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A Meshless Method for Solving 1D Heat Equation on a Semicircle

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Abstract: In this work, we present the one-dimensional heat equation solving by the isogeometric method, a meshless method using Galerkin Method. This equation has been solved on a semicircle, a curve on $\mathbb{R}^2$. The basis of approximation used for this paper, is the B-splines basis. We define univariate B-splines. We look at their properties as well as b-splines curves. We calculate the numerical solution of the heat equation using the principle of Galerkin's method. The numerical solution is calculated, using the parametrization of the domain and using the numerical integration of Gauss. Solving this partial differential equation leads to solving a system of differential equations. This system will be solved using the classic fourth-order Runge-Kutta method and a CFL condition. Numerical tests have been presented to show the efficiency of this method.

Keywords: B-splines, Isogeometric Method, Galerkin Method, Heat Equation, Parametrization, a CFL Condition

1. Introduction

The heat equation was first formulated and studied by Joseph Fourier in the early 19th century (in 1807). Fourier mathematically modeled thermal conduction, describing how temperature changes in a solid body over time. His theory, published in 1822, marked a revolution in the field of mathematical modeling of physical phenomena, particularly in thermodynamics [1, 2].

Heat transfer problems can be found useful for many practical engineering applications, such as electric heat conduction in the packaging and metallurgy industries, laser beam heat generation and process simulation in automotive and aerospace industries, and thermal conduction and heat insulation in pipelines for heat-supply [3–5].

The recent technological advancements in computers have made numerical methods an essential tool for analyzing physical systems. Many algorithms, mathematical concepts, and programs have been proposed aiming at obtaining accurate simulations of physical systems, especially in the field of engineering [6, 7].

Several traditional numerical methods such as the finite element method (FEM), finite volume method (FVM), have been effectively utilized to address heat transfer problems. However, the mesh generation requirement in FEM and FVM, limit their full applicability in engineering applications. The inconsistency between the actual physical geometry and finite element based models introduces two significant disadvantages. First, in addition to the inherent inaccuracy of the numerical solution, the inconsistency between the CAD geometry and the analysis model introduces additional errors [8–11]. This issue is particularly pronounced in problems with curved boundaries or in physical problems that are highly sensitive to boundary representation [11]. Second, as the analysis model is refined, the domain it represents gradually becomes closer to the original geometry. However, it remains inconsistent across different refinement levels. This inconsistency can impact the accuracy and convergence of the solution of the physical problem. .

Isogeometric methods (IGA) provide efficient, accurate and high-order alternative solutions and attracted extensive attentions over the past decade [12]. The main idea of IGA is that the basis functions used for defining the geometry will also be applied as the basis for downstream analysis, which consequently leads to the use of an accurate representation for both the geometry and analysis solutions [8–11, 13]. Isogeometric method has also obtained a lot of attention lately because of their low computational cost in numerical simulation with reasonable accuracy. Isogeometric Analysis (IgA), introduced in the seminal paper [8–11], is an evolution of the classical finite element methods. IgA uses spline functions, both to represent the computational domain and to approximate the solution of the partial differential equation that models the problem of interest [14]. This integration of geometric modeling and numerical simulation facilitates the transition from Computer Aided Design to analysis.

The utilization of IGA for solving diffusion-type equations was initially introduced by Vuong and al. [15]. This approach presented several advantages over standard finite element methods and was swiftly incorporated into various commercial software packages [16]. An adaptive version of this approach was also recently outlined by Yu and al. [17]. NURBS functions have also been adapted for the boundary element method as demonstrated by An and al. [18] and Takahashi and al. [19] and implemented by Yu and al. [20] for solving transient heat conduction problems with heat sources. The potential of IGA in heat transfer problems has been further explored in subsequent publications. This technique has been employed to analyze heat transfer within polymer blends in microstructures [21] and axial-flux permanent magnets [22]. The coupling of IGA and a control volume approach has enabled the determination of temperature variations in nonuniformly heated plates [23].

In 2015, U. Langer, S.E. Moore, M. Neumüller have presented an isogeometric error estimate for parabolic equations in 2D [24].

In 2019, M. Montardini, M. Negri, G. Sangalli and M. Tani, solved parabolic problems, using least-squares isogeometric method. Joaquín Cornejo Fuentes, Giancarlo Sangalli, Mattia Tani, Thomas Elguedj, David Dureisseix, Arnaud Duval solved nonlinear transient heat transfer problems with the isogeometric method in 2025, in two dimensions [25].

In 2020, Gabriele Loli, Monica Montardini, Giancarlo Sangalli and Mattia Tani solved parabolic equations in three dimensions, using the isogeometric method. They focus on the preconditioning of a Galerkin space-time isogeometric discretization of the heat equation [13].

Rachid Bouna, Nouh Izem, M. Shadi Mohamed and Mohammed Seaid solved heat equation in three dimensions, with the isogeometric method, on march 2025. They propose a class of high-order time integration schemes combined with high-order IsoGeometric Analysis (IGA) in three space dimensions. The combined methods offer robust solutions of nonlinear heat diffusion in three-dimensional composites that pose numerical challenges [26].

