where there might be singularities that need additional tools like stratification, surgery theory, or blowup techniques of singular manifolds to be dealt with. [112–114]

6. Topological Isomorphism of Codimension-1 Foliations of Black Hole Spacetime and Non-orientable Surfaces: A Hypothesis

Referring to [4] Section 5, we propose that the overall topology of black hole spacetime is isomorphic to a Möbius strip or a class of non-orientable surfaces.[24] We propose this for the following reasons:

Suppose we consider a manifold with smooth foliations of a 3D disconnected Möbius strip but with twists. The twist partitions the Möbius strip into three regions: Region A' exterior to the twist, Region B' as the twisting region of the Möbius strip, and Region C' interior to the twisting region. We find the following similarities with the topology of black hole spacetime:

1. Both region A' in the Möbius strip and region A

[24] The reason for non-orientability can be found in equation (32) of this paper.

(timelike in black hole spacetime) are smooth regions.

2. Just as sections from the null hypersurface to timelike and spacelike regions are infinite-dimensional, similarly in the Möbius strip, when glued across twisting regions, there may be conical singularities. Thus, if we can establish that as a bundle, the Möbius strip has a singularity at the twisting point, we can embed both the bundle structure of spacetime isomorphically to bundles isomorphic to the Möbius strip. Thus, each foliation of submanifold B is smooth, like the twisting region of the Möbius strip, but gluing a family of Möbius strips together along with gluing of the twisting region may generate conical singularities; similarly, foliations of null hypersurfaces might be smooth as codimension-2 surfaces but gluing them together generates sections that are infinite-dimensional because of something similar to conical singularities in the gluing process. Using this similarity, we conjecture that all foliations of null hypersurfaces are smooth, but as a submanifold, the null hypersurface has singularities [112–114].
3. The third reason is that just as the Möbius strip changes orientation, we see a similar reversal of orientation when considering mapping from timelike region A to spacelike region C. The change of sign of the determinant of the metric for black hole spacetime when transitioning from timelike region to spacelike region C is an indicator of spacetime as a whole being a non-orientable manifold.

Because a non-orientable surface when glued with a family of its isomorphism classes generates singularities, the singular nature of sections of the spacetime manifold when equipped with a bundle structure can feature such similarity. This is also in alignment with the idea of stratification of spacetime, each of which as a family of surfaces is smooth [60–64]. One of the problems identified with foliation of spacetime in [4] is that because of varying redshift across foliation of spacetime, there is no global notion of constant time slices while foliating spacetime into submanifolds with three subregions: $(a \in A, a \subset A)$, $(b \in B, b \subset B)$, and $(c \in C, c \subset C)$, with a, b, and c as codimension-1 subsurfaces glued together. It is yet to be verified if this isomorphism with non-orientable surfaces implies an isomorphism among topologies of submanifolds, thus providing a way for identifying the topology of black hole spacetime across strata B and C. It is yet to be verified if gluing fibres of the Möbius strip or a non-orientable 3D surface generates infinite-dimensional sections of the twisting region. This isomorphism is only an approximation to study topology of black hole spacetime regions B and C because we have already hinted at the Finsler nature of black hole spacetime, which implies a Finsler nature of the 3D Möbius strip. It is not yet known if the singularity with sections is identical to conical singularities, but it is supposed that sections can be smoothened out via surgery theory or blowing up surfaces to have similar behavior [108, 109].

Implications

1. This suggests that as there is no universal parameter for globally foliating spacetime into fibres (because of varying redshift of the time factor while traversing spacetime submanifold), the conical singularity presents a similar kind of obstruction to foliating non-orientable surfaces globally [65–71].
2. The topology of smooth differentiable surfaces is studied via Morse theory; it suggests that to understand topology of spacetime, we need stratified Morse theory with topological properties of spacetime also being inherent while studying Morse theory on non-orientable surfaces with a conical singularity like the Möbius strip [50–53]. Given the topology of the Möbius strip or similar isomorphism classes, we can make several inferences regarding the fate of spacetimes with gravity:
3. Suppose spacetime manifold admits horizontal and transverse foliations[25] or sections of the bundle being horizontal and vertical [56–59]. As gravity dilates time for Schwarzschild spacetime, thus twisting the fibres, there are some implications of horizontal fibres close to the null hypersurface when evolved via group action (that projects evolution of submanifold along vertical fibres), rising vertically to smaller heights than further horizontal bundles. This generates smaller vertical evolution of submanifold near the null hypersurface[26] and larger evolution vertically of sections further away from the null hypersurface (less gravitating region). This causes the whole slice to undergo bending towards the horizon and relatively raising regions further away from the horizon. One implication is that because of bending of nearby horizontal sections, the twisting region and interior region shrink in size, and thereby topologically, the exterior region away from the horizon expands. Thus, using this picture, we can model evolution of black hole spacetime as shrinking the interior of the horizon and expanding horizontal sections. Thus, we can suggest this topological phenomenon as an explanation of expansion of the universe and shrinking of black hole studied via Hawking radiation as a topological phenomenon. Using this, we can study the fate of an evaporating black hole as shrinking of horizon due to bending of fibres and expansion of universe alongside as topological phenomena. This expansion of universe can be attributed as a topological gravity effect and shrinking as a local gravitational effect.
4. There are many eventual fates of black hole spacetime: either the universe splits or tears due to stretching of fibres at the horizon?we call this splitting of universe into sub-universes. Alongside, as the union of horizon and interior shrinks, gravity becomes weaker; thus dilation of time becomes weaker. We label this phenomenon as leveling or reducing gravitational effect on fibres, thereby making the universe more

