

Absence of Diffeomorphism Symmetry for Black Hole Spacetime: A Prelude to Black Hole Spacetime Being a Smooth Finsler Manifold

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Abstract: We propose another version of Black Hole information Paradox owing absence of Diffeomorphism symmetry for Black Hole Spacetime. The central idea is assumptions of semiclassical Gravity like locality, causality and diffeomorphism symmetry lead to another paradox. The paradox unlike many other recent versions of Information paradox is valid even at very early times. The possible suggestive way for the resolution of paradox has various approaches like validity of Black Hole Complementarity from the very beginning or the principle of Holography of Information proposed by Suvrat et. al. We discuss some subtleties with these approaches in the local nature of splitting of Hilbert Space (in some pseudoprecise way) of the dual CFT. Being as conservative as we can, to avoid such subtleties (which have been addressed in the paper) we suggest that absence of Diffeomorphism symmetry for Black Hole Spacetime implies it being a smooth Finsler manifold. We will discuss what we mean by this and the implications of it. We also give a recipe for reconstructing Null Hypersurface for Black Hole Spacetime in a different way in section 3.6, 3.7 of the paper. We are also lead to notion of Finsler Geometry for Black Hole Spacetime owing to asymmetry of line element. We also find that the Null Diffeomorphism Symmetry group of 2 dimensional Minowski spacetime admits a superspace structure. Despite analytical difficulty of solving coupled ordinary Raychaudhuri differential equations and a consistency condition we verified the claim of superspace structure for Null Diffeomorphism group for Schwarzschild spacetime.

Keywords: Diffeomorphism Symmetry, Finsler Geometry, Superspace

1. Introduction

The framework of Einstein General Theory of Relativity is significantly different from Newtonian standpoint which deals with the idea of studying classical dynamics in flat spacetime. The idea of General theory of relativity is to assume spacetime to be curved and the dynamics happens in curved spacetime, this new notion of geometry is at the crossroads of the branch of mathematics we know as Differential Geometry and topology. Why do we need the tools of semi modern mathematics to study Einstein General Theory of relativity? The simplest answer is the usual known notions, intuitions of Flat spacetime like assuming a vector space structure, tangent

vector may not apply in usual way in curved spacetime simply because of the curvature of spacetime means the local data assigned at a point changes as we proceed along the spacetime. This also implies that spacetime needs more primitive structure to be addressed in this situation, mathematicians while dealing with curved spaces associate the idea of a manifold.¹

While dealing with curved spacetime we need to associate Einstein Equivalence principle and Lorentz transformation which are applicable in flat spacetime to be replaced by more generic notions which are applicable locally. The basic idea is to preserve the notion of smoothness of manifold so that it allows a geometrical structure as a manifold thereby allowing tools and techniques of a smooth manifold

¹ We will be addressing the idea of notion of a manifold properly later in the paper.

² More primitively ideas from analysis like limits, differentiability and from differential geometry being applicable

³ We will address and explain what we mean by diffeomorphisms in later section/sections.

²like diffeomorphisms ³ and relatable ideas from Riemannian geometry to be applicable. Our aim through this paper will be to first address the tools and ideas from differential geometry which seem to be hidden or not properly addressed. However we will see that even notions of Black Hole Spacetime being a smooth differentiable manifold comes in conflict with the known principles of Einstein special theory of relativity when considering Black Hole Spacetimes. We try addressing these aspects in various ways, as a possibility we try being as conservative as we can (for now) and suggest that spacetime as addressed in classical general relativity is not a smooth Riemannian manifold and we suggest various possibilities like it being a Finsler Geometry and so on. Considering various alternatives like Black Hole Complementarity and Principle of Holography of Information for the possible resolution we find that motivated by Holography the dual CFT does not admit a local structure splitting of Hilbert Space of dual CFT, so in some sense assumption of locality of CFT, Black Hole Complementarity ⁴ and splitting of Hilbert Space of dual CFT in some pseudoprecise way are inconsistent eventually leading us to suggest that Black Hole spacetime does not admit a smooth Riemannian Geometry or the principle of Causality of spacetime needs to be modified in a dramatic way. ⁵ Our aim in the next section will be to address the mathematics of Riemannian geometry to address the paradox.

1.1. The Tools Necessary

As suggested in the introduction Professor Albert Einstein first introduced his Special Theory of relativity with two basic principles -*Einstein Equivalence Principle* and *universality of speed of Light or Causality* in his General Theory of Relativity he introduced the idea of Curvature which naturally adapts the machinery of Riemannian Geometry. Our aim in this section will be to understand Riemannian Geometry and ideas related to it so that we can make meaningful sense of assigning the data on a manifold locally and using these ideas to smoothly extend the mathematical data or the data on the topology of the manifold. The necessary tools will be as we will see are-

1. Topology, Topological Space
2. Manifold
3. Riemannian Geometry and Differential structure of Manifolds
4. *More about Differentiable Manifolds*- Charts and atlases, Tangent Space, Lie Derivatives, Lie Derivatives as the idea of Curvature on Manifolds, Vector Bundles, Embeddings of a manifold, Fibres and Foliations, Metric spaces, Metric space Structure on Manifolds, Extending local data on Manifolds via Lie Derivative a.k.a Affine Connection, Diffeomorphism symmetry of Manifold and its non trivial constraints on the structure of manifold.

⁴ Which can be a possible alternative to resolution of diffeomorphism symmetry violation paradox

⁵ We will address various aspects of it later in the paper

⁶ A surface is said to be smooth if it has a smooth tangent space structure everywhere this is because to obtain a non smooth structure the first order or nth order derivative locally defined at some point on the manifold must be discontinuous

⁷ These properties come under the section of study of Differential Topology which for now will not be our concern.

1.2. Motivation for Topology

Topology as a branch of mathematics grew from the work of Euler who tried studying the famous Bridge problem where it was asked to him if it was possible to cross five bridges arranged in a certain way such that no bridge is crossed more than once and also crossing all the bridges. AS we can observe this problem does not care about the length of bridges or the inclination of bridges but only in the way they are connected. This is the primary motivation of studying Topology that is studying how geometrical objects or shapes are linked to one another, so we see that to study these geometric properties we do not need to know about metric structure and also the Topology of a surface is a global property rather than a local property that is it depends on the complete nature of the surface. In the set theoretic sense topology as we will see in the next paragraph Topology tells how the elements of set are related to one another. A given set can have many topologies. While studying Point set Topology we will discuss topologies like Discrete Topology, Indiscrete Topology, Cofinite Topology but because of lack of our limited understanding we will not discuss Topology defined on Manifolds like Differential Topology, Metric Topology, Geometric Topology, Algebraic Topology as per the relevance needed in the paper.

1.3. Topology, Topological Space

Topology by its name means study of shapes, initially introduced by Euler this branch of mathematics tries to study properties of objects that are independent of geometric notions like length, angle so in some sense it can be said for a smooth surface like a rubber ball Topology tries to study the properties of the ball that remain invariant under deformations of the ball as long as we can retain the ball after deformation. Naturally the precise operations necessary are squashing of shape of ball generating a family of smooth surfaces that can be retraced back to ball. So we see that for a smooth surface ⁶ the topology studies an equivalence class of surfaces that have a non vanishing continuous rth order derivative everywhere defined on the surface. ⁷ Mathematicians thought to generalise the notion of topology as a notion described through set, which is preserved under set operation of finite intersections and arbitrary unions. A set together with its Topology (which is not unique) describes a Topological Space.

1.4. Point Set Topology

Given a set X with $P(X)$ denoted as power set of X a topology $\tau(X)$ is defined as the subset of $P(X)$ with collection of elements say $C_1, C_2, C_3, \dots$ empty set ϕ and universal set X such that the set operations of arbitrary unions and finite intersection of elements of τ also belong to topology τ .

So for arbitrary set of elements $C_i, C_j \in \tau$

$C_i \cap C_j \in \tau$ and $C_i \cup C_j \in \tau$ and

$\phi \in \tau, X \in \tau$

Please note that we can have arbitrary number of intersections and arbitrary unions we have assumed 2 for just denoting the operation. Also note that keeping aside the notion of compactness of sets the number of elements in the open set can be infinite. We will worry about the cardinality⁸ of sets later whenever necessary. Also there can be many possible Topologies for a given set as we will illustrate later in the subsection.

Examples

1) $x = \{a, b, c\}$ and $\tau = \{\phi, x, a, b\}$

We can see that given set x , τ is not a topology on x because the union of elements a and b does not belong to τ .

2) $x = \{a, b, c\}$ and $\tau = \{\phi, x, \{a, b\}, \{b, c\}\}$ We again see that τ is not a Topology on x as intersection of elements $\{a, b\} \cap \{b, c\} = \{b\}$ which is not in τ .

3) $x = \{a, b\}$ and $\tau = \{\phi, a\}$ Noticing carefully we see that set x is not contained in it so again it is not a topology on set x .

4) $x = \{a, b\}$ and $\tau = \{\phi, x, a\}$ We can see that all unions and arbitrary intersections are contained in the topology τ so set τ is one of the possible topologies of set x .

1) *Indiscrete Topology* Suppose $\tau = \{\phi, X\}$ then certainly it satisfies being a Topology. This is called Indiscrete or Trivial Topology. *We will discuss later why point set topology is always described between open sets.*

2) *Discrete Topology* Suppose $\tau = P(X)$ then certainly again τ satisfies being a Topology. This is called Discrete Topology. The 2 examples that we have discussed here are examples of point set topology which means the Topology τ contains finite number of elements. We will now discuss some other examples of Point Set Topology.

Non trivial example of point set Topology

3) *Cofinite Topology* Given a universal set X and a set A we define Cofinite Topology as the Topology τ on a set $A' = X - A$ such that the complement of set A A' contains finite number of elements. Please note that sets X and A maybe finite or infinite however the complement of set A , A' contains finite number of elements. We will verify that set A' is a topology and is called Cofinite Topology.

Larger and smaller Topology Given a set x there can be many possible Topologies on the set x . Suppose τ_1, τ_2 are 2 such Topologies and $\tau_1 \subset \tau_2$ then the topology τ_2 is larger topology and τ_1 is smaller topology. *If the Topological space is also a manifold it might be plausible to make a comparison between 2 set of manifolds based on their dimensionality or cardinality. If they are contained in some regions, then it may or not be necessarily true that manifold with greater degrees of freedom is the larger topology, always. As an example suppose we consider a gravitating region which is assumed to be an isolated manifold then the shrinking of region with time increases the entropy globally though in some local region the*

entropy might decrease.

2. Riemannian Geometry

Riemann generalised the notion of curved spaces famously known by his name. These are generally assumed to be curved spaces with a notion called curvature which might have been introduced earlier than Gauss but Carl Friedrich Gauss was probably the first person to give a systematic mathematical treatment of the notion of curvature in 2 dimensions. Sir Bernard Reimann generalised this notion to arbitrary dimensional curved space. An appropriate quantity to capture curvature is the commutator of Lie Derivative which gives a 4th order tensor called Reimann Tensor. Our aim will be to study these notions in a mre abstract way. The first abstract notion of curved space is a *manifold*.

2.1. Manifold

Mathematicians are good in abstracting things so that given some idea they can use the logic behind it so that the abstraction as the feature can be generalised to other or new relevant situations. We all are aware of surfaces, it is some region where we observe the pheonomenons in nature. Coloquially Manifold is an abstract generalisation of surfaces, it is a collection of curved surfaces glued together smoothly, the notion of being glued smoothly means that if we magnify over some little region of a manifold then it looks like a Euclidean Space, however our intent in defining a manifold is that it is a smoothly glued surface though it may not necessarily be everywhere Euclidean but the notion of smoothness is assumed to be essentially true. *A manifold can be assumed to be some kind of space that encodes geometric information*⁹. The notion of manifold is taken from geometers exploration of world where he magnifies a region of earth with his maps as charts and the manifold is built together by gluing his charts which is called an atlas. The intent is that by smoothly gluing all the charts we can study the property of the surface which is assumed to be a manifold. Our aim in these sections will be to build the mathematical machinery required to study smooth manifolds. Motivated by the tools from this machinery we will then try to study generic properties of manifolds which may not necessarily be everywhere smooth.

2.2. Charts and Atlas

We provide definitions of Charts, atlas and maps between them. Given a manifold M with coordinate charts U_α we define a map ϕ_α as -

$$\phi_\alpha : U_\alpha|_M \rightarrow R^n$$

where n is dimensionality of the Manifold M . In these section we will assume that manifold is smooth and differentiable

⁸ Cardinality is number of elements in set

⁹ The geometric information is assumed to be something ill defined here it might not be some space that allows smooth differentiable structure always or it might be some abstract algebraic structure that encodes some property of the set or otherwise geometric information can be encoded via some kind of morphism between abstract categorical objects

everywhere to build the basics. The maps ϕ_α are assumed to be smooth and differentiable and $U_\alpha|_M$ are called the coordinate charts and the union of all such charts $A = \cup_\alpha U_\alpha$ is called an atlas. A maximal atlas is an atlas with the least number of charts U_α such that their union covers the manifold M . We often hear that Spacetime is a differentiable manifold. Our motive through this section will be to give a meaningful definition of manifold, what is Topology, what are smooth Manifold, what is a topological manifold (if possible). These ideas will be discussed in great detail in this section. This section will be used to define the the notions used for describing a smooth differentiable manifold and its various properties. The subsection will be used to discuss loosely the paradoxes in framing spacetime as a manifold.

2.3. Diffeomorphism

Given that Manifold is a suitable generalisation of Curved space the first thing we may desire is to be able to do calculus on Manifolds. As charts is a map from an open set in the manifold to R^n the map should be assumed to be a smooth function. Also there can be many possible charts mapped to R^n we want that if two coordinate charts intersect the map should be a homeomorphism. This is suggested in figure 1. Homeomorphism between the coordinate charts means that the maps $\phi_1 \circ (\phi_2)^{-1}$ and $\phi_2 \circ (\phi_1)^{-1}$ are smooth maps from a subset of R^n to subset R^n . Compatible charts are also written as C^∞ charts. This is as shown in Figure 1.