In 2023, G. Loli, G. Sangalli, and P. Tesini, use the isogeometric method with smooth splines, to enhance the stability and the accuracy of the method, in one dimension and in two dimensions. They solved the problem in one dimension, on the interval [0;1]. In our work, we want to solve the heat problem in 1D, not on a polygonal geometry but rather on a non-polygonal geometry. The solution will be found on a curve in $\mathbb{R}^2$, using the parametrization of the domain [27].

In this work, we present the isogeometric method. Then, we calculate the numerical solution for the heat equation using this method. Finally, we carry out numerical tests, followed by a discussion.

2. Isogeometric Method

In order to solve a partial differential equation on a domain with isogeometric method, we work in a subspace of finite dimension. So we need a basis. The basis used in this work, is the B-splines basis, specifically the univariate B-splines basis.

2.1. Univariate B-splines

Definition 1. Let $x_1 \leq x_2 \leq \cdots \leq x_m$ be an increasing sequence of reals, B-splines functions of degree k are defined by Cox-de Boor-Mansfield recursion formula [28] :

$$\begin{cases} \text{For} \quad 1 \leq i \leq m-1 & \\ N_{i,0}(t) \quad = 1 & if \quad t \in [x_i, x_{i+1}[\\ N_{i,0}(t) \quad = 0 & otherwise \end{cases} \quad (1)$$

$$\begin{cases} \text{For} \quad k \geq 1 \quad and \quad 1 \leq i \leq m-k-1 \\ N_{i,k}(t) \quad = \dfrac{t - x_i}{x_{i+k} - x_i} N_{i,k-1}(t) + \dfrac{x_{i+k+1} - t}{x_{i+k+1} - x_{i+1}} N_{i+1,k-1}(t) \end{cases} \quad (2)$$

With the convention $\frac{x}{0} := 0$ for all real number x

The set $(x_i)_{i=1}^{m} \quad (1 \leq i \leq m)$ is called knots vector.

Equations (1) et (2) show that the choice of knots vector has a significant influence on the B-spline basis functions. Open knots vectors are used to solve partial differential equations [30]. The open knots vectors for a B-splines basis of degree k are defined by :

$$X = [\underbrace{x_1 \quad \cdots \quad x_1}_{\text{k+1 times}} \quad x_{k+2} \cdots \quad \underbrace{x_m \quad \cdots \quad x_m}_{\text{k+1 times}}]$$

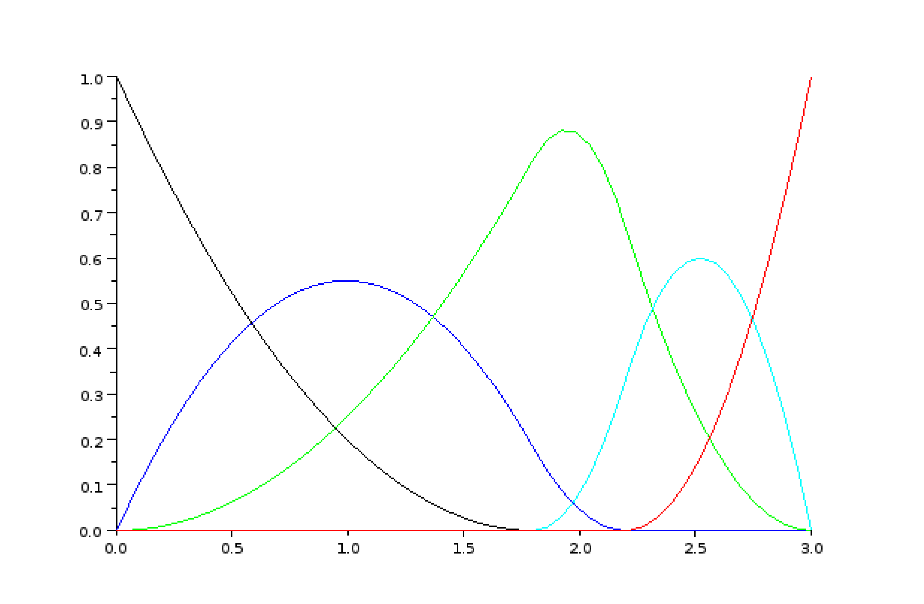

Figure 1. *Basis B spline of degree 2.*

$$X = [\quad 0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 3 \quad 3 \quad] \quad k = 2 \quad m = 8 \quad n + 1 = 5 \quad [29].$$