[25]Just as studied in Ehresmann connections

[26]Larger gravitating region causes dilation of time; hence, pictorially, fibres near the horizon evolve to lesser heights.

homogeneously distributed with gravity. Thus, topology of spacetime starts turning itself into a more homogeneous and isotropic space. Thus, the eventual fate of gravitating universe is splitting of universe into baby universes or otherwise a more expanded homogeneous single universe.

7. Framework for Further Study

There are a plethora of questions that need further investigation, such as:

1. To prove that as submanifolds, A, B, or C are smooth submanifolds.
2. Do A, B, or C encode the structure of a homogeneous space so we can apply ideas from Lie theory to study such spaces? From our basic investigation so far, we have no hint whether we can study such spaces using Lie theory. Since mappings are non-invertible (unlike in Lie groups, which are smooth differentiable manifolds), we are led to ideas of groupoids (which are generalizations of Lie groups) and topological semigroups[27] (which are semigroups only in the topological sense) as basic ingredients of our study.
3. Investigating the topology of regions A, B, or C in black hole spacetime.
4. Proving that as submanifolds, regions A, B, or C are stable with reference to perturbation, as the set of allowed values of line element ds^2 along submanifolds A, B, or C contains a very small interval of allowed values with reference to the domain of allowed values of line element, which is from $-\infty$ to $+\infty$ as a manifold for black hole spacetime. Since the null hypersurface contains only a single point (the origin) for line element ds^2, it is a question of investigation whether the null hypersurface is stable as a geometric structure with reference to perturbations of values allowed for geometry of line element for values other than zero. Proving $ds^2 = 0$ is a question of investigation for us. Using isomorphism of topologies of non-orientable surfaces, we can study Morse theory of non-orientable surfaces to deduce dynamical evolution of black hole spacetimes.
5. Studying finite singularities of entire black hole spacetime using algebraic varieties A_1, A_2, and A_3 and their singularities.
6. The degenerate projection map and singular sections encoding black hole spacetime as a principal G-bundle cause obstruction to associating the spacetime manifold with a bundle structure and hence require studying fibre bundles that are singular. This leads us to understanding singular foliations and singular fibre bundles or Fibre bundles on Infinite dimensional spaces or studying topology and singularities of generic algebraic varieties A_1, A_2, and A_3.
7. Investigating generic forms for mapping groups $D_{A\to B}$, $H_{C\to B}$, $\sigma_{C\to A}$, and $\tau_{A\to C}$ is a question of investigation. Thus, we are led to understanding mapping groups [78] across strata between regions A, B, and C.
8. Since Einstein equations are not valid for null hypersurfaces, we are in a more generalized arena where ideas from Riemannian geometry need generalization to a more general setting such that the Ricci tensor does not vanish. One such setting is to embed submanifolds with a principal G-bundle structure and study generalizations of affine connections like Ehresmann connections on fibre bundles or fibre subbundles. Thus, a lot of new machinery like bundle theory, generalized connections, stratification, topological semigroups or groupoids, or infinite-dimensional Lie groups (if black hole spacetime as a manifold admits Lie group structure)[28] or as groupoids or infinite-dimensional topological semigroups, or something else unknown to us[29] [92, 93, 101–103, 116–120].

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Author Contributions

Shreyansh Singh: Conceptualization, Formal analysis, Investigation, Methodology, Visualization, Writing - original drat

Conflicts of Interest

The author declares that he has no conflicts of interest.

[27] And not as groups because mappings are non-invertible.

[28] We need to prove or verify that as a manifold, singular sections as infinite-dimensional spaces admit Lie group structure. Interestingly, one of Hilbert's 23 problems is to investigate whether any generic manifold contains isometries and geometric structure of a Lie group. Prof. Terence Tao book Hilbert's 5th problem and related topics is a very good reference for going through it.

[29] In its full generality we presume studying the problem as studying Topology of real algebraic varities , a more refined way connecting the problem is to study the space via Sheaves and Grothendieck topology.It is beyond our expertise to say anything about it and we consider it as a problem to be worked on in future.

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