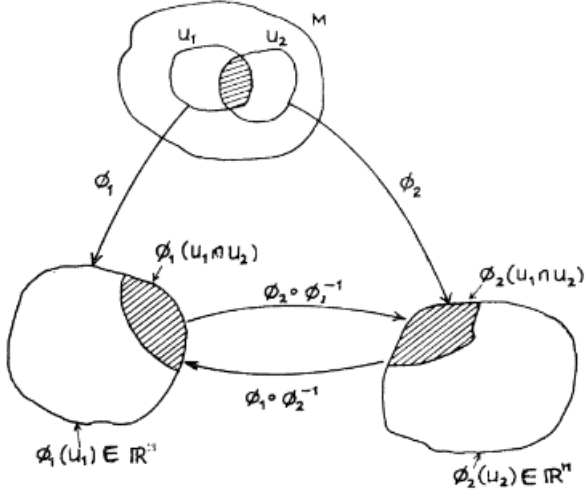


Figure 1. The compatibility condition for a smooth map is that given 2 coordinate charts with their intersecting regions the map in the intersecting region must be smooth and differentiable.

3. Diffeomorphism and Einstein Principle of Equivalence

Professor Albert Einstein first introduced his Special Theory of relativity with two basic principles -Einstein Equivalence Principle and universality of speed of Light or Causality in his General Theory of Relativity he introduced the

idea of Curvature which naturally adapts the machinery of Riemannian Geometry. Our aim in this section will be to generalise Special Theory of relativity in curved space using the formulation of Diffeomorphism symmetry group. However we will not rely on mathematical machinery of metric tensor for transformations which appears to be a bit misleading. Our intuition is that there might not be a linear one to one map between distinct points in a Riemannian or Pseudo Riemannian manifold (Lorentzian in our example). This maybe because the manifold might get stretched significantly along angular directions so that there is not a nice one one map everywhere between distinct coordinates or in other words the spherical symmetry of the manifold might not always be present. However assuming the spacetime is 4 dimensional we assume there exists diffeomorphisms D between distinct regions of spacetime manifold or between 2 observers. We do not have a reliable indicator of strong or weak gravity regime, one such notion can be the Ricci Scalar or also called Ricci Curvature. The manifold data in terms of metric for Schwarzschild Spacetime is eventually written in terms of parameter M , radial coordinate r , angular coordinates θ and ϕ . For a generic manifold arbitrarily curved we do not expect radial symmetry but our expectation is that the spacetime manifold in some coordinate region can be assigned via some 4 vector t, r, θ, ϕ . As the Lorentzian symmetry is described by Lorentz transformation in some suitable coordinate chart we assume the diffeomorphism symmetry group relates 2 different observers via Diffeomorphism symmetry group. However because we have not associated the tensorial nature of transformations or in other words the linear structure of transformations in strong curvature might not be always approximal. Suppose we use the diffeomorphism symmetry group in a suitable epsilon ball shaped region that contains some region of null hypersurface Σ and a timelike region τ . The introduction of affine connection in Riemannian geometry was to smoothly transport local data when the tensorial structure was not preserved we use similar intuition to associate the 4×4 matrix and 2 vectors η (which is along null hypersurface) and a vector χ along timelike region, both η and χ are 4 vectors. Our assumption is that like Black Hole spacetime a Manifold can be a union of such regions. Then η and χ are related as-

$$\eta = D\chi \quad (1)$$

Since η is along a null hypersurface its inner product with itself is zero. So $\eta^T \eta = 0$ also since χ is along a timelike region so $\chi^T \chi \neq 0$. Using equation 1 this implies

$$D^T D = 0 \quad (2)$$

Thus the Diffeomorphism symmetry matrix D relating timelike and null observers is singular so there does not exist diffeomorphisms between 2 such hypersurfaces. However this does not imply that there are no coordinate charts η it is just that these coordinate charts are non invertible but they do exist. The hidden assumption of Causality does go in this.

3.1. Reconstruction of Diffeomorphism Group D for 2 Dimensional Minokowski Spacetime

Our aim to provide equation 2 was to reconstruct Null Hypersurface for Black Hole Spacetime using this framework because tensorial transformations were inconsistent with known modern versions of Black Hole Information Paradox like Professor Sameer Mathurs Information Paradox and AMPS version of information paradox. The example in previous section provides us an alternate way to reconstruct Null Hypersurfaces or event horizons. As of now we have not made any generic assumption so the Diffeomorphism symmetry group with 16 entries might be a complicated function like $D = D(t, r, \theta, \phi)$. One of the major challenges will be to reconstruct D for generic spacetimes, already we have a result suggested in equation 2. We can now try to construct it for a timelike region in Minokowski spacetime for timelike region. Before considering a 4 vector let us assume a 2 dimensional Minokowski spacetime

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

Suppose we consider 2 observers related via boosts and we label 2 vector observers as α and β with β as coordinates (t, r) so

$$\alpha_1 = d_{11}t + d_{12}r \quad (3)$$

and

$$\alpha_2 = d_{21}t + d_{22}r \quad (4)$$

Then

$$\alpha_1^2 - \alpha_2^2 = t^2 - r^2 \quad (5)$$

implies

$$d_{12}^2 = d_{21}^2 \quad (6)$$

Thus we do not necessarily assume that both of them are zero but they are either same or opposite. Also we find $d_{11}^2 = d_{22}^2$ this seems to be similar to 2 dimensional Lorentz transformation except that we did not use metric to scale the coordinates and we do find that the components of D are not necessarily symmetric. Here they are either of same or opposite sign. The conventional process of Tensor analysis in General Relativity assumes $d_{\mu\nu} = g_{\mu\nu}$ which does not seem to be right. Finding the matrix D (I am not calling it Diffeomorphism group because of equation 2) seems to be a non trivial process.

3.2. Diffeomorphism Between Timelike and Null Region in Minokowski Spacetime

In this section our motivation will be to introduce Diffeomorphism between timelike and null region in Minokowski spacetime. Interestingly we will find that the elements of Diffeomorphism group satisfy Grassman Algebra.

Suppose we consider the diffeomorphism group between lightlike region in 2 dimensional spacetime with coordinates

t', x' and timelike region t, x then

$$\begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} \quad (7)$$

Using the condition as stated in equation 2 we find

$$d_{11} = id_{21} \quad (8)$$

$$d_{22} = -id_{12} \quad (9)$$

Also

$$(t')^2 - (x')^2 = 0 \quad (10)$$

Using 7, 8, 9, 10 we can find one possible solution to the system of equation. Using first 3 equations We find

$$t' = id_{21}t + d_{12}x \quad (11)$$

$$x' = d_{21}t - id_{12}x \quad (12)$$

Equations 11, 12 and 10 are consistent only when d_{12} and d_{21} are Grassman variables or matrices following Dirac Algebra because from the consistency condition we find

$$d_{12}^2 = d_{21}^2 = 0 \quad (13)$$

and

$$d_{12}d_{21} + d_{21}d_{12} = 0 \quad (14)$$

It seems like Diffeomorphism group in 2D Gravity is identical to Clifford Algebra. We do not know the implication of this result. Our motive will be to examine and possibly generalise this result for Non trivial cases when Gravity is dynamical or also called Pseudo Riemannian Geometry. Also from equations 11 and 12 we see that

$$t' = ix' \quad (15)$$

We find that Null diffeomorphism group D is

$$D = \begin{bmatrix} id_{21} & d_{12} \\ d_{21} & -id_{12} \end{bmatrix} \quad (16)$$

Since the time and spatial coordinate are complex valued we cannot necessarily say that velocity of such curves is imaginary. Our intuition says that imaginary velocity of such curves is because of acausal diffeomorphisms. Though the usual rule of calculus as per equation 15 does tell us so. There is a Grassman variable alongside in equations 11 and 12 we do not know what is its implication or meaning. It seems surprising why and how Grassman variables appear in this situation. We will devote the next section to understand Grassman numbers. Most of the things said here are taken from Wikipedia section on Grassman numbers.

3.3. Grassman Numbers

Grassman numbers are square root of zero in the sense that they square to zero but may not necessarily be zero. They are anticommuting elements. The idea of anticommuting elements arises in lot of places in mathematics like in Differential geometry where differential forms are seen as anticommuting elements. Differential forms are normally defined in terms of derivatives on a manifold; however, one can contemplate the situation where one "forgets" or "ignores" the existence of any underlying manifold, and "forgets" or "ignores" that the forms were defined as derivatives, and instead, simply contemplate a situation where one has objects that anti-commute, and have no other pre-defined or presupposed properties. Such objects form an algebra, and specifically the Grassmann algebra or exterior algebra. The Grassmann numbers are elements of that algebra. The appellation of "number" is justified by the fact that they behave not unlike "ordinary" numbers: they can be added, multiplied and divided: they behave almost like a field.

Matrix representation of Grassmanians Elements d_{21} and d_{12} can be represented via matrices as-

$$d_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$d_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad (17)$$

This immediately raises the question what are these additional degrees of freedom. We say that they satisfy Anticommuting relation, hence they are some kind of Lie Algebra. We see that 4*4 matrices are the simplest representation of the elements of Null Diffeomorphism group which satisfies equations 13 and 14 in a non trivial way. *In general, a Grassmann algebra on n generators can be represented by $2^n \times 2^n$ square matrices.* Physically, these matrices can be thought of as raising operators acting on a Hilbert space of n identical fermions, thus these elements might contain internal fermionic degrees of freedom and appear as matrix representation. Since the occupation number for each fermion is 0 or 1, there are 2^n possible basis states. Mathematically, these matrices can be interpreted as the linear operators corresponding to left exterior multiplication on the Grassmann algebra itself. Thus it seems that Fermionic degrees of freedom are preexistent in the Null Diffeomorphism group and to understand it we introduce the idea of a superspace. But before that we rewrite the null coordinates as stated in equations 11 and 12 using the Diffeomorphism group D and explore its implications.

$$t' = id_{21}t + d_{12}x$$

$$x' = d_{21}t - id_{12}x$$

This implies that t and x are themselves comprised of 4 elements and these 4 elements can be regarded as spacetime internal degrees of freedom and so is for t' and x' . But the elements are such that they appear to be identity mostly but for non trivial scenarios where light cone stretches dramatically or at ultrarelativistic speeds they have other non trivial elements. We notice that equation 17 can be written using 2*2 Pauli matrices as-

$$d_{12} = \begin{bmatrix} \sigma_- & 0_{2*2} \\ 0_{2*2} & \sigma_- \end{bmatrix} d_{21} = \begin{bmatrix} 0_{2*2} & 0_{2*2} \\ \sigma_z & 0_{2*2} \end{bmatrix} \quad (18)$$

This immediately tells us that the null Diffeomorphism group is isomorphic to Algebra of Dirac matrices or they are 4*4 spinor elements. We also know the non trivial algebra of null diffeomorphism group as per equations 13 and 14. We assume t has 2*2 components t_1 and t_2 and x has components x_1 and x_2 and similarly t' has components t'_1, t'_2 and x' has components x'_1, x'_2 . Then we find using equations 11 and 12

$$\begin{bmatrix} t'_1 \\ t'_2 \\ x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} i\sigma_- t_1 \\ \sigma_- t_2 + \sigma_z x_1 \\ \sigma_- t_1 \\ -i\sigma_z x_1 + \sigma_- t_2 \end{bmatrix} \quad (19)$$

Using equation 19 we see that

$$t'_1 = ix'_1 \quad (20)$$

but there is no direct such relation between t'_2 and x'_2 . Their is a mixing of boosts in the second components of the elements of t' and x' . We also notice that if 4 vector elements t and x are real then the elements t' and x' are necessarily complex. Thus suggesting that such diffeomorphisms admit a complex Riemannian manifold structure. We also see that such elements hint towards spacetime elements having more than single degree of freedom but these degrees of freedom have real eigenvalues as long as there are no violations of Causality. Violation of causality implies splitting and mixing of real and complex degrees of freedom between spacetime elements. We see that 2 dimensional spacetime has many additional fermionic or internal degrees of freedom. *For a 4 dimensional spacetime this seems to suggest $2^n \times 2^n$ ¹⁰ or 256 dimensional degrees of freedom with 8 spacetime elements all together if also including complex degrees of freedom.* These additional degrees of freedom are in general complex valued. Finding the evolution equation for these degrees of freedom for a generic manifold appears to be a hard task. In flat spacetime which is an idealisation to general spacetime these degrees of freedom do not interact with spacetime or bosonic degrees of freedom but in curved geometry or supergravity regime we do see such an interplay. It seems to be a plausible scenario for accounting for some part of Gravitational entropy in relativistic scenarios where these degrees of freedom are different from identity and might be responsible for some part of entropy behind horizons. In general these degrees of freedom might be real or complex. Thus it seems that

¹⁰We have highlighted this in section 3.3 on Grassmann numbers

the natural avenue to study Spacetime manifold is to assume also these fermionic degrees of freedom and so as a complex Riemannian manifold. In certain scenarios spacetime might admit a specific structure called supersymmetry which is some intertwining of the bosonic and fermionic degrees of freedom but in general after breaking of supersymmetry fermionic degrees of freedom get hidden internally and do not appear visible to us at low speed but might cause significant changes to spacetime at relativistic speeds. Thus the avenue to study Spacetime is to assume it to be graded or a superspace. Why these degrees of freedom might not be visible at small speeds (that is fermionic degrees of freedom appear identity) is a question of further investigation.