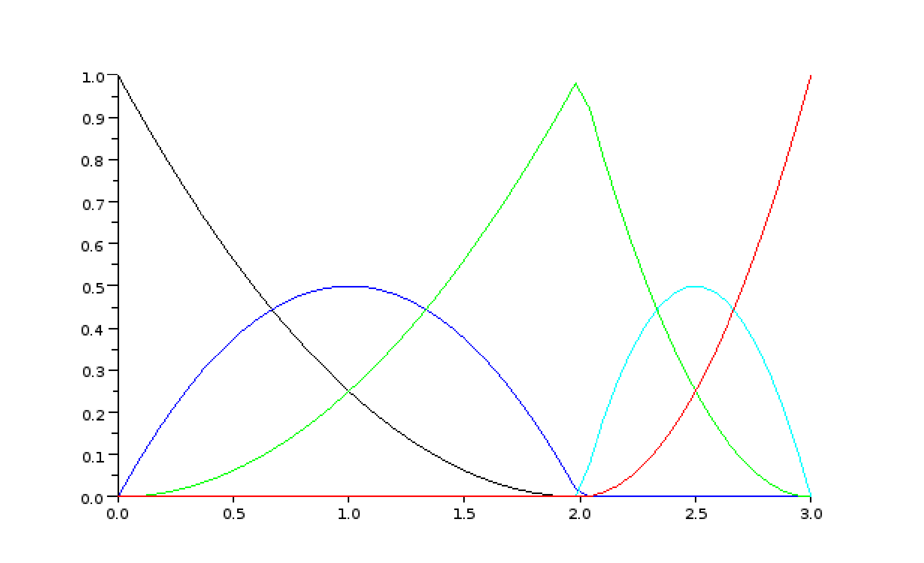

Figure 2. *Basis B spline of degree 2.*

$$X = [\quad 0 \quad 0 \quad 0 \quad 2 \quad 2 \quad 3 \quad 3 \quad 3 \quad] \quad k = 2 \quad m = 8 \quad n + 1 = 5 \quad [29].$$

Definition 2. Let $(x_i)_{i=1}^{m} \quad (1 \leq i \leq m)$ be a knots vector. Let n be a non-zero natural integer et let $(P_i)_{i=0}^{n}$ be a sequence of points of $\mathbb{R}^d$ $(d = 1, 2)$. We call b-spline curve of degree k (d'ordre $k+1$) and of control points P_i, $i = 0, \ldots, n$, the function P defined from the interval $[x_1 \, , \, x_m]$ into $\mathbb{R}^d$ by :

$$P(t) = \sum_{i=0}^{n} P_i N_{i,k}(t), \quad x_1 \leq t \leq x_m, \; 0 \leq k \leq n$$

The set of points $(P_i)_{i=0}^{n}$ are the vertices of a polygon called control polygon of the curve P.

2.2. B-Splines Functions Properties

Let $(n+1)$, be the control points number, let k be the B-splines function degree $N_{i,k}$ and let m be the knots number. Let $X = [x_1 \quad x_2 \cdots \quad x_m]$ be a knots vector.

1. The $N_{i,k}$ support is a subset of $[x_i, x_{i+k+1}]$ [10, 30, 31]
2. $\forall t \in [x_1, x_m]$, $N_{i,k}(t) \geq 0, \forall i$ [10, 30, 31]
3. $\forall j \in \{k+1, \ldots, m-k-1\}$, $\forall t \in [x_j, x_{j+1}[$, $\sum_{i=1}^{n+1} N_{i,k}(t) = 1$ [31]
4. If an interior knot vector x_i has multiplicity p_i, then $N_{i,k}$ is C^{k-p_i} at $t = x_i$, and C^{∞} elsewhere. [32]
5. Let a and b be respective minimal and maximal values of X.
   If X is an open knots vector of degree k, then

$$N_{i,k}(a) = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad N_{i,k}(b) = \begin{cases} 1 & \text{if } i = n+1 \\ 0 & \text{otherwise} \end{cases}$$

Remark 1. From the property 4, we can infer that increasing knot multiplicity decreases in this knot regularity and increasing degree increases regularity [31].

Heat equation is solved in the physical domain Ω, which is defined through a geometrical mapping F :

$$F : \widehat{\Omega} = [0; 1] \rightarrow \Omega$$

$$\varepsilon \longmapsto x = F(\varepsilon) = \sum_{i=1}^{n+1} C_i \widehat{N}_{i,k}(\varepsilon)$$

With $\widehat{\Omega}$, the parametric domain. The C_i are the so-called control points and $(n+1)$ is the number of B-splines functions.

We assume throughout that F is invertible, with smooth inverse F^{-1}.

We introduce V_h, the space spanned by B-splines basis functions in Ω :

$$V_h := span\{N_{i,k} \circ F^{-1}\}_{i=1}^{n+1}.$$

3. Numerical Solution of the Heat Equation

3.1. Problem Formulation

The purpose of this section is to numerically solve the heat equation using the isogeometric method :

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \nabla.(C \nabla u) = f(x,t), x \in \Omega, t \in [0; T] \\ u(x, 0) = u_0(x), x \in \Omega, \text{ with } \Omega \subset \mathbb{R}^2 \\ u = 0 \text{ on } \partial\Omega \times [0; T] \end{cases}$$

Where C is the diffusion coefficient, $C \in \mathbb{R}$.

$X = (X_i)_{i=1}^{m}$ is an open knots vector, with m the total number of knots.

$X_d = (\chi_i)_{i=1}^{N_d}$ is the set of distinct knots of X.

Let be $\Omega_i = \{x \in \mathbb{R}^d; x = F([\chi_i, \chi_{i+1}]), \}, 1 \leq i \leq N_d - 1$

Thus $\exists!$ $q(i) \in \{1, \ldots, m\}$ such as $\chi_i = X_{q(i)}$ where $q(i) = \sum_{k=1}^{i} n_k$,

And n_i is the multiplicity of χ_i in X.

Let be $\Omega = \bigcup_{i=1}^{N_d - 1} \Omega_i$, with $\Omega_i \cap \Omega_s = \emptyset, \forall i \neq s$.

$(N_{j,p})_{j \in I_i}$ is a b-spline basis of degree p,

Where $I_i = \{k \in \{0, \ldots, n\}; SuppN_{k,p} \cap \Omega_i \neq \emptyset\}$,

$SuppN_{j,p} = F([\chi_j; \chi_{j+p+1}])$.