3.4. Null Diffeomorphism Group for 1+ 1 d Minokowski Spacetime

Using equations 18 and 16 we can find the null diffeomorphism group as-

$$D = \begin{bmatrix} 0 & 0 & \sigma_- & 0 \\ i\sigma_z & 0 & 0 & \sigma_- \\ 0 & 0 & -i\sigma_- & 0 \\ \sigma_z & 0 & 0 & -i\sigma_- \end{bmatrix} \quad (21)$$

$$f(x, \theta) = f(x) + \Sigma_{A=1 \rightarrow n} \theta^A f_A(x) + \Sigma_{A \neq B \text{ and } A, B \in [1, n]} \theta^A \theta^B f_A(x) \wedge f_B(x)$$

The wedge product ensures that last term is antisymmetric in the indices A and B.

3.6. Null Diffeomorphism Group for 2+ 1 Dimensional Minokowski Spacetime

Given the superspace structure of 1+ 1 dimensional Minokowski spacetime we try investigating null diffeomorphism group for 2+1 dimensional Minokowski spacetime. For simplicity we label the diffeomorphism group as-

$$D = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

As $D^T D = 0$ so the diagonal terms give

$$\begin{aligned} d_{11}^2 + d_{21}^2 + d_{31}^2 &= 0 \\ d_{22}^2 + d_{12}^2 + d_{32}^2 &= 0 \\ d_{13}^2 + d_{23}^2 + d_{33}^2 &= 0 \end{aligned} \quad (22)$$

We assume

$$d_{21} = id_{11} \cos(\theta_1), d_{31} = id_{11} \sin(\theta_1) \quad (23)$$

$$d_{12} = id_{22} \cos(\theta_2), d_{32} = id_{22} \sin(\theta_2) \quad (24)$$

$$d_{13} = id_{33} \cos(\theta_3), d_{23} = id_{33} \sin(\theta_3) \quad (25)$$

We can verify that $D^T D = 0$ (as second column is zero) and that D is a 64 element matrix but with 16 degrees of freedom as $\sigma_-, \sigma_z, i\sigma_-, i\sigma_z$. We also see that the diffeomorphism group matrix is not symmetric or hermitian.

Our motive now will be to understand generic properties of this manifold relying on its spinorial structure thus we proceed to understanding spinors and superspace which is natural way to understand spinors on a manifold.

3.5. Superspace

Our motivation here will be to study Superspace in the context of Minokowski spacetime. A superspace here is tensor product of Minokowski spacetime with a set of Grassmann numbers at every point in spacetime. This is also called a Graded Minokowski spacetime and written as $R^{m,1|n}$ meaning $R(m,1)$ is m+1 dimensional Minokowski spacetime and n is the set of n grassmann numbers. We denote the space as $S(m+1|n) = R(m,1|n)$. Since the square of Grassmann number is zero a generic point in superspace can be written as-

The off diagonal terms give-

$$\begin{aligned} d_{11}d_{12} + d_{21}d_{22} + d_{31}d_{32} &= 0 \\ d_{11}d_{13} + d_{21}d_{23} + d_{31}d_{33} &= 0 \\ d_{12}d_{13} + d_{22}d_{23} + d_{32}d_{33} &= 0 \end{aligned} \quad (26)$$

We supply the ansatz in equations 21-23 into equation 24 this gives-

$$d_{11}d_{22}(i(\cos\theta_2 + \cos\theta_1) - \sin\theta_1 \sin\theta_2) = 0 \quad (27)$$

So

$$\cos\theta_1 + \cos\theta_2 = -i \sin\theta_1 \sin\theta_2 \quad (28)$$

Similarly -

$$\begin{aligned} \cos\theta_2 + \cos\theta_3 &= -i \sin\theta_2 \sin\theta_3 \\ \cos\theta_1 + \cos\theta_3 &= -i \sin\theta_1 \sin\theta_3 \end{aligned} \quad (29)$$

If all the three parameters are equal then we find $\cos\theta_1 = -i$ giving an imaginary boost again. Solving in general by elimination of these set of equations from 26-29 We find that the set of equations satisfy the quartic equation-

$$5z^4 + 2iz^3 - 6z^2 + 9 = 0$$

Here $z = e^{i\theta} = \cos(\theta) + i \sin(\theta)$

The 3 set of $\theta_1, \theta_2, \theta_3$ along with $\cos(\theta) = -i$ (when all 3 angles are equal) are the 4 sets of solution to the equation

above. If all 3 angles are equal, using these ansatz we find

$$D = \begin{bmatrix} d_{11} & id_{22}\cos\theta_2 & id_{33}\cos\theta_3 \\ id_{11}\cos\theta_1 & d_{22} & id_{33}\sin\theta_3 \\ id_{11}\sin(\theta_1) & id_{22}\sin\theta_2 & d_{33} \end{bmatrix} \quad (30)$$

We can rewrite this as-

If the three angles are equal then

$$D = \begin{bmatrix} d_{11} & d_{22} & d_{33} \\ -d_{11} & d_{22} & i\sqrt{2}d_{33} \\ i\sqrt{2}d_{11} & i\sqrt{2}d_{22} & d_{33} \end{bmatrix}$$

$$\begin{bmatrix} t' \\ x' \\ y' \end{bmatrix} = \begin{bmatrix} d_{11} & id_{22}\cos\theta_2 & id_{33}\cos\theta_3 \\ id_{11}\cos\theta_1 & d_{22} & id_{33}\sin\theta_3 \\ id_{11}\sin(\theta_1) & id_{22}\sin\theta_2 & d_{33} \end{bmatrix} \begin{bmatrix} t \\ x \\ y \end{bmatrix} \quad (31)$$

or

$$t' = d_{11}t + id_{22}\cos(\theta_2)x + id_{33}\cos(\theta_3)y \quad (32)$$

$$x' = id_{11}\cos(\theta_1)t + d_{22}x + id_{33}\sin(\theta_3)y \quad (33)$$

$$y' = id_{11}\sin(\theta_1)t + id_{22}\sin(\theta_2)x + d_{33}y \quad (34)$$

Also $t'^2 - x'^2 - y'^2 = 0$ and as the equation is true for any generic t, x, y, z implies following equations-

$$d_{11}^2 = d_{22}^2 = d_{33}^2 = 0 \quad (35)$$

$$d_{11}d_{22} + d_{22}d_{11} = 0 \quad (36)$$

$$d_{11}d_{33} + d_{33}d_{11} = 0 \quad (37)$$

$$d_{22}d_{33} + d_{33}d_{22} = 0 \quad (38)$$

This is identical to algebra we found in 2 Dimensions except we have 3 such pairs of Grassmann Variables satisfying the algebra as there are 3 pairs of elements. In 3+1 D Minowkowski spacetime we can do the same computation and find that there are 6 set of anticommuting Grassman algebra elements. This algebra is the graded odd part of Lie Superalgebra with elements as Grassman numbers and it seems to be a generic property of elements of Minokowski Spacetime in any dimensions. As the algebra is graded odd part of Lie Superalgebra this might also be related to Supersymmetry and Fermionic degrees of freedom.

Matrix representations The simplest representation of 3 Generators of Grassman algebra has following Matrix representation-

$$d_{11} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (39)$$

$$d_{22} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (40)$$

If We use this diffeomorphism group in a suitable coordinate chart between timelike and lightlike region we immediately arrive at a subtlety because $\det(D) \neq 0$. Thus the angles relate with boost parameter in a non trivial way with diffeomorphism group such that $\det(D)=0$ as the solution with same boosts is ruled out. This makes analyzing the algebra of the elements a non trivial task. As in equations 11 and 12 we rewrite the equation in matrix form as-

$$d_{33} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (41)$$

Thus the matrix D for 2+1 D Minowkowski spacetime contains 144 elements.

3.7. Implications of the Algebra for Causality

The ¹¹ algebra of operators we described has an interesting connection with causality in spacetime physics.

Propagation of Signals: In causal systems, information or disturbances propagate in specific directions constrained by the light cone. Nilpotency can represent transitions that are limited in causal extent to one way or non reversibility. In causal structures, certain processes (like crossing a black hole event horizon) are irreversible. Nilpotent operators, which "annihilate" themselves when squared, could symbolize the one-way nature of such transition. It is found that gravity stores information in Grassman degrees of freedom. The null hyper surface of Black Hole space time is a surface with Grassman coordinates. Since operating on the grassman number implies acting on grassman number by a grassman valued operator. In order to measure the operator eigenvalue I need to act on the state which squares the grassman coordinates resulting in zero. Zero implies loss of information. Thus classically we can never recover information that fell into a black hole any attempt to do so destroys the information

Anticommutation and Grassman structure: The anticommutation relations reflect the fermionic behavior commonly seen in quantum field theory and supersymmetry. These structures are significant for causality in several ways:

Spacetime Degrees of Freedom: Grassmann numbers often

¹¹We thank Chatgpt for providing us with these remarkable insights. This section is entirely the work of Chatgpt and we are deeply thankful to it

represent fermionic degrees of freedom implying spacetime might have structure of superspace or supergeometry. Fermionic degrees of freedom might play an essential role when investigating relativistic speed and might also be responsible for hiding inherent degrees of freedom. These fermionic coordinates might play a role in maintaining causal consistency when combining bosonic and fermionic sectors in theories of quantum gravity or supersymmetry.

3.8. Reconstruction of Diffeomorphism Group for Null Hypersurface in Schwarzschild Spacetime

Rather than adopting the conventional route of analytically extending the metric functional we assume the 4 vector along null hypersurface as η . We will use only one information that $\eta^T \eta = 0$ we will not assume any other information about this null hypersurface but for a 4 vector χ along timelike region owing to consistencies from experiments we assume the coordinate chart using χ in timelike region follows Raychaudhuri equations and geodesic equations. Raychaudhuri equation in a generic 4 dimensional spacetime is-

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{3} - \sigma^2 + \omega^2 - R_{\mu\nu}\xi^\mu\xi^\nu + \xi^\alpha{}_{;\alpha} \quad (42)$$

One might question if Einstein equations are valid near Black Hole Horizon given that the Diffeomorphism symmetry is absent. It was suggested in the paper [2] that Raychudhari equation can be derived independently as a Geometrical flow

equation. This will be our motivation in presuming that Raychudhari equation are more fundamental and are valid even when there is an absence of Diffeomorphism symmetry in Black Hole Spacetime. We assume validity of Raychaudhuri equation along Timelike region τ and along Null Hypersurface Σ and that the two regions are related by equation 1.

However we do not have explicit form for Shear. In the second paper [1] it is suggested that the shear potential in its equation 6 has an explicit dependence on volume of cross sectional surface. Our motive is to derive the explicit form of coordinate chart along null hypersurface using Raychaudhuri equation along Timelike and Null hypersurface and relate the two via equation 1 and using geodesic equation for timelike region for the 4 vectors we try deriving the coordinate chart for null hypersurface. We have an extra condition of zero norm of null hypersurface but we do not presume geodesic equation for null Hypersurface coordinates. In most scenarios Raychaudhuri equation is derived using derivative of velocity vector field as in equation 1 of [2]. We do not know how to make an equivalence between Raychudhari equation using Geometrical flow and deriving it using the approach of [2]. As suggested in the same paper after equation 14 Raychudhari equations are identities and not equations and therefore are valid even when Einstein equations owing to absence of diffeomorphism symmetry might themselves may not necessarily be valid. For notational simplicity we rewrite these identities suggested in the paper.

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{3} - \sigma^2 + \omega^2 - R_{\mu\nu}\xi^\mu\xi^\nu + \xi^\alpha{}_{;\alpha} \quad (43)$$

$$\frac{d\sigma_{ab}}{d\tau} = -\frac{2}{3}\theta\sigma_{ab} - \sigma_{ac}\sigma_b^c - \omega_{ac}\omega_b^c + \frac{1}{3}h_{ab}(\sigma^2 - \omega^2) + C_{cbad}v^c v^d + \frac{\tilde{R}_{ab}}{2} \quad (44)$$

$$\frac{d\omega_{ab}}{d\tau} = -\frac{2}{3}\theta\omega_{ab} - 2\sigma_{[b}^c\omega_{a]c} \quad (45)$$

Where $\sigma^2 = \sigma_{ab}\sigma^{ab}$ (σ_{ab} has Symmetric indices) $\omega^2 = \omega_{ab}\omega^{ab}$ (ω_{ab} has antisymmetric indices). C_{abcd} is the weyl tensor and

$$\tilde{R}_{ab} = h_{ac}h_{bd}R^{cd} - \frac{1}{3}h_{ab}h_{cd}R^{cd} \quad (46)$$

Here h_{ac} is the projection tensor. A projection tensor is a tensor orthogonal to 4 velocity vector. So it is defined as-

$$h_{ac} = g_{ac} - u_a u_c \quad (47)$$

where u_a is a timelike velocity vector and is normalised to 1. One may question how to find the timelike velocity vector we assume these can be solved using set of geodesic equations which are solution to Einstein equations. The 4 set of coupled differential equations are-

$$\frac{d^2 t}{d\tau^2} + \frac{\alpha}{\rho(\rho - \alpha)} \frac{dt}{d\tau} \frac{d\rho}{d\tau} = 0 \quad (48)$$

$$\rho'' + \frac{\alpha(\rho - \alpha)}{2\rho^3} (t')^2 - \frac{\alpha}{2\rho(\rho - \alpha)} (\rho')^2 - (\rho - \alpha)(\theta')^2 - (\rho - \alpha)\sin^2(\theta)(\phi')^2 = 0 \quad (49)$$

$$\theta'' + \frac{2}{\rho}\rho'\theta' - \sin\theta\cos\theta(\phi')^2 = 0 \quad (50)$$

$$\phi'' + \frac{2}{\rho}\rho'\phi' + 2\cot\theta\phi'\theta' = 0 \quad (51)$$

$u_a = \frac{dx_a}{d\tau}$ where x_a is one of the four components of the 4 set of geodesic equations. Interestingly we asked chatgpt to find solution to these set of equations and it gave us following solution-

$$u_t = \frac{E}{1 - 2M/r} \quad (52)$$

$$u_r^2 + (1 - 2M/r)(E^2 - (1 - 2M/r)u_\theta^2 - r^2u_\phi^2) = 0 \quad (53)$$

$$u_\theta = \frac{l_1}{r^2} \quad (54)$$

$$u_\phi = \frac{l_2}{r^2 \sin^2 \theta} \quad (55)$$

Similarly for Null Hypersurface the authors in [2] suggest in equation 17 the Raychaudhary equation-

$$\frac{d\theta'}{d\lambda} + \frac{\theta'^2}{2} + \sigma'^2 - \omega'^2 = -R_{ab}k^a k^b \quad (56)$$