$SuppN_{j,p} \cap F([\chi_i; \chi_{i+1}]) \neq \emptyset \iff j \in \{q(i) - p, \ldots, q(i)\}$.

$$\forall i \in \{1, \ldots, N_d - 1\}, N_{j,p}^{i} x = \begin{cases} N_{j,p} x \text{ if } x \in F([\chi_i; \chi_{i+1}]) \\ \\ 0 \text{ otherwise} \end{cases}$$

$$u(x,t) = \sum_{i=1}^{N_d - 1} u^i(x,t) \text{ where } u^i(x,t) = \begin{cases} u_{/\partial\Omega_i} \text{ on } \bar{\Omega}_i \\ 0 \text{ otherwise} \end{cases}$$

$$\text{With } u^i(x,t) = \sum_{j \in I_i} \alpha_j^i(t) N_{j,p}^i(x), \forall i \in \{1, \dots, N_d - 1\}.$$

We look for $u(t) \in V_h, \forall v_h \in V_h$:

$$\int_{\Omega_i} \frac{\partial u}{\partial t}(x,t) v_h(x) d\Omega_i - \int_{\Omega_i} \nabla.(C\nabla u) v_h(x) d\Omega_i = \int_{\Omega_i} f(x,t) v_h(x) d\Omega_i$$

$$\text{Where } V_h = \{v_h : v_h^i \in L^2(\Omega_i); \nabla.(C\nabla v_h^i) \in L^2(\Omega_i), v^i_{h/\partial\Omega} = 0\}$$

For $v_h = N_{k,p}^i(x)$, we have :

$$\int_{\Omega_i} \frac{\partial u}{\partial t}(x,t) N_{k,p}^i(x) d\Omega_i - \int_{\Omega_i} \nabla.(C\nabla u) N_{k,p}^i(x) d\Omega_i = \int_{\Omega_i} f(x,t) N_{k,p}^i(x) d\Omega_i$$

$$\int_{\Omega_i} \nabla.(C\nabla u) N_{k,p}^i(x) d\Omega_i = \int_{\partial\Omega_i} (C\nabla u) N_{k,p}^i(x) d\Omega_i - \int_{\Omega_i} (C\nabla u) \nabla N_{k,p}^i(x) d\Omega_i$$

Knowing that $suppN_{k,p}^i \subset \Omega_i, \int_{\partial\Omega_i} (C\nabla u) N_{k,p}^i(x) d\Omega_i = 0.$
Thus, we have :

$$-\int_{\Omega_i} \nabla.(C\nabla u) N_{k,p}^i(x) d\Omega_i = \int_{\Omega_i} (C\nabla u) \nabla N_{k,p}^i(x) d\Omega_i$$

$$\text{Since } u^i(x,t) = \begin{cases} u_{/\partial\Omega_i} \text{ on } \bar{\Omega}_i \\ 0 \text{ otherwise} \end{cases}$$

$$\int_{\Omega_i} \nabla.(C\nabla u) N_{k,p}^i(x) d\Omega_i = -\int_{\Omega_i} (C\nabla u^i) \nabla N_{k,p}^i(x) d\Omega_i$$

We can deduce that :

$$\begin{aligned} -\int_{\Omega_i} \nabla.(C\nabla u) N_{k,p}^i(x) d\Omega_i &= \int_{\Omega_i} (C\nabla u^i) \nabla N_{k,p}^i(x) d\Omega_i \\ &= \sum_{j \in I_i} \alpha_j^i(t) \int_{\Omega_i} (C\nabla N_{j,p}^i(x)) \nabla N_{k,p}^i(x) d\Omega_i \quad (3) \end{aligned}$$

In addition, we have :

$$\int_{\Omega_i} \frac{\partial u}{\partial t}(x,t) N_{k,p}^i(x) d\Omega_i = \sum_{j \in I_i} \frac{d}{dt} \alpha_j^i(t) \int_{\Omega_i} N_{j,p}^i(x) N_{k,p}^i(x) d\Omega_i \quad (4)$$

From equalities (3) and (4), we deduce the following equalities :

$$\sum_{j \in I_i} \frac{d}{dt} \alpha_j^i(t) \int_{\Omega_i} N_{j,p}^i(x) N_{k,p}^i(x) d\Omega_i + \sum_{j \in I_i} \alpha_j^i(t) \int_{\Omega_i} (C\nabla N_{j,p}^i(x)) \nabla N_{k,p}^i(x) d\Omega_i = \int_{\Omega_i} f(x,t) N_{k,p}^i(x) d\Omega_i.$$

$$M^i \frac{d}{dt} \alpha^i(t) + R^i \alpha^i(t) = F^i$$

With $M^i = (M^i_{kj})_{\substack{k,j\in I_i\\ 1\le i\le N_d-1}}$

$$M^i_{kj} = \int_{\Omega_i} N^i_{j,p}(x) N^i_{k,p}(x) d\Omega_i$$

$$R^i = (R^i_{kj})_{\substack{k,j\in I_i\\ 1\le i\le N_d-1}}$$

$$R^i_{kj} = C\int_{\Omega_i} \nabla N^i_{j,p}(x)\nabla N^i_{k,p}(x) d\Omega_i$$

$$F^i = (F^i_k)_{\substack{k\in I_i\\ 1\le i\le N_d-1}}$$

$$F^i_k = \int_{\Omega_i} f(x,t) N^i_{k,p}(x) d\Omega_i$$

$\frac{d}{dt}\alpha^i(t) = (\mathrm{inv}\, M^i)(F^i - R^i\alpha^i(t))$ because M^i is invertible