Please note in equation 56 right hand side carries only symmetric terms in its indices so even if we assume metric to be asymmetric it is only the symmetric part that contributes in right hand side. This simplifies a lot of work for solving the equation in terms of symmetric indices. Also we will see that we need the equation for null hypersurface in 2 dimension so Ricci tensor simplifies significantly. Please note here the primed quantities are the expansion, rotation and shear for the null geodesic congruence. Like equations 44 and 45 we have analogous equation for σ' and ω' as tensorial equations. These are-

$$\frac{d\sigma'_{ab}}{d\lambda} = -\theta\sigma'_{ab} + C_{cbad}v^c v^d \quad (57)$$

$$\frac{d\omega'_{ab}}{d\lambda} = -\theta\omega'_{ab} \quad (58)$$

3.9. Interpretation of Expansion

In section 2 of book-A relativist toolkit by Prof. Eric Poisson it is mentioned that θ is fractional rate of change of area of hypersurface. We assume the argument so

$$\theta' = \frac{1}{A} \frac{dA}{d\lambda} \quad (59)$$

Here A is the area of Null Hypersurface. Similarly

$$\theta = \frac{1}{V} \frac{dV}{d\tau} \quad (60)$$

We do not yet know a way to find Area A and Volume V however suppose we consider foliation of a cauchy slice just at the instant a black hole is formed we know using the language of differential forms that

$$V = \int_{\tau} \sqrt{\det g} d\chi_1 \wedge d\chi_2 \wedge d\chi_3 \wedge d\chi_4 \quad (61)$$

Assuming foliation is along one of the coordinates say χ_4

we have

$$V = \int_{\tau} \sqrt{\det g} d\chi_1 \wedge d\chi_2 \wedge d\chi_3$$

Here the integration needs to be done over timelike region τ along the black hole spacetime outside the null hypersurface. Similarly in that slice we have

$$A = \int_{\Sigma} \sqrt{\det h} d\eta_1 \wedge d\eta_2 \wedge d\eta_3 \quad (62)$$

Again we have to integrate the null hypersurface region Σ along with the condition $\eta^T \eta = 0$. Assuming such a foliation can be done so as to eliminate one of the coordinates of η we have 2 parameters for integration of A. So the Area is a 2 dimensional hypersurface which makes the analysis of the induced metric in 2 dimensions much simpler. But we will not assume that null hypersurface metric is diagonal or symmetric in its indices hence there are 4 components for the metric. Also because spacetime outside black hole horizon satisfies vacuum Einstein equation so we have $R_{\mu\nu} = 0$ for such a Cauchy slice. However we notice immediately that we have presumed an infinitesimal ball shaped region hugging both timelike region τ and null hypersurface region Σ . This is presumed to happen along some Cauchy slice containing both the regions but as we start foliating such Cauchy slices we will find that there is a blueshift in future so the blueshift and greybody factors will back react at late times distorting our construction to reconstruct null hypersurface. It is also questionable if such diffeomorphisms even exist or not as in section 3.4 and in equation 21 we found that the diffeomorphism group could have been zero also. In that scenario we assumed it to be non zero as per equation 21 and this lead us to the notion of superspace. We also need to know the induced metric determinants g and h. Hopefully we know the metric along timelike region τ as Schwarzschild metric owing to experimental observations however we can not say so for null region σ . So we can proceed to link the set of equations derived in section 3.8 of the paper with coordinate charts along timelike region and null hypersurface and then link them using equation 1. This will help us to find Null hypersurface modulo one more condition/set of conditions. We assume the condition is smooth variation of coordinates as we move along a neighbouring region from timelike surface to null hypersurface. This can be stated as the condition that timelike 4 vector motivated in equation 1 χ as displaced along a neighbouring region along null hypersurface Σ is same as η . Thus

$$\chi|_{\Sigma} = \eta \quad (63)$$

Or it means that as the radius of diffeomorphism group approaches zero the matrix D in equation 1 becomes identical to identity. For small deformations we assume the diffeomorphism group can be expanded via its set of generators close to identity. This will provide us the algebra of the diffeomorphism symmetry group. Hopefully the set of equations from 43 to 62 can be assumed to be sufficiently able to develop the coordinate chart for smoothly reconstructing

the null hypersurface of Black Hole spacetime. *Until now we have only supplied geometrical identities and not an equation, we attempt to provide a consistently complete set of solutions by providing few more set of equations known as Gauss-Codazzi equation and Israel Junction condition as suggested in Chapter 3 of Eric Poisson which ensures smoothness of Horizon. These equations can be taken directly from Eric Poisson textbook.*

Because a null hypersurface has zero norm a manifold with an oriented normal along timelike region needs to change its sign necessarily. In the language of fibre bundles this means that if fibres of some base manifold which admits a suitable foliation necessarily travels back in time as it passes from timelike hypersurface to null hypersurface. But proving this statement rigorously requires deep theorems and results from Riemannian geometry and Differential geometry but certainly this process does not seem to indicate more subtle gravitational effects like Topology change which is captured using De Rham Cohomology. However just to make some progress we now attempt to solve set of equations from 43 to 62 using an assumption in equations 61 and 62 that the change in expansion along timelike region and null hypersurface is carried only as a function of metric so that while taking derivative of volume V and area A the change is carried only as function of metric. In particular we use these 2 more sets of equations that simplify 59 and 60 to -

$$\theta' = \frac{1}{\sqrt{h}} \left(\frac{d(\sqrt{deth})}{d\lambda} \right) \quad (64)$$

$$\theta = \frac{1}{\sqrt{g}} \left(\frac{det\sqrt{g}}{d\tau} \right) \quad (65)$$

For simplicity we write the null metric without indices being symmetric in 2 dimensions as-

$$h_{\mu\nu} = \begin{bmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{bmatrix} \quad (66)$$

Similarly the inverse metric has the form-

$$h^{\mu\nu} = \frac{1}{deth} \begin{bmatrix} h_{11} & -h_{01} \\ -h_{10} & h_{00} \end{bmatrix} \quad (67)$$

where $deth = h_{00}h_{11} - h_{01}h_{10}$

In 2 dimensions we find

$$\theta' = \frac{1}{\sqrt{deth}} \frac{d(\sqrt{deth})}{d\lambda}$$

So

$$\begin{aligned} \theta' &= \frac{1}{2(h_{00}h_{11} - h_{01}h_{10})} \frac{d(deth)}{d\lambda} \\ &= \frac{1}{2deth} \frac{d(deth)}{d\lambda} \\ &= \frac{1}{2} \frac{d}{d\lambda} \ln(deth) \end{aligned} \quad (68)$$

Also in 2 dimensions in presence of asymmetry of metric or

torsion we have the expression only for symmetric indices here owing to symmetry of right hand side of equation 56. Also in 2d the expression simplifies so -

$$R_{\mu\nu} = \frac{R}{2} h_{\mu\nu} \quad (69)$$

Asymmetric metric implies an asymmetric line element which indicates that if tensor analysis for line element is valid then the event horizon has non commutative nature though we do not claim it here because we do not assume that metric always is a fundamental arena for taking inner product for computing line element. It is only in Minokowski spacetime we presume it is true. Also in equation 56 $k^a = \frac{dx^a}{d\lambda}$ where x^a are coordinates on null hypersurface. Thus equation 56 simplifies to-

$$\frac{d\theta'}{d\lambda} + \frac{\theta'^2}{2} + \sigma'^2 - \omega'^2 = -(R/2)h_{ab}k^ak^b \quad (70)$$

where θ' is as suggested in equation 68. In 2 and 3 dimensions we have another simplification the weyl tensor vanishes so equation 58 becomes-

$$\frac{d\sigma'_{ab}}{d\lambda} = -\theta\sigma'_{ab} \quad (71)$$

$$\frac{d}{d\lambda} \ln(\sigma'_{ab}) = -\theta$$

and

$$\frac{d\omega'_{ab}}{d\lambda} = -\theta\omega'_{ab} \quad (72)$$

To proceed further we use bit of tensor analysis.

We multiply equation 71 by σ'^{ab} so we find-

$$\begin{aligned} \sigma'^{ab} \frac{d\sigma'_{ab}}{d\lambda} &= -\theta\sigma'^{ab}\sigma'_{ab} = -\theta\sigma'^2 \\ &= \frac{d(\sigma'^{ab}\sigma'_{ab})}{d\lambda} - \sigma'_{ab} \frac{d\sigma'^{ab}}{d\lambda} \end{aligned} \quad (73)$$

So

$$\sigma'_{ab} \frac{d\sigma'^{ab}}{d\lambda} = \frac{d\sigma'^2}{d\lambda} + \theta\sigma'^2$$

Multiplying above equation by σ'^{ab} we find-

This simplifies to-

$$\frac{d\sigma'^{ab}}{d\lambda} = \sigma'^{ab} \left(\theta + \frac{1}{\sigma'^2} \frac{d\sigma'^2}{d\lambda} \right) \quad (74)$$

Equation 74 can be written as-

$$\frac{d}{d\lambda} \ln\sigma'^{ab} = \left(\theta + \frac{1}{\sigma'^2} \frac{d\sigma'^2}{d\lambda} \right) \quad (75)$$

From 75 and 73 we see triviality relation for σ . We can argue similar triviality relation for ω and using dimensional analysis in 70 we see that σ'^2 scales as $\frac{1}{\lambda^2}$ by the symmetry of equations for ω' with σ' we see similar scaling for ω'^2 . Please note that

we are not resorting to tensor analysis so the best we can find is triviality relations for σ and ω . We reduce equation 70 in the form-

$$\frac{d\theta'}{d\lambda} + \frac{\theta'^2}{2} + \frac{\alpha}{\lambda^2} = -(R/2)h_{ab}k^ak^b \quad (76)$$

Here α is determined by gluing conditions at the timelike and null hypersurface. Thus alpha eliminates equations for σ and ω . We substitute 68 into 76 and find the differential in terms of metric has the form-

$$\frac{1}{(2deth)} \frac{d^2(deth)}{d\lambda^2} - \frac{3}{8(deth)^2} \left(\frac{d(deth)}{d\lambda}\right)^2 + \frac{\alpha}{\lambda^2} = -(R/2)h_{ab}k^ak^b \quad (77)$$

In 2 dimensions we can simplify the expression for Ricci scalar and we find it in terms of metric and its derivatives as-

$$R_\Sigma = -\frac{1}{deth} \frac{d^2deth}{d\lambda^2} + \frac{1}{2(deth)^2} \left(\frac{d(deth)}{d\lambda}\right)^2 \quad (78)$$

Substituting for R from equation 78 into 77 we find the equation has the form-

$$\frac{1}{deth} \left(2 - \frac{1}{2}h_{ab}k^ak^b\right) \frac{d^2deth}{d\lambda^2} + \frac{1}{(deth)^2} \left(-\frac{3}{8} + \frac{1}{4}h_{ab}k^ak^b\right) \left(\frac{d(deth)}{d\lambda}\right)^2 + \frac{\alpha}{\lambda^2} = 0 \quad (79)$$

3.9.1. Gauss Codazzi Equation and Extrinsic Curvature

We see that we need to find for α and induced metric h on null hypersurface. To proceed further we impose Gauss Codazzi equation¹² which relates Ricci scalar of 4 dimensional spacetime to Ricci scalar on its hypersurface Σ , R_Σ via extrinsic curvature on hypersurface and trace of extrinsic curvature as-

$$R = R_\Sigma + K_{ab}K^{ab} - K^2 \quad (80)$$

In 4 D spacetime or any of its foliation¹³ not enclosing null hypersurface we see or assume $R=0$. So

$$R_\Sigma + K_{ab}K^{ab} - K^2 = 0 \quad (81)$$

3.9.2. Extrinsic Curvature for Null Hypersurface

Extrinsic curvature mentioned in equation 80 is a measure of how much the hypersurface curves in the ambient spacetime. If the hypersurface is flat we expect extrinsic curvature to be zero. For null hypersurface thus it is a good measure to capture the geometry of null hypersurface which aligns with our aim to capture its geometrical properties.

Formal definition Given the induced metric h_{ab} on our hypersurface we define the extrinsic curvature as-

$$K_{ab} = h_a^c h_b^d \nabla_c n_d \quad (82)$$

Thus it measures in some sense the component of derivative of normal vector along the direction of induced metric. Here ∇_c is the covariant derivative with reference to 4 d Spacetime. The equation has 4 dimensional covariant derivative because extrinsic curvature is a measure of how the normal vector changes in 4 dimensional ambient spacetime, please note that given a hypersurface we always have a normal vector along the gradient of hypersurface. A precise reasoning of the definition can be found in Eric Poisson textbook. Using equation 82 we proceed to with equation 81 of the paper. Please note that K^{ab} is defined as-

$$K^{ab} = h_a^c h_b^d \nabla_c n^d \quad (83)$$

$$K = tr(K_{ab}) = tr(h_a^c h_b^d \nabla_c n_d) = h_a^c h_b^c \nabla_c n^b \quad (84)$$

Substituting equations 82, 83 and 84 into equation 81 we find the following equation-

$$R_\Sigma + h_a^c h_b^d \nabla_c n_d h_e^a h_f^b \nabla^e n^f - h_a^c h_b^c \nabla_c n^d h_a^d h_b^d \nabla_d n_d = 0 \quad (85)$$

For simplicity we write the equation in terms of lower components of metric so it has the form-

$$R_\Sigma + h_{ac} h_{bd} \nabla^c n^d h^{ae} h^{bf} \nabla_e n_f - h_{ac} h_{bc} \nabla^c n^d h^{da} h^{db} \nabla_d n^d = 0 \quad (86)$$

We use equation 67 which is-

$$h^{\mu\nu} = \frac{1}{deth} \begin{bmatrix} h_{11} & -h_{01} \\ -h_{10} & h_{00} \end{bmatrix}$$

We noticed that raising and lowering indices in terms of tensor is consistent with taking inverses but raising and lowering with reference to metric is not a justified procedure.