[11]

3.2. Numerical Integration

$$R^i_{kj} = C\int_{\Omega_i} \nabla N^i_{j,p}\nabla N^i_{k,p} d\Omega_i \quad (5)$$

Let's set $x = (x_1, x_2)$. $\nabla N^i_{j,p}\begin{pmatrix}\frac{\partial N^i_{j,p}}{\partial x_1}(x) & \frac{\partial N^i_{j,p}}{\partial x_2}(x)\end{pmatrix}$ and $\nabla N^i_{k,p}\begin{pmatrix}\frac{\partial N^i_{k,p}}{\partial x_1}(x) & \frac{\partial N^i_{k,p}}{\partial x_2}(x)\end{pmatrix}$ So we have :

$$\nabla N^i_{j,p}\nabla N^i_{k,p} = \frac{\partial N^i_{j,p}}{\partial x_1}(x)\frac{\partial N^i_{k,p}}{\partial x_1}(x) + \frac{\partial N^i_{j,p}}{\partial x_2}(x)\frac{\partial N^i_{k,p}}{\partial x_2}(x)$$

The relationship (5) becomes :

$$R^i_{kj} = C\int_{\Omega_i}\left(\frac{\partial N^i_{j,p}}{\partial x_1}(x)\frac{\partial N^i_{k,p}}{\partial x_1}(x) + \frac{\partial N^i_{j,p}}{\partial x_2}(x)\frac{\partial N^i_{k,p}}{\partial x_2}(x)\right) d\Omega_i$$

Let F be the parametrization of the domain $\Omega \subset \mathbb{R}^2$.

$$F : \widehat{\Omega} \to \Omega \subset \mathbb{R}^2$$

$$\varepsilon \longmapsto x = (x_1, x_2) = F(\varepsilon)$$

$$x = F(\varepsilon) \Longrightarrow d\Omega_i = \|F'(\varepsilon)\|_2 d\varepsilon.$$

With $\|F'(\varepsilon)\|_2 = \sqrt{(x_1'(\varepsilon))^2 + (x_2'(\varepsilon))^2}$.

$N^i_{j,p}(x) = N^i_{j,p}(F(\varepsilon))$
$N^i_{j,p}(x) = \widehat{N}^i_{j,p}(\varepsilon)$ because $N^i_{j,p}\circ F = \widehat{N}^i_{j,p}$.

So we obtain :

$$\frac{\partial N^i_{j,p}(x)}{\partial x_1} = \frac{d\widehat{N}^i_{j,p}}{d\varepsilon}\times\frac{d\varepsilon}{dx_1}.$$
$$\frac{\partial N^i_{j,p}(x)}{\partial x_1} = \frac{d\widehat{N}^i_{j,p}}{d\varepsilon}\times\frac{1}{\frac{dx_1}{d\varepsilon}} \text{ with } \frac{dx_1}{d\varepsilon} \neq 0.$$

$$\frac{\partial N^i_{k,p}(x)}{\partial x_1} = \frac{d\widehat{N}^i_{k,p}}{d\varepsilon}\times\frac{d\varepsilon}{dx_1}.$$
$$\frac{\partial N^i_{k,p}(x)}{\partial x_1} = \frac{d\widehat{N}^i_{k,p}}{d\varepsilon}\times\frac{1}{\frac{dx_1}{d\varepsilon}} \text{ with } \frac{dx_1}{d\varepsilon} \neq 0.$$

$$\frac{\partial N^i_{k,p}(x)}{\partial x_2} = \frac{d\widehat{N}^i_{k,p}}{d\varepsilon}\times\frac{d\varepsilon}{dx_2}.$$
$$\frac{\partial N^i_{k,p}(x)}{\partial x_2} = \frac{d\widehat{N}^i_{k,p}}{d\varepsilon}\times\frac{1}{\frac{dx_2}{d\varepsilon}} \text{ with } \frac{dx_2}{d\varepsilon} \neq 0.$$

$$\frac{\partial N^i_{j,p}(x)}{\partial x_2} = \frac{d\widehat{N}^i_{j,p}}{d\varepsilon}\times\frac{d\varepsilon}{dx_2}.$$
$$\frac{\partial N^i_{j,p}(x)}{\partial x_2} = \frac{d\widehat{N}^i_{j,p}}{d\varepsilon}\times\frac{1}{\frac{dx_2}{d\varepsilon}} \text{ with } \frac{dx_2}{d\varepsilon} \neq 0.$$

The relationship (5) becomes :

$$R^i_{kj} = C\int_{\Omega_i}\frac{d\widehat{N}^i_{j,p}}{d\varepsilon}\frac{d\widehat{N}^i_{k,p}}{d\varepsilon}\times\left(\frac{1}{(\frac{dx_1}{d\varepsilon})^2} + \frac{1}{(\frac{dx_2}{d\varepsilon})^2}\right)\|F'(T(v^i))\|_2 d\varepsilon \quad (6)$$