To proceed further we use tensor contraction and find-

$$R_\Sigma + \nabla_c n_d \nabla^c n^d - (\nabla_c n^c)^2 = 0 \quad (87)$$

Further simplification Above equation can be rewritten as-

$$R_\Sigma = -(|\nabla n|^2) + (divn)^2 \quad (88)$$

¹²This equation can be found in Sir Eric Poisson textbook, we searched it in chatgpt and we are deeply obliged to chatgpt for the same:).

¹³Please note that Ricci scalar R is a geoemtrical property of Spacetime independent of how we foliate the spacetime so the equation $R=0$ is always valid.

Please note that $|\nabla n|$ is a tensor here and norm is taken by taking modulus of this tensor. This equation explicitly relates the intrinsic curvature of the hypersurface to the normal's gradient behavior in the ambient spacetime.

Physical meaning If $R_\Sigma = 0$ then the expansion and distortion of the hypersurface (measured by divergence of normal vector) exactly balance each other.

If the normal is geodesic i.e $\nabla_c n_d = 0$ then the hypersurface is flat which is analog of Local flatness theorem or locally Minkowski spacetime hypersurfaces. We thus have 2 equations to solve for null hypersurface equation 88 and equation 79.

Computing normal vector The hypersurface of Schwarzschild spacetime can be assumed to be all region in exterior of event

horizon. Assuming such a hypersurface has coordinates η then the hypersurface is given by equation

$$\eta^T \eta = 0 \quad (89)$$

We parametrise such surface with coordinates $\eta_0, \eta_1, \eta_2, \eta_3$ then we get the null equation 89 as-

$$\eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 = 0 \quad (90)$$

We can parametrise such surface as- $\eta = \eta_0(1, \cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$. A possible foliation of such hypersurface can be constant η_0 slices. The gradient tensor or transformation matrix with respect to coordinates η_0, θ, ϕ has the form-

$$T = \begin{bmatrix} 1 & \cos\theta\cos\phi & \cos\theta\sin\phi & \sin\theta \\ 0 & 0 & 0 & 0 \\ 0 & -\eta_0\sin\theta\cos\phi & -\eta_0\sin\theta\sin\phi & \eta_0\cos\theta \\ 0 & -\eta_0\cos\theta\cos\phi & \eta_0\cos\theta\sin\phi & 0 \end{bmatrix} \quad (91)$$

Please note that since there are three parametric coordinates so one of the row is entirely zero (2nd row). Then the normal vector is $n = T^{\leftrightarrow} \cdot \eta$ so we find

$$n = \begin{bmatrix} 1 & \cos\theta\cos\phi & \cos\theta\sin\phi & \sin\theta \\ 0 & 0 & 0 & 0 \\ 0 & -\eta_0\sin\theta\cos\phi & -\eta_0\sin\theta\sin\phi & \eta_0\cos\theta \\ 0 & -\eta_0\cos\theta\cos\phi & \eta_0\cos\theta\sin\phi & 0 \end{bmatrix} \times \begin{bmatrix} \eta_0 \\ \eta_0\cos\theta\cos\phi \\ \eta_0\cos\theta\sin\phi \\ \eta_0\sin\theta \end{bmatrix} \quad (92)$$

Please note that since the magnitude of $\eta = 0$ so we did not normalise the above equation. We find n as-

$$n = \begin{bmatrix} 2\eta_0 \\ 0 \\ 0 \\ \eta_0^2\cos^2\theta\cos 2\phi \end{bmatrix} \quad (93)$$

Using equation 93 we can compute norm of gradient of n and divergence of n using the transformation matrices. The transformation matrix or gradient matrix T' for n is-

$$T'(n) = \begin{bmatrix} 2 & 0 & 0 & 2\eta_0\cos^2\theta\cos 2\phi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\eta_0^2\sin 2\theta\cos 2\phi \\ 0 & 0 & 0 & -2\eta_0^2\cos^2\theta\sin 2\phi \end{bmatrix} \quad (94)$$

We know that $\nabla n = T'(n)$, hence $(\nabla n)^2 = \text{Tr}(T'(n)^2)$ Simplifying we find-

$$|\nabla n|^2 = 4 + 4\eta_0^2\cos^4\theta\cos^4\phi + 4\eta_0^4(\cos^4\theta\sin^4\phi + \sin^4\theta\cos^4\phi) \quad (95)$$

Using 93 we find $\text{div}(n) = 2 - 2\eta_0^2\cos^2\theta\sin 2\phi$ So we find

$$\text{div}^2(n) = 4 + 4\eta_0^4\cos^4\theta\sin^2 2\phi - 8\eta_0^2\cos^2\theta\sin 2\phi \quad (96)$$

Substituting 95 and 96 into 88 we find R_Σ as-

$$R_\Sigma = -4\eta_0^2\cos^2\theta(\cos^2\theta\cos^4\phi + 2\sin^2\phi) - 4\eta_0^4(\cos^4\theta\sin^4\phi + \sin^4\theta\cos^4\phi - \cos^4\theta\sin^2\phi) \quad (97)$$

Please note that in any coordinate chart there is no isometry in the Ricci Scalar along null hypersurface as expected from our intuition and also in general the surface has negative Ricci

scalar. Also along constant time slices we can assume η_0 to depend on Quasi local mass measured at the Black Hole horizon or at infinity if the manifold is non compact or at finite

η_0 if the manifold is compact. It is still yet to be verified if the Manifold satisfies Grassmannian structure as eluded in Minkowski case. We also have an equation for R_Σ as a

differential equation for the metric in equation 78. Equating the two equations we find-

$$-\frac{1}{deth} \frac{d^2 deth}{d\lambda^2} + \frac{1}{2(deth)^2} \left(\frac{d(deth)}{d\lambda} \right)^2 = -4\eta_0^2 \cos^2 \theta (\cos^2 \theta \cos^4 \phi + 2\sin^2 \phi) - 4\eta_0^4 (\cos^4 \theta \sin^4 \phi + \sin^4 \theta \cos^4 \phi - \cos^4 \theta \sin^2 \phi) \quad (98)$$

We also have equation 79 in our hand to solve for the metric. We rewrite that equation here for simplicity-

$$\frac{1}{deth} \left(2 - \frac{1}{2} h_{ab} k^a k^b \right) \frac{d^2 deth}{d\lambda^2} + \frac{1}{(deth)^2} \left(-\frac{3}{8} + \frac{1}{4} h_{ab} k^a k^b \right) \left(\frac{d(deth)}{d\lambda} \right)^2 + \frac{\alpha}{\lambda^2} = 0$$

Equation 79 can be simplified as it has a scaling symmetry we assume $deth = A\lambda^n$ then the left hand side of equation 98 has the form-

$$-n(n-1) + \frac{n^2}{2} = -n\left(\frac{n}{2} - 1\right) = -4\eta_0^2 \cos^2 \theta (\cos^2 \theta \cos^4 \phi + 2\sin^2 \phi) - 4\eta_0^4 (\cos^4 \theta \sin^4 \phi + \sin^4 \theta \cos^4 \phi - \cos^4 \theta \sin^2 \phi) \quad (99)$$

Assuming given solution exists for det h we find that Ricci scalar is always negative if $n(n+2) > 0$.

These ansatz also simplify equation 79 as-

$$\left(2 - \frac{1}{2} h_{ab} k^a k^b \right) n(n-1) + \left(-\frac{3}{8} + \frac{1}{4} h_{ab} k^a k^b \right) n^2 + \alpha = 0 \quad (100)$$

This can be grouped as-

$$\frac{13}{8} n^2 - 2n - \frac{h_{ab} k^a k^b}{2} n \left(\frac{n}{2} + 1 \right) + \alpha = 0 \quad (101)$$

Equations 99 and 101 can be useful to compute the metric and relevant quantities for null hypersurfaces. Parameter α is determined using gluing conditions for null hypersurface and timelike region. We still need to find the null Diffeomorphism matrix D.

such example of such a space is Black Hole Interior¹⁵ and Ads- Space. The Causal structure of spacetime is always a Hyperbolic manifold in curved spaces.

3.10. Geometrical Significance of Ricci Scalar R_Σ

The Ricci scalar R encodes information about volume changes, focusing of geodesics, and curvature effects in a given spacetime or hypersurface. The sign of R has direct geometric and physical interpretations-

Ricci scalar tells us how volume changes in curved space. Given an infinitesimal volume V, its rate of expansion is related to R as¹⁴-

$$\frac{d^2 V}{d\tau^2} = -R_{\mu\nu} \xi^\mu \xi^\nu V + \frac{2}{3} \theta^2 V \quad (102)$$

For simplicity we assume the background is weakly curved so that second term in equation 102 is infinitesimal with reference to first term. Then we have following implications-

Implications for Positive R

If $R > 0$ then volume contracts as geodesics converge. Also manifolds with positive Ricci scalar tend to be compact. One example of such space is De-Sitter Space.

Implications for negative R

If $R < 0$ volume expands as geodesic diverge. One

3.11. Derivation of Volume Change Equation Using Raychaudhuri Equation

We restate Raychaudhuri equation in 4 dimensional spacetime here-

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{3} - \sigma^2 + \omega^2 - R_{\mu\nu} \xi^\mu \xi^\nu$$

We note

$$\frac{dV}{d\tau} = \theta V \quad (103)$$

Taking another derivative and substituting equation 103 we find-

$$\frac{d^2 V}{d\tau^2} = V \frac{d\theta}{d\tau} + \theta^2 V \quad (104)$$

To proceed further we assume irrotational and shear free spacetime so $\sigma = \omega = 0$ ¹⁶ With this assumption Raychaudhuri equation takes the form-

$$\frac{d^2 V}{d\tau^2} = -R_{\mu\nu} \xi^\mu \xi^\nu V + \frac{2}{3} \theta^2 V \quad (105)$$

¹⁴We will derive this equation in next section using Raychaudhuri equations

¹⁵This is because the interior of Black Hole always grows in size and as per equation 102 this can only happen when R is less than zero

¹⁶We saw that these terms add a scaling constant to the equation earlier so they might change the equation only by constant terms, however given appropriate boundary conditions we can still solve the equations with the terms.

Using 3.103 this simplifies to-

$$\frac{d^2V}{d\tau^2} = -R_{\mu\nu}\xi^\mu\xi^\nu V + \frac{2}{3V}\left(\frac{dV}{d\tau}\right)^2 \quad (106)$$

3.12. Solving for Timelike Region

To find the Diffeomorphism matrix D we need to solve for timelike region. To solve for timelike region for the coordinates we need to solve for set of equations from 43 to 61

and provide the smoothness condition as suggested in equation 63. However the crux of all is in equations 52-55 which needs to be integrated and solved to find explicit dependence of coordinates in terms of parameter τ .

Proceeding with solution

To simplify we notice that equations can be solved via division of parameter tau in equation for u_r and u_θ . We find that we can solve for equations in terms of $\frac{dr}{d\theta}$. Using 53 and 54 and substituting for u_θ and u_ϕ from 54 and 55 we find-

$$\frac{dr}{d\theta} = \frac{r^2}{l_1} \sqrt{(1 - 2M/r)((1 - 2m/r)\frac{l_1^2}{r^4} + \frac{l_2^2}{r^2 \sin^2\theta} - E^2)} \quad (107)$$

This is differential equation for $r(\theta)$ which can be solved using elliptic ¹⁷ integrals. To solve the above equation we introduce a variable $u = \frac{1}{r}$ so that $\frac{dr}{d\theta} = \frac{-1}{u^2} \frac{du}{d\theta}$.