We are going to define a function :
T : $v^i \in [-1;1] \longmapsto \varepsilon \in [\chi_i;\chi_{i+1}]$ such as $T(-1) = \chi_i$ and $T(1) = \chi_{i+1}, 1 \le i \le (N_d - 1)$

$$T(v^i) = \frac{1}{2}(\chi_{i+1} - \chi_i)v^i + \frac{1}{2}(\chi_{i+1} + \chi_i)$$

$$T'(v^i) = \frac{1}{2}(\chi_{i+1} - \chi_i)$$

The relationship (6) becomes :

$$R^i_{kj} = \frac{C(\chi_{i+1}-\chi_i)}{2}\int_{-1}^{1}\frac{d\widehat{N}^i_{j,p}(T(v^i))}{d\varepsilon}\frac{d\widehat{N}^i_{k,p}(T(v^i))}{d\varepsilon}\left(\frac{1}{\left(\frac{dx_1(T(v^i))}{d\varepsilon}\right)^2}+\frac{1}{\left(\frac{dx_2(T(v^i))}{d\varepsilon}\right)^2}\right)\|F'(T(v^i))\|_2 dv^i$$

Using the Gaussian quadrature method, R^i_{kj} becomes :

$$R^i_{kj} = \sum_{r=pos1(i)}^{pos2(i)} l^i_r \frac{d\widehat{N}^i_{j,p}(T(v^i_r))}{d\varepsilon}\frac{d\widehat{N}^i_{k,p}(T(v^i_r))}{d\varepsilon}\left(\frac{1}{\left(\frac{dx_1(T(v^i_r))}{d\varepsilon}\right)^2}+\frac{1}{\left(\frac{dx_2(T(v^i_r))}{d\varepsilon}\right)^2}\right)\|F'(T(v^i_r))\|_2$$

With $l^i_r = \frac{C(\chi_{i+1}-\chi_i)}{2}\omega^i_r$ then $pos1(i) = (i-1)Npgs + 1, \quad pos2(i) = iNpgs$ and $1 \le i \le N_d - 1$. $Npgs$ is the number of Gaussian points per segment, ω^i_r and ε^i_r are the Gaussian weights and nodes, respectively.

$$\begin{aligned}
F^i_k &= \int_{\Omega_i} f(x,t)N^i_{k,p}(x)d\Omega_i\\
F^i_k &= \int_{\hat{\Omega}_i} f(F(\varepsilon),t)N^i_{k,p}(F(\varepsilon))\|F'(\varepsilon)\|_2 d\varepsilon\\
F^i_k &= \int_{-1}^{1}\frac{(\chi_{i+1}-\chi_i)}{2} f(T(v^i),t)\widehat{N}^i_{k,p}(T(v^i))\|F'(T(v^i))\|_2 dv\\
F^i_k &= \sum_{r=pos1(i)}^{pos2(i)} l^i_r f(F(T(v^i_r)),t)\widehat{N}^i_{k,p}(T(v^i_r))\|F'(T(v^i_r))\|_2
\end{aligned}$$

With $l^i_r = \frac{C(\chi_{i+1}-\chi_i)}{2}\omega^i_r$ then $pos1(i) = (i-1)Npgs + 1, \quad pos2(i) = iNpgs$ and $1 \le i \le N_d - 1$. $Npgs$ is the number of Gaussian points per segment, ω^i_r and ε^i_r are the Gaussian weights and nodes, respectively.

$$\begin{aligned}
M^i_{kj} &= \int_{\Omega_i} N^i_{k,p}(x)N^i_{j,p}(x)d\Omega_i\\
&= \int_{\widehat{\Omega_i}} \widehat{N}^i_{j,p}(\varepsilon)\widehat{N}^i_{k,p}(\varepsilon)\|F'(\varepsilon)\|_2 d\varepsilon\\
&= \int_{-1}^{1}\frac{(\chi_{i+1}-\chi_i)}{2}\widehat{N}^i_{j,p}(T(v^i))\widehat{N}^i_{k,p}(T(v^i))\|F'(T(v^i))\|_2 dv^i\\
M^i_{kj} &= \sum_{r=pos1(i)}^{pos2(i)} l^i_r \widehat{N}^i_{j,p}(T(v^i_r))\widehat{N}^i_{k,p}(T(v^i_r))\|F'(T(v^i_r))\|_2
\end{aligned}$$

With $l^i_r = \frac{C(\chi_{i+1}-\chi_i)}{2}\omega^i_r$ then $pos1(i) = (i-1)Npgs + 1, \quad pos2(i) = iNpgs$ and $1 \le i \le N_d - 1$. $Npgs$ is the number of Gaussian points per segment, ω^i_r and ε^i_r are the Gaussian weights and nodes, respectively.