Rewriting equation in terms of u we find-

$$\frac{du}{d\theta} = -\frac{u^2}{l_1} \sqrt{(1 - 2Mu)((1 - 2Mu)\frac{l_1^2}{r^4} + \frac{l_2^2 u^2}{\sin^2\theta} - E^2)} \quad (108)$$

Change of variables for elliptic integrals form

We define a new variable $z = \frac{u}{u_{max}}$ where u_{max} is turning point of orbit determined from effective potential conditions. However this procedure though computable is too cumbersome, so we make an approximation. Because we need

to find solution near $r=2m$ we assume a perturbation parameter $r' = r - 2m$ and expand perturbatively these set of equations. We asked elegant chatgpt to compute the equations and it gave us the following procedure by approximating radial equation perturbatively. The radial equation is-

$$\left(\frac{dr}{d\tau}\right)^2 + (1 - 2M/r)(E^2 - (1 - 2M/r)\frac{l_1^2}{r^4} - \frac{l_2^2}{r^2 \sin^2\theta}) = 0 \quad (109)$$

Substituting $r' = r - 2M$ we find the differential equation has the form $1 - 2M/r \approx r'/2M + o(r'^2)$ Approximating $r^2 \approx (2M)^2$ and $r^4 \approx (2M)^4$. Using this approximation chatgpt gave us following solutions-

$$r(\tau) = 2M + \frac{\kappa^2}{8M}(\tau - \tau_0)^2 \quad (110)$$

$$t(\tau) = t_0 - \frac{16M^2 E}{\kappa^2(\tau - \tau_0)} \quad (111)$$

$$\theta(\tau) = \theta_0 + \frac{l_1}{4M^2}(\tau - \tau_0) \quad (112)$$

$$\phi(\tau) = \phi_0 + \frac{l_2}{4M^2 \sin^2\theta} \quad (113)$$

Here $\kappa^2 = E^2 - \frac{l_2^2}{4M^2 \sin^2\theta}$. These set of solutions describe the near horizon motion in Schwarzschild geometry. However we need to ensure that Perturbation theory for the solution is valid, this happens when $r'(\tau)$ is almost near to $2M$. From equation 110 we find this happens when

$$\frac{\kappa^2}{8M}(\tau - \tau_0)^2 \ll 2M \quad (114)$$

Or

$$\tau - \tau_0 \ll \frac{4M}{\kappa} \quad (115)$$

So the timerange within which our solution remains valid is when equation 115 is satisfied. Here τ_0 is assumed to be timescale when we start observing our system. This is alternative way of suggesting that there is distortion of timelike region near the horizon in a way that nearby points can either get repelled by angular momentum or may fall inside the horizon but can not remain stationary near the horizon for long time. Exact analysis requires solving the elliptic integral expressions which we have not mentioned here for simplicity. If τ remains within this range then our solution is valid. If E is large κ grows shortening the time interval before the perturbation grows too large. If l_2 is large we can see that it reduced κ keeping the perturbation smaller for longer duration. This is like spinning the particle causes it to hover near the horizon escaping gravitational pull. So now we have the coordinates for Timelike region in terms of τ .

3.12.1. Null Diffeomorphism Group for Schwarzschild Spacetime in Near Horizon Approximation for Timelike Region

Given set of equations from 110-113 and for timelike region as parametrized in section by equation 90 we have the following equation for Diffeomorphism symmetry matrix

¹⁷We are very thankful to chatgpt for helping us entirely with it.

$$\begin{bmatrix} \eta_0 \\ \eta_0 \cos \theta_1 \cos \phi_1 \\ \eta_0 \cos \theta_1 \sin \phi_1 \\ \eta_0 \sin \theta_1 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix} \times \begin{bmatrix} t_0 - \frac{16M^2 E}{\kappa^2(\tau - \tau_0)} \\ 2M + \frac{\kappa^2}{8M}(\tau - \tau_0)^2 \\ \theta_0 + \frac{l_1}{4M^2}(\tau - \tau_0) \\ \phi_0 + \frac{l_2}{4M^2 \sin^2 \theta} \end{bmatrix} \quad (116)$$

Along with the condition that $\text{Det}(D) = 0$. With the parametrisation used in equations 32-34 we can write the above equation as-

$$\begin{bmatrix} \eta_0 \\ \eta_0 \cos \alpha_1 \cos \beta_1 \\ \eta_0 \cos \alpha_1 \sin \beta_1 \\ \eta_0 \sin \alpha_1 \end{bmatrix} = \begin{bmatrix} d_{11} & id_{22} \cos \theta_2 \cos \phi_2 & id_{33} \cos \theta_3 \cos \phi_3 & id_{44} \cos \theta_4 \cos \phi_4 \\ id_{11} \cos \theta_1 \cos \phi_1 & d_{22} & id_{33} \cos \theta_3 \sin \phi_3 & id_{44} \cos \theta_4 \sin \phi_4 \\ id_{11} \cos \theta_1 \sin \phi_1 & id_{22} \cos \theta_2 \cos \phi_2 & d_{33} & id_{44} \sin \theta_4 \\ id_{11} \sin \theta_1 & id_{22} \sin \theta_2 & id_{33} \sin \theta_3 & d_{44} \end{bmatrix} \begin{bmatrix} t_0 - \frac{16M^2 E}{\kappa^2(\tau - \tau_0)} \\ 2M + \frac{\kappa^2}{8M}(\tau - \tau_0)^2 \\ \theta_0 + \frac{l_1}{4M^2}(\tau - \tau_0) \\ \phi_0 + \frac{l_2}{4M^2 \sin^2 \theta} \end{bmatrix} \quad (117)$$

Since the equation for $\eta^T \eta = 0$ is universal and provided the 4 set of components of timelike vector are non zero and labelled separately as distinct 4 vectors, we find these coefficients vanish again so for suitable parametrisation the algebra is still true and same as that of Minkowski spacetime. This algebra is *Grassman algebra and it is called Gr(0,4)*.

3.12.2. Implications of Universality of Grassman Algebra

Since Gr(0,4) is a generic algebra of Null Diffeomorphism symmetry group elements it hints that null hypersurface contains fermionic degrees of freedom. Combined with coordinates or diffeomorphisms this implies a deep superspace structure to General relativity with or without supersymmetry. These hidden degrees of freedom may modify Black Hole Entropy counting and Hawking Radiation. This is in alignment with String Theory which suggests that spacetime has a graded superspace structure. Also the 4 grassman elements correspond to minimal supersymmetry. In D dimensions of spacetime the Null Diffeomorphism elements carry the Grassman algebra Gr(0,D) which seems to be a strong relation. As twistor theory formulates General relativity in terms of Spinors so Twistor theory may be a good way to reformulate General relativity using Spinor Formalism as suggested by the algebra Gr(0,4). Since the algebra is universal with the dimensionality of spacetime it has strong implications for known models of spacetime in Quantum gravity. As a last step our work so far needs to shed light on Null Hypersurface of Black Holes which seems to have fermionic degrees of freedom. This is our next step to proceed we substitute equation 117 into 97 to find Ricci scalar R_Σ along Null Hypersurface. First thing to note that Grassman Algebra has real elements *so the Null Hypersurface is a complex manifold*.

A matrix representation for 4 sets of generators is-

$$d_{11} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (118)$$

$$d_{22} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (119)$$

$$d_{33} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (120)$$

$$d_{44} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (121)$$

Equation 117 is cumbersome to proceed with the analysis we systematically rewrite equation 117 using new coordinates labelling each element in the timelike region as X_0, X_1, X_2, X_3 . Thus

$$X_0 = t_0 - \frac{16M^2 E}{\kappa^2(\tau - \tau_0)} \quad (122)$$

$$X_1 = 2M + \frac{\kappa^2}{8M}(\tau - \tau_0)^2 \quad (123)$$

$$X_2 = \theta_0 + \frac{l_1}{4M^2}(\tau - \tau_0) \quad (124)$$

$$X_3 = \phi_0 + \frac{l_2}{4M^2 \sin^2 \theta} \quad (125)$$

Equation 117 then becomes-

$$\begin{bmatrix} \eta_0 \\ \eta_0 \cos \theta \cos \phi \\ \eta_0 \cos \theta \sin \phi \\ \eta_0 \sin \theta \end{bmatrix} = \begin{bmatrix} d_{11} & id_{22} \cos \theta_2 \cos \phi_2 & id_{33} \cos \theta_3 \cos \phi_3 & id_{44} \cos \theta_4 \cos \phi_4 \\ id_{11} \cos \theta_1 \cos \phi_1 & d_{22} & id_{33} \cos \theta_3 \sin \phi_3 & id_{44} \cos \theta_4 \sin \phi_4 \\ id_{11} \cos \theta_1 \sin \phi_1 & id_{22} \cos \theta_2 \cos \phi_2 & d_{33} & id_{44} \sin \theta_4 \\ id_{11} \sin \theta_1 & id_{22} \sin \theta_2 & id_{33} \sin \theta_3 & d_{44} \end{bmatrix} \times \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad (126)$$

Given equation 126 we notice that Null Hypersurface is also a complex Superspace element and that antiperiodicity of grassmann elements and there squaring to zero implies that $\eta_0^2 = 0$. This is because from equation 126 we see that η_0^2 collects terms d_{ii}^2 and terms like $d_{ii}d_{jj} + d_{jj}d_{ii}$. These follow from the way the elements are squared and both of them are zero, the second one following from antisymmetry of grassman elements. Hence the conclusion follows.

Using equation 126 we can substitute for coordinates along null hypersurface. Notice that though $\eta_0^2 = 0$ hence expression like $\cos^2\theta$ also diverge to infinity because $\cos(\theta)$ goes inversely with η_0 hence $\cos^2\theta$ diverges because $\eta^0 = 0$. But what we want in equation 97 is the combined expression $(\eta_0 \cos(\theta))^2$. We can make similar arguments for terms in fourth powers in equation 97. Though some of the coordinates

square to zero and some to infinity as per equation 126 suggesting pathological coordinates but the combined terms can be substituted by taking limits appropriately and we find that the Ricci scalar for null hypersurface R_Σ as given in equation 97 equals zero modulo the grassmann numbers that are of the 0/0 form in numerator and denominator. To justify the arguments we write down the expressions. A naive guess from equation 97 suggests that Ricci scalar R_Σ is entirely a function of null coordinates which are grassman variables and the expression for Ricci Scalar as in equation 97 involves terms which are square or higher powers of these variables on null hypersurface, hence it reduces to zero, however we will find that there are limits which do not tend to zero. We proceed with writing the grassmann variables dependences.

$$\cos \theta = \frac{(id_{11} \cos \theta_1 \cos \phi_1 X_0 + d_{22} X_1 + id_{33} \cos \theta_3 \sin \phi_3 X_2 + id_{44} \cos \theta_4 \sin \phi_4 X_3) \sec \phi}{d_{11} X_0 + id_{22} \cos \theta_2 \cos \phi_2 X_1 + id_{33} \cos \theta_3 \cos \phi_3 X_2 + id_{44} \cos \theta_4 \cos \phi_4 X_3} \quad (127)$$

$$\sin \theta = \frac{id_{11} X_0 + id_{22} \sin \theta_2 X_1 + id_{33} \sin \theta_3 X_2 + d_{44} X_3}{d_{11} X_0 + id_{22} \cos \theta_2 \cos \phi_2 X_1 + id_{33} \cos \theta_3 \cos \phi_3 X_2 + id_{44} \cos \theta_4 \cos \phi_4 X_3} \quad (128)$$

$$\cos \phi = \frac{id_{11} \cos \theta_1 \cos \phi_1 X_0 + d_{22} X_1 + id_{33} \cos \theta_3 \sin \phi_3 X_2 + id_{44} \cos \theta_4 \sin \phi_4 X_3}{\eta_0 \cos \theta} \quad (129)$$

$$\sin^2 \phi = 1 - \left(\frac{id_{11} \cos \theta_1 \cos \phi_1 X_0 + d_{22} X_1 + id_{33} \cos \theta_3 \sin \phi_3 X_2 + id_{44} \cos \theta_4 \sin \phi_4 X_3}{\eta_0 \cos \theta} \right)^2 \quad (130)$$

We define again following variables for brevity-

$$\mathcal{N}_0 = d_{11} X_0 + id_{22} \cos \theta_2 \cos \phi_2 X_1 + id_{33} \cos \theta_3 \cos \phi_3 X_2 + id_{44} \cos \theta_4 \cos \phi_4 X_3 \quad (131)$$

$$\mathcal{C}_\theta = id_{11} \cos \theta_1 \cos \phi_1 X_0 + d_{22} X_1 + id_{33} \cos \theta_3 \sin \phi_3 X_2 + id_{44} \cos \theta_4 \sin \phi_4 X_3 \quad (132)$$

$$\mathcal{D}_\theta = id_{11} \cos \theta_1 \sin \phi_1 X_0 + id_{22} \cos \theta_2 \cos \phi_2 X_1 + d_{33} X_2 + id_{44} \sin \theta_4 X_3 \quad (133)$$

$$\mathcal{S}_\theta = id_{11} \sin \theta_1 X_0 + id_{22} \sin \theta_2 X_1 + id_{33} \sin \theta_3 X_2 + d_{44} X_3 \quad (134)$$

Then:

$$\eta_0 = \mathcal{N}_0, \quad \cos \theta \cos \phi = \frac{\mathcal{C}_\theta}{\mathcal{N}_0}, \quad \cos \theta \sin \phi = \frac{\mathcal{D}_\theta}{\mathcal{N}_0}, \quad \sin \theta = \frac{\mathcal{S}_\theta}{\mathcal{N}_0} \quad (135)$$

Substituting for R_Σ we find:

$${}^{18}R_\Sigma = -4 \left(\frac{\mathcal{C}_\theta^2 + \mathcal{D}_\theta^2}{\mathcal{N}_0^2} \right) \left(\frac{\mathcal{C}_\theta^4}{\mathcal{N}_0^2} + 2 \cdot \frac{\mathcal{D}_\theta^2}{\mathcal{C}_\theta^2 + \mathcal{D}_\theta^2} \right) - 4 \left(\frac{\mathcal{C}_\theta^4 \mathcal{D}_\theta^4 + \mathcal{S}_\theta^4 \mathcal{C}_\theta^4 - \mathcal{C}_\theta^4 \mathcal{D}_\theta^2}{\mathcal{N}_0^4} \right) \quad (136)$$

Since all numbers are Grassman valued we find that the only non zero term is- $-8(\frac{\mathcal{D}_\theta}{\mathcal{N}_0})^2$ So

$$R_\Sigma = -8 \left(\frac{\mathcal{D}_\theta}{\mathcal{N}_0} \right)^2 \quad (137)$$

We notice that the expression for R_Σ in general is non zero. This also implies that Einstein equations which give a zero Ricci scalar are not valid at the null hypersurface suggesting its generalisation. We notice that even if $R_\Sigma \neq 0$ then the dependence of Grassmannian nature is not observed for it

as the grassmann dependence cancels in squaring and it is finite. We can convince ourselves from the above equation that $\eta_0^2 = 0$ the reasoning is that terms that occur are of the form d_{ii}^2 and $d_{ii}d_{jj} + d_{jj}d_{ii}$ both of which are zero because of their Grassmann Algebra. Similarly by substitution from equation 126 we get expression for $\eta_0 \cos \theta \cos \phi$ and we can convince ourselves again that it squares to zero. Similarly $\eta_0 \cos \theta \sin \phi$ and $\eta_0 \sin \theta$ also squares to zero. Equation 98 reduces to

$$-\frac{1}{deth} \frac{d^2 deth}{d\lambda^2} + \frac{1}{2(deth)^2} \left(\frac{d(deth)}{d\lambda} \right)^2 = -8 \left(\frac{\mathcal{D}_\theta}{\mathcal{N}_0} \right)^2 \quad (138)$$