3.3. Treatment of the Initial Condition

Knowing that $u(x,0) = u_0(x)$, we obtain the following equalities :

$$\int_{\Omega_i} u(x,0)N^i_{k,p}(x)d\Omega_i = \int_{\Omega_i} u_0(x)N^i_{k,p}(x)d\Omega_i$$

$$\int_{\Omega_i} u^i(x,0)N^i_{k,p}(x)d\Omega_i = \int_{\Omega_i} u_0(x)N^i_{k,p}(x)d\Omega_i$$

$$\sum_{j\in I_i}\alpha^i_j(0)M^i_{kj} = \int_{\Omega_i} u_0(x)N^i_{k,p}(x)d\Omega_i$$

$$M^i\alpha^i(0) = B^i$$

Where $B^i = (B^i_k)_{\substack{k\in I_i\\ 1\le i\le N_d-1}}$ and $B^i_k = \int_{\Omega_i} u_0(x)N^i_{k,p}(x)d\Omega_i$

We infer that $\alpha^i(0) = (invM^i)B^i$ because M^i is invertible [11], with $(invM^i)$ the inverse matrix of M^i.

$$B_k^i = \int_{\Omega_i} u_0(x) N_{k,p}^i(x) d\Omega_i = \int_{\widehat{\Omega}_i} u_0(F(\varepsilon)) \widehat{N}_{k,p}^i(\varepsilon) \|F'(\varepsilon)\|_2 d\varepsilon$$
$$= \int_{-1}^{1} u_0(F(T(v^i))) \widehat{N}_{k,p}^i(T(v^i)) \|F'(T(v^i))\|_2 dv^i$$
$$B_k^i = \sum_{r=pos1(i)}^{pos2(i)} l_r^i u_0(F(T(v_r^i))) \widehat{N}_{k,p}^i(T(v_r^i)) \|F'(T(v_r^i))\|_2$$

With $l_r^i = \frac{C(\chi_{i+1}-\chi_i)}{2}\omega_r^i$ then $pos1(i) = (i-1)Npgs + 1, \quad pos2(i) = iNpgs$ and
$1 \leq i \leq N_d - 1$. $Npgs$ is the number of Gaussian points per segment, ω_r^i and ε_r^i are the Gaussian weights and nodes, respectively.

3.4. Treatment of the Dirichlet Condition

Given that $u = 0$ on $\partial\Omega \times [0, T]$, we have $\alpha_1^1(t) = \alpha_{p+1}^{Nd-1}(t) = 0$. [11]

Solving the heat equation is equivalent to solving the following system :

$$\begin{cases} \dfrac{d}{dt}\alpha^i(t) = (invM^i)(F^i - R^i\alpha^i(t)) \\ \alpha^i(0) = (invM^i)B^i \\ \alpha_1^1(t) = 0 \\ \alpha_{p+1}^{Nd-1}(t) = 0 \end{cases}$$

Under the CFL condition $|C|\dfrac{\Delta t}{(\Delta x)^2} \leq \dfrac{1}{2}$ [33].

Let $t_i = t_0 + i\Delta t$ for $i = 0, \ldots, N$ be a regular grid of points,
with step size Δt such that :
$\Delta t = \dfrac{T - t_0}{N}$ where $t_0 = 0$.

Let $G(t_i; \alpha^i(t)) = (invM^i)(F^i - R^i\alpha^i(t))$.

We will solve the above system using the RUNGE KUTTA method of order 2. We therefore have :

$$\begin{cases} \alpha^i(t+h) = \alpha^i(t) + \frac{1}{2}(K_1^i + K_2^i) \\ \alpha^i(0) = (invM^i)B^i \\ \alpha_1^1(t) = 0 \\ \alpha_{p+1}^{Nd-1}(t) = 0 \\ K_1^i = \Delta t G(t; \alpha^i(t)) \\ K_2^i = \Delta t G(\tilde{t}, \tilde{\alpha^i}(t)) \\ \text{With } \tilde{t} = t + h \text{ and } \tilde{\alpha^i}(t) = \alpha^i(t) + K_1^i \end{cases}$$

4. Numerical Tests

These numerical tests were carried out using Scilab.

Test 1. $u(x, y, t) = \sin(\pi x)\sin(\pi y)\cos(\pi t), (x, y) \in \Omega$, *the half-circle with center* $(0, 0)$ *and radius* $1, t \in [0; 1]$. *We set* $t_0 = 0, t_{41} = 1$. *We have* $\Delta t = \dfrac{1-0}{41-1}$, $\Delta t = 0.025$.

$u_0(x, y) = \sin(\pi x)\sin(\pi y)$.

We have : $f(x, y, t) = -\pi\sin(\pi x)\sin(\pi y)\sin(\pi t) + 2\pi^2\sin(\pi x)\sin(\pi y)\cos(\pi t)$.

The parameterization of Ω is given by :

1. An open knots vector of degree 2 :
   X=[0 0 0 0.125 0.25 0.375 0.5 0.625 0.75 0.875 1 1 1]
2. The control points :
   $A_1(1; 0)$,
   $A_2(1; 0.4142136)$,
   $A_3(0.7071068; 0.7071068)$,
   $A_4(0.4142136; 1)$,
   $A_5(0; 1)$,
   $A_6(0; 1)$,
   $A_7(-0.4142136; 1)$,
   $A_8(-0.7071068; 0.7071068)$,
   $A_9(-1; 0.4142136)$,
   $A_{10}(-1; 0)$.

For $t = 0.2$, we compute the rate of $\log\left(\|u - u_h\|_{L^2(\Omega)}\right)$ in function of $log(h)$.

Table 1. *Results of experience 1.*

Space step	$log(h)$	$\log\left(\|\mathbf{u} - \mathbf{u}h\|L^2(\Omega)\right)$	Rate
$h = 0.5$	-0.6931472	-2.5385606	-
$\frac{h}{2}$	-1.3862944	-3.85554028	1.90000
$\frac{h}{4}$	-2.0794415	-5.21419870	1.96013
$\frac{h}{8}$	-2.7725887	-6.68519568	2.12220
$\frac{h}{16}$	-3.4657359	-8.1581334	2.12500

For $t = 0.6$, we compute the rate of $\log\left(\|u - u_h\|_{L^2(\Omega)}\right)$ in function of $log(h)$.