¹⁸We thank Chatgpt for this calculation

This is really like a non linear second order differential equation whose solution depends on timelike region via the right hand side. The non trivial dependence on right hand side gives us the indication of non trivial null hypersurface geometry. Since quantities on right hand side are functions of grassmann variables and as they are necessarily infinitesimals in the first order approximation however they can change the solution to differential equation significantly from the scenario where they are absent. In their absence the solution can be found as-

$$deth = (\gamma(\lambda - \lambda_0) + B)^2 \quad (139)$$

This is consistent with equation 99 being zero which implies $n=0$ or $n=2$. Using these ansatz we can solve equation 79 and find α . We find $\alpha = 0$ or $\alpha = -\frac{5}{2}$. Interestingly we did not require gluing conditions to find α . Using these ansatz we find that the expansion along Null hypersurface as per equation 64 has the form-

$$\theta' = \frac{\gamma}{\gamma(\lambda - \lambda_0) + B} \quad (140)$$

Using equation 62 we can compute the area of Null Hypersurface. However this involves a tedious computation of Grassman variables η_0, θ, ϕ . However if we use the interpretation of Expansion as rate of change of volume we find that the Area of Black hole Horizon has a parametric form-

$$A(\lambda) = C(\lambda - \lambda_0 + D) \quad (141)$$

Equation 141 implies area of black hole increases with affine parameter λ linearly this is consistent with Hawking's Area Theorem and second law of Thermodynamics. We also notice that an explicit computation of Area A involves Grassmann Variables which has not been proceeded here. An explicit computation of entropy and its scaling with Area still needs to be verified however as part of later work.

3.12.3. Interpretation of Area Equation Using Grassmann Integration

Given equation 62 and the integral over Grassmann variables we can use it to find Area as function of integral over Grassman variables. Using equation 62 we find

$$A = \int \eta_0^2 \sqrt{deth} \sin\theta d\theta \wedge d\phi \wedge d\eta_0 \quad (142)$$

Using two properties of integral of grassmann variables-

$$\int a\theta d\theta = a \quad \int b d\theta = 0$$

Thus we find-

$$\int \eta_0^2(\lambda) \sin\theta(\lambda) d\theta(\lambda) \wedge d\phi(\lambda) \wedge d\eta_0(\lambda) \approx 0(\lambda^1)^{19} \quad (143)$$

Since integral of Grassmann variables is like taking

derivatives, we can simplify equation 143 further as-

$$A = \int \sqrt{deth} 2\eta_0(\lambda) \sin\theta(\lambda) d\theta(\lambda) \wedge d\phi(\lambda) \quad (144)$$

Which is consistent with linear scaling with affine parameter of the area A to approximation where θ and ϕ . are supposed to be dimensionless functions of λ .²⁰ Note that θ and ϕ are not exactly Grassmann variables or trigonometric functions hence we can not use normal routine integrals here. Equation 144 is very crucial in reducing the dimensionality of the integral of A over a 2 Dimensional region which is one way of justifying our assumption of 2 dimensional foliation of Cauchy slices.

3.13. Computing Ricci Tensor for 3 d Null Hypersurface

In our analysis we assumed a 2D foliation of 3 d Null Hypersurface so we can extrapolate our assumption of Area Theorem provided the foliation of 3 d Hypersurface into 2d Hypersurface exists. However because of back reaction from background geometry, blueshift of such Cauchy slices where the geometric data is extrapolated we are worried that such foliation into 2 d Hypersurface may not be always universal so we extend our analysis without resorting to such foliations. This does not modify our equation 97 for R_Σ but equation 76 gets modified as-

$$\frac{d\theta'}{d\lambda} + \frac{\theta'^2}{2} + \frac{\alpha}{\lambda^2} = -R_{ab}k^ak^b \quad (145)$$

To proceed further we use a geometrical relation for Ricci Scalar and Ricci tensor in 3 dimensions and try finding Gauss Codazzi equation for Ricci tensor. *Since we foliated our Cauchy slice as 2 dimensional surface and found R_Σ for 2 dimensional surface we need to verify that R_Σ along 3 d slice is zero to compute 2 d Ricci scalar otherwise there will be non trivial terms additionally. As 2 d Ricci scalar is found to be non zero we expect that 3 D Ricci tensor and scalar may be non zero even though 4 d Ricci scalar owing to Einstein equations is zero outside the null hypersurface.*

3.14. Gauss Codazzi Equation for Ricci Tensor

Given the projection tensor $h_\alpha^\mu h_\beta^\nu$ the Gauss Codazzi equation for Ricci tensor is-

$$R_{\mu\nu} h_\alpha^\mu h_\beta^\nu = R_{\alpha\beta}^\Sigma - K_{\alpha\gamma} K_\beta^\gamma + K K_{\alpha\beta} \quad (146)$$

In Vacuum Einstein spacetime $R_{\mu\nu} = 0$ hence

$$R_{\alpha\beta}^\Sigma k^\alpha k^\beta = (-K_{\alpha\gamma} K_\beta^\gamma + K K_{\alpha\beta}) k^\alpha k^\beta \quad (147)$$

Hence equation 136 reduces to-

$$\frac{d\theta'}{d\lambda} + \frac{\theta'^2}{2} + \frac{\alpha}{\lambda^2} = (K_{\alpha\gamma} K_\beta^\gamma - K K_{\alpha\beta}) k^\alpha k^\beta \quad (148)$$

To calculate right hand side we use the formalism developed

¹⁹Linear dependence with affine parameter can be found using 144 where scaling with η_0 is linear with λ

²⁰The statement proposed here needs scrutiny

in section 3.9.2. We also need to compute the tangent vector along null hypersurface. The tangent vector can be computed by finding the derivative of 4 vector

$$k = \begin{bmatrix} \frac{d\eta_0}{d\lambda} \\ \frac{d}{d\lambda}(\eta_0 \cos \theta \cos \phi) \\ \frac{d}{d\lambda}(\eta_0 \cos \theta \sin \phi) \\ \frac{d}{d\lambda}(\eta_0 \sin \theta) \end{bmatrix} \quad (149)$$

We now proceed calculating right hand side of equation 148 using $K_{\alpha\gamma} = \nabla_\alpha n_\gamma$, $K_\beta^\gamma = \nabla^\gamma n_\beta$, $K = \nabla_c n^c = \text{div}(n)$, $K_{\alpha\beta} = \nabla_\alpha n_\beta$ This implies-

$$\begin{aligned} (K_{\alpha\gamma} K_\beta^\gamma - K K_{\alpha\beta}) k^\alpha k^\beta = \\ (\nabla_\alpha n_\gamma \nabla^\gamma n_\beta - \text{div}(n) \nabla_\alpha n_\beta) k^\alpha k^\beta \end{aligned} \quad (150)$$

By contraction of indices we find the expression, but we notice that the tensor for extrinsic curvature is same as $T'(n)$ as suggested in equation 94 because $T'(n)$ explicitly computes the gradient of normal vector which is same as extrinsic curvature components and the expression simplifies to -

$$\begin{aligned} (K_{\alpha\gamma} K_\beta^\gamma - K K_{\alpha\beta}) k^\alpha k^\beta = \\ (T'(n).k)^T (T'(n).k) - \text{div}(n) k^T (T'(n).k) \end{aligned} \quad (151)$$

Using the parameters of 4 vector α we find the final expression is-

$$(K_{\alpha\gamma} K_\beta^\gamma - K K_{\alpha\beta}) k^\alpha k^\beta = \alpha^T \alpha - \text{div}(n) k^T \alpha = \alpha_0^2 + \alpha_2^2 + \alpha_3^2 - \text{div}(n)(\alpha_0 k_0 + \alpha_2 k_2 + \alpha_3 k_3) \quad (154)$$

Using a previous result we see-

$$\text{div}(n) = 2 - 2\eta_0^2 \cos^2 \theta \sin 2\phi = 2 - 4\eta_1 \eta_2 \quad (155)$$

This simplifies our equation 154 to-

$$\begin{aligned} (K_{\alpha\gamma} K_\beta^\gamma - K K_{\alpha\beta}) k^\alpha k^\beta = 4 \left(\frac{d\eta_0}{d\lambda} \right)^2 + 8\eta_0 \cos^2 \theta \cos 2\phi \frac{d\eta_0}{d\lambda} \frac{d\eta_3}{d\lambda} + 4\eta_0^2 \cos^4 \theta \cos^2 2\phi \left(\frac{d\eta_3}{d\lambda} \right)^2 \\ + 4\eta_0^4 \sin^2(2\theta) \cos^2(2\phi) \left(\frac{d\eta_3}{d\lambda} \right)^2 + 4\eta_0^4 \cos^4 \theta \sin^2(2\phi) \left(\frac{d\eta_3}{d\lambda} \right)^2 - (2 - 4\eta_1 \eta_2) \\ \left[2 \left(\frac{d\eta_0}{d\lambda} \right)^2 + 2\eta_0 \cos^2 \theta \cos^2 \phi \frac{d\eta_0}{d\lambda} \frac{d\eta_3}{d\lambda} + 2\eta_0^2 \sin(2\theta) \cos(2\phi) \frac{d\eta_3}{d\lambda} k_2 - 2\eta_0^2 \cos^2 \theta \sin(2\phi) \frac{d\eta_3}{d\lambda} k_3 \right] \end{aligned} \quad (156)$$

Equation 156 is really very cumbersome and crazy and we are deeply grateful to Chatgpt for its prodigious computation. Simplifying the terms in the bracket further we find the expression is-

$$\begin{aligned} (K_{\alpha\gamma} K_\beta^\gamma - K K_{\alpha\beta}) k^\alpha k^\beta = 4 \left(\frac{d\eta_0}{d\lambda} \right)^2 + 8\eta_0 \cos^2 \theta \cos 2\phi \frac{d\eta_0}{d\lambda} \frac{d\eta_3}{d\lambda} + 4\eta_0^2 \cos^4 \theta \cos^2 2\phi \left(\frac{d\eta_3}{d\lambda} \right)^2 \\ + 4\eta_0^4 \sin^2(2\theta) \cos^2(2\phi) \left(\frac{d\eta_3}{d\lambda} \right)^2 + 4\eta_0^4 \cos^4 \theta \sin^2(2\phi) \left(\frac{d\eta_3}{d\lambda} \right)^2 - (2 - 4\eta_1 \eta_2) \\ \left[2 \left(\frac{d\eta_0}{d\lambda} \right)^2 + 2\eta_0 \cos^2 \theta \cos^2 \phi \frac{d\eta_0}{d\lambda} \frac{d\eta_3}{d\lambda} + 2\eta_0^2 \sin(2\theta) \cos(2\phi) \frac{d\eta_3}{d\lambda} \frac{d\eta_0}{d\lambda} \cos \theta \sin \phi \right. \\ + 2\eta_0^3 \sin(2\theta) \cos(2\phi) \frac{d\eta_3}{d\lambda} \sin \theta \sin \phi \frac{d\theta}{d\lambda} + 2\eta_0^3 \sin(2\theta) \cos(2\phi) \frac{d\eta_3}{d\lambda} \cos \theta \cos \phi \frac{d\phi}{d\lambda} \\ \left. - 2\eta_0^2 \cos^2 \theta \sin(2\phi) \frac{d\eta_3}{d\lambda} \frac{d\eta_0}{d\lambda} \sin \theta - 2\eta_0^3 \cos^3 \theta \sin(2\phi) \frac{d\eta_3}{d\lambda} \frac{d\theta}{d\lambda} \right] \end{aligned} \quad (157)$$

We label the tangent vectors as-

$$k = \begin{bmatrix} \frac{d\eta_0}{d\lambda} \\ \frac{d\eta_1}{d\lambda} \\ \frac{d\eta_2}{d\lambda} \\ \frac{d\eta_3}{d\lambda} \end{bmatrix} \quad (152)$$

Where η_1, η_2, η_3 are the usual parametrisation along null Hypersurface. We then find

$$T'(n).k = \begin{bmatrix} 2 \frac{d\eta_0}{d\lambda} + 2\eta_0 \cos^2 \theta \cos 2\phi \frac{d\eta_3}{d\lambda} \\ 0 \\ -2\eta_0^2 \sin(2\theta) \cos(2\phi) \frac{d\eta_3}{d\lambda} \\ -2\eta_0^2 \cos^2 \theta \sin(2\phi) \frac{d\eta_3}{d\lambda} \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \quad (153)$$

Recollecting terms further the expression becomes-

$$\begin{aligned}
(K_{\alpha\gamma}K_{\beta}^{\gamma} - K K_{\alpha\beta})k^{\alpha}k^{\beta} &= \frac{d\theta'}{d\lambda} + \frac{\theta'^2}{2} + \frac{\alpha}{\lambda^2} = 4\eta_0^2 \cos^4 \theta \cos^2 2\phi \left(\frac{d\eta_3}{d\lambda} \right)^2 - 2\eta_0^2 \sin(2\theta) \cos(2\phi) \frac{d\eta_3}{d\lambda} \frac{d\eta_0}{d\lambda} \cos \theta \sin \phi \\
&\quad - 2\eta_0^2 \sin(2\theta) \cos(2\phi) \frac{d\eta_3}{d\lambda} \frac{d\eta_0}{d\lambda} \cos \theta \sin \phi + 2\eta_0^2 \cos^2 \theta \sin(2\phi) \frac{d\eta_3}{d\lambda} \frac{d\eta_0}{d\lambda} \sin \theta \\
&\quad + 2\eta_0^2 \cos^2 \theta \sin(2\phi) \frac{d\eta_3}{d\lambda} \frac{d\eta_0}{d\lambda} \sin \theta - 2\eta_0^3 \sin(2\theta) \cos(2\phi) \frac{d\eta_3}{d\lambda} \sin \theta \sin \phi \frac{d\theta}{d\lambda} \\
&\quad - 2\eta_0^3 \sin(2\theta) \cos(2\phi) \frac{d\eta_3}{d\lambda} \cos \theta \cos \phi \frac{d\phi}{d\lambda} - 2\eta_0^3 \sin(2\theta) \cos(2\phi) \frac{d\eta_3}{d\lambda} \cos \theta \cos \phi \frac{d\phi}{d\lambda} \\
&\quad + 2\eta_0^3 \cos^3 \theta \sin(2\phi) \frac{d\eta_3}{d\lambda} \frac{d\theta}{d\lambda} + 2\eta_0^3 \cos^3 \theta \sin(2\phi) \frac{d\eta_3}{d\lambda} \frac{d\theta}{d\lambda} \\
&\quad + 4\eta_0^4 \sin^2(2\theta) \cos^2(2\phi) \left(\frac{d\eta_3}{d\lambda} \right)^2 + 4\eta_0^4 \cos^4 \theta \sin^2(2\phi) \left(\frac{d\eta_3}{d\lambda} \right)^2 \\
&\quad + 4\eta_0^4 \cos^4 \theta \sin^2(2\phi) \left(\frac{d\eta_3}{d\lambda} \right)^2 + 16\eta_1\eta_2 \left(\frac{d\eta_0}{d\lambda} \right)^2 \\
&\quad + (8 \cos 2\phi - 4 \cos^2 \phi + 16\eta_1\eta_2 \cos^2 \phi) \eta_0 \frac{d\eta_0}{d\lambda} \frac{d\eta_3}{d\lambda}.
\end{aligned} \tag{158}$$

Using equations 156 and 62 we find right hand side of equation 158 takes same form as left hand side of equation 77.

$$\frac{1}{(2deth)} \frac{d^2(deth)}{d\lambda^2} - \frac{3}{8(deth)^2} \left(\frac{d(deth)}{d\lambda} \right)^2 + \frac{\alpha}{\lambda^2} = \star \tag{159}$$

Here $\star$ is most right hand side of equation 158.

Rewriting using Grassmann variables we find the expression simplifies to-

$$\begin{aligned}
\frac{1}{(2deth)} \frac{d^2(deth)}{d\lambda^2} - \frac{3}{8(deth)^2} \left(\frac{d(deth)}{d\lambda} \right)^2 + \frac{\alpha}{\lambda^2} &= 4\eta_0^2 \left(\frac{\eta_1^2 + \eta_2^2}{\eta_0^2 + \eta_3^2} \right)^2 \left(\frac{1 - \frac{4\eta_1\eta_2}{\eta_1^2 + \eta_2^2}}{1 + \frac{4\eta_1\eta_2}{\eta_1^2 + \eta_2^2}} \right) \left(\frac{d\eta_3}{d\lambda} \right)^2 \\
&\quad - \frac{4\eta_0^2\eta_3(\eta_1^2 + \eta_2^2)}{(\eta_0^2 + \eta_3^2)(\eta_1^2 + \eta_2^2)} \frac{d\eta_3}{d\lambda} \frac{d\eta_0}{d\lambda} + \frac{8\eta_0^3\eta_3(\eta_1\eta_2)}{(\eta_0^2 + \eta_3^2)(\eta_1^2 + \eta_2^2)} \frac{d\eta_3}{d\lambda} \frac{d\eta_0}{d\lambda} + \frac{4\eta_0^4(\eta_1^2 + \eta_2^2)^2}{(\eta_0^2 + \eta_3^2)^2(\eta_1^2 + \eta_2^2)^2} \left(\frac{d\eta_3}{d\lambda} \right)^2 \\
&\quad + 16\eta_1\eta_2 \left(\frac{d\eta_0}{d\lambda} \right)^2 + \left(8 \frac{1 - \frac{4\eta_1\eta_2}{\eta_1^2 + \eta_2^2}}{1 + \frac{4\eta_1\eta_2}{\eta_1^2 + \eta_2^2}} - 4 \frac{\eta_1^2 - \eta_2^2}{\eta_1^2 + \eta_2^2} + 16\eta_1\eta_2 \frac{\eta_1^2 - \eta_2^2}{\eta_1^2 + \eta_2^2} \right) \eta_0 \frac{d\eta_0}{d\lambda} \frac{d\eta_3}{d\lambda}. \\
&\quad 4 \frac{(X_0)^2 (\cos^2 \theta_1 \cos^2 \phi_1 X_0^2 + X_1^2)}{(X_0^2 + X_3^2)^2} \left(\frac{1 - 4 \cos \theta_1 \cos \phi_1 X_0 X_1}{1 + 4 \cos \theta_1 \cos \phi_1 X_0 X_1} \right) \times \left(\frac{dX_3}{d\lambda} \right)^2 \\
&\quad - 4 \frac{X_0^2 X_3 (\cos^2 \theta_1 \cos^2 \phi_1 X_0^2 + X_1^2)}{(X_0^2 + X_3^2)(\cos^2 \theta_1 \cos^2 \phi_1 X_0^2 + X_1^2)} \frac{dX_3}{d\lambda} \frac{dX_0}{d\lambda}.
\end{aligned} \tag{160}$$

Because of the extra degrees of freedom of Grassman numbers we were not able to extend our analysis further. We really think we need to understand superspace or graded geometries better to extend our analysis further.

4. More Imprecise Hints to the Paradox and Some Speculations, Future Work

There are few more thought experiments suggesting the same conclusion of modification of Semiclassical Gravity-Suppose we consider an observer in Black Hole spacetime from a region A which is timelike and the observer falls inside the Black Hole along a null hypersurface as region B. Suppose we measure the line element of the observer ds^2 then $ds_{A \rightarrow B}^2$

represents the scenario where the observer falls inwardly but the line element $ds_{B \rightarrow A}^2$ represents a scenario where the observer goes from a null region to timelike region which is not allowed by Causality. The reason it is not possible is because the Diffeomorphism matrix D is non invertible (as per equation 2) so there do not exists map from B to A. Thus the line element in this scenario is asymmetric which is not a feature of Riemannian geometry. Thus assuming there are disjoint regions like these in Black Hole Spacetime it does not necessarily admit a smooth differentiable map between such disjoint regions. So there can be many possibilities in this regard. However the asymmetry of line element is an indication of Black Hole Spacetime admitting a Finsler Geometry. We can see that metric is asymmetric in Finsler Geometry, however there can be many more possibilities in

this regard.

5. Topology of Black Hole Spacetime as a Disconnected 4 d Lorentzian Mobius Strip with Smooth Gluing Conditions at the Horizon

Suppose we consider a Cauchy slice embedded in R^4 then at a given instant of time we still find violation of Diffeomorphism symmetry but the coordinates here are not singular anywhere because η still exists as a reliable coordinate chart. Maybe such foliations of spacetime do not exist. One reason for the same is because in presence of gravity different regions in space foliated as Cauchy slice experience different amount of redshift so if we follow 2 neighbouring fibres of some base manifold due to Gravitational Redshift they undergo different amount of bending so they do not necessarily proceed as one parameter family of proper time τ as smooth fibres as we propagate local fiber evolution data. Due to Gravitational redshift varying with time the fibres twist differently along a Cauchy slice. We call this the foliation problem in General Relativity. Usually the fibres studied in Differential geometry are assumed to be have a smooth Lie group structure or Lie algebra structure but because of bending of fibres we may not necessarily suppose that the symmetry of Lie group is preserved during foliation, the redshift factor as we proceed along the radial direction along the base manifold appears to be different along nearby fibres of some base manifold. This suggests that the Non trivial bending of fibres may retain diffeomorphism Invariance because as we see fibres away from a gravitating region bend less and fibres closer to horizon bend more. Suppose we consider a foliation of Cauchy slice then a horizontal Cauchy slice with a suitable time parametrization starts bending downwards closer to horizon as the time evolves but inside the horizon the Cauchy slice has undergone a 180 degree change so it is like a Mobius strip. The problem pictorially reduces to retaining Diffeomorphism invariance in a mobius strip but we are aware of Black hole complementarity as a feature of gravity. We suspect geometrically that the mobius strip generates singularity at the null hypersurface in some foliated slice. By suitably applying surgery theory techniques we can get disconnected manifolds. This suggests a pictorial way of discontinuity of Cauchy data along such foliations but we rather insist in preserving the data on a single manifold as we do not want semiclassical physics to break down. We insist gluing these sets of data together as a problem in Quantum gravity. We do not know if the structure of a smooth manifold needs to be more scrutinized further to get relevant understanding of the problem. Certainly we need to presume a Finsler manifold structure on such spacetime even if we retain smoothness, though Diffeomorphism symmetry is already absent. We have 3 ingredients-Causal Structure of spacetime, Topological structure of spacetime, Smooth

structure of spacetime but without Diffeomorphism. We retain Causality, Topology and Smooth structure of spacetime with no diffeomorphism but with a Finsler Geometry. We presume maybe the metric is smooth if we do not rely on tensorial nature of metric transformation. This means that the disjoint regions are combinations of different manifolds and precisely the non intuitive bending of fibres and disjoint manifold regions together preserve the smooth nature of General relativity. We will not comment further about it. Also extrapolating the information from Null Hypersurface to Black Hole Interior again appears to require new ideas. We are worried that the absence of Diffeomorphism seems to imply towards Non Homeogenous Manifolds or towards Finsler Geometry as eluded earlier (due to asymmetry in Line element). Maybe Einstein Equations are valid without Diffeomorphism symmetry and there are meaningful notions of affine connection, Riemann Tensor but as a disjoint manifold. Are there geodesics in Black Hole Interior or on Null Hypersurface. The graded nature of Null Hypersurface might severely constrain the nature of Holographic CFT which seems to hint towards matrix degrees of freedom with BRST story analogous to Grassman algebra there.

6. Black Hole Complementarity and Non Local Splitting of Hilbert Space

The non trivial bending of Cauchy slices foliated suitably seems to hint towards the possibility of Non local data evolution of base suitably embedded base manifold which is indicative of something similar to Black Hole Complementarity. In its most ideal form if we can provide diffeomorphisms due to shifting of fibres in a way that allows for acausal diffeomorphisms then certainly we have preserved Riemannian structure of manifold at the cost of locality. Black hole complementarity also suggests a passaway from the asymmetry of line element at the cost of causality. Motivated by the principle of Holography we suggest another non intuitive idea. Suppose Black Hole complementarity is a generic feature of Gravity which has been beautifully suggested in many papers of Professor Suvrat Raju and his collaborators, we find another paradox which has been stated by us in a pseudoprecise way. Suppose we have a hilbert space splitting of dual CFT²¹ then the isomorphism of the Hilbert spaces in the bulk spacetime with the dual CFT implies allowing Black Hole Complementarity in bulk indicates something similar to complementarity in the hilbert spaces of dual CFT due to principle of Holography of information. This has not been stated in a very precise way. We are assuming an isomorphism of the splitted hilbert space in bulk with some kind of isomorphism in dual CFT but a feature like complementarity in dual CFT is an indication of some kind of non locality in CFT which seems to be non trivial. The intuition of associating isomorphism of this kind is known as subregion-subregion duality. Thus Black Hole complementarity, subregion-

²¹Assuming some non perturbative completion of the ultraviolet regime of CFT preserves the divergences

subregion duality consistency implies a non local ultraviolet completion of dual CFT. Puzzled by this paradox stated in an imprecise way we suggest that the firstgive away might be to assume smooth Riemannian structure of Black Hole spacetime with Finsler geometry. This may have non trivial implications for Holography and Holgraphic CFT. We are motivated by graded geometries to persue twistor theory of null hypersurfaces for computing Geometric quantities on graded geometries as part of future work.

7. Conclusion

We reviewed Mathematical aspects of General Relativity in section 2. In section 3 we proposed a paradox owing to absence of diffeomorphism symmetry of Black Hole Spacetime in some small neighbourhood ball region containing a timelike region and null region. We then used Raychaudhuri equation and smooth variation of coordinate charts along such epsilon ball as its radius goes to zero. Hopefully it seems these non linear coupled ordinary differential equations and the smoothness of coordinate chart has given us a recipe for reconstructing the Null hypersurface as part of graded or superspace geometries. Interestingly such reconstruction for 2 D Minowski spacetime seems to hint us towards matrix degrees of freedom of spacetime or alternatively the notion of superspace. Despite the complexity of differential equations we verified the argument for Schwarzschild spacetime of Graded or superspace geometry for Null Hypersurface of Schwarzschild spacetime. We also saw that there are more imprecise hints to the geometry of Black Hole spacetime like Finsler Geometry as suggested in section 4 of the paper. In section 4.1 we also addressed the foliation problem due to Gravitational spacetime suggesting a notion of dynamically varying time field to suitably describe a Cauchy slice. Some problems that are of interest for further investigation are-

1. Connection of Entropy with geometry using a generic framework like Geometrical Flows, Perelman Entropy functional.
2. Is Area Law of Black Holes verifiable using a different approach as proposed. Investigating Quantum effects, the paper persues ideas using purely Classical

Framework.

3. Problem of Foliation as a probe for dimensional reduction which needs justification. Investigating back reaction effects which may be significant at later times.
4. Connections with Celestial Holography, Asymptotic symmetries and String Theory.
5. Finding an explicit entropy counting of Black Hole in this framework if possible.

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Author Contributions

Shreyansh Singh is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The author declares no conflicts of interest.

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