Table 2. Results of experience 1.

Space step	$log(h)$	$\log\left(\|\mathbf{u}-\mathbf{u_h}\|_{\mathbf{L^2(\Omega)}}\right)$	Rate
$h = 0.5$	-0.6931472	-2.9889553	-
$\frac{h}{2}$	-1.3862944	-4.3406776	1.950123
$\frac{h}{4}$	-2.0794415	-5.6992468	1.960001
$\frac{h}{8}$	-2.7725887	-7.1555352	2.10098
$\frac{h}{16}$	-3.4657359	-8.6288667	2.125568

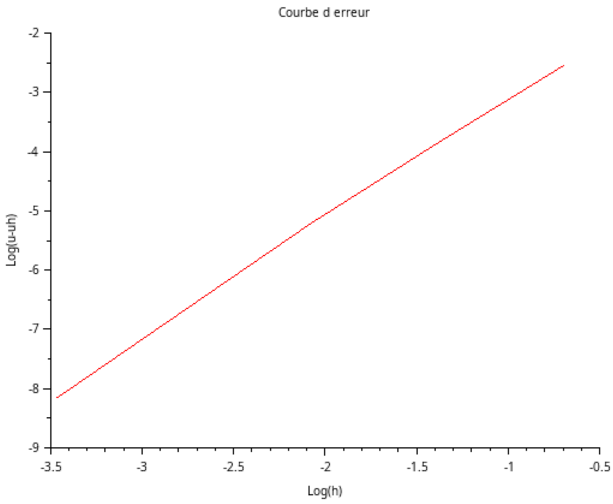

Figure 3. Error curve for $t = 0.2$.

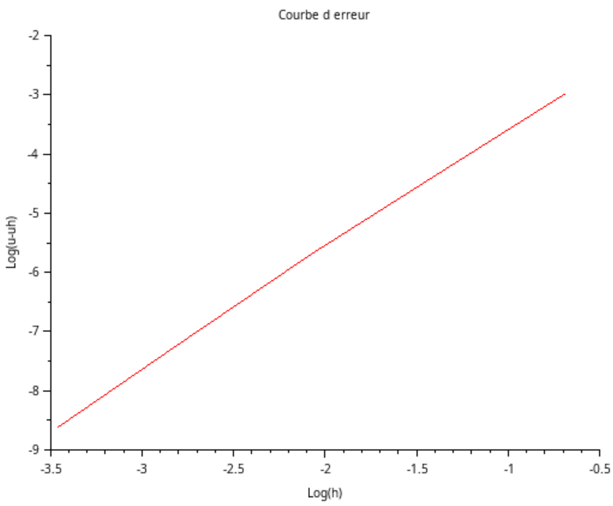

Figure 4. Error curve for. $t = 0.6$.

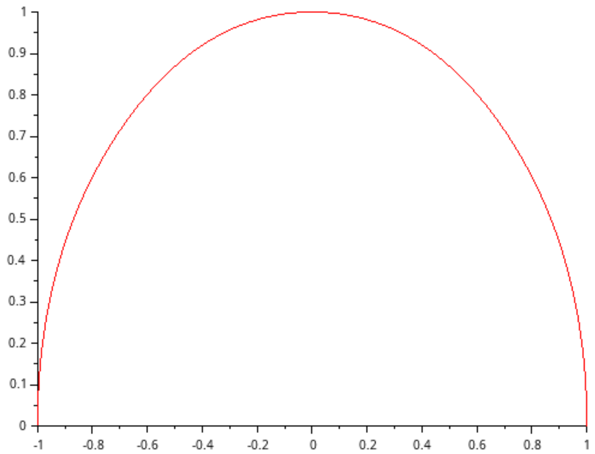

Figure 5. Half-circle with center $(0, 0)$ *and radius* 1.

5. Discussion

In this study, a mesh refinement has been performed for a B-spline basis of degree two. To evaluate the spatial order of convergence for the isogeometric method employed, mesh refinement was conducted at specific values of t. For each time step h, we computed $log(h)$ and $\log\left(\|u - u_h\|_{L^2(\Omega)}\right)$ subsequently determining the slope of $\log\left(\|u - u_h\|_{L^2(\Omega)}\right)$ as a function of $log(h)$. At $t = 0.2$, the spatial order of convergence was calculated and found to be approximately 2.125. The same procedure was applied for $t = 0.6$, yielding similar results, confirming that the spatial order of the method remains around 2.125.

6. Conclusion

In this study, Isogeometric Analysis is applied to solve the one-dimensional heat equation defined on a curve in $\mathbb{R}^2$. The problem is specifically addressed on a semicircular domain. The approximation is based on B-spline basis functions, and particular emphasis is placed on parametrization to enhance the effectiveness of the isogeometric approach. Numerical experiments were conducted under a CFL condition to ensure the stability of the numerical scheme. Looking ahead, this method will be extended to solve other partial differential equations, such as the Burgers equation on curves in $\mathbb{R}^2$. within a one-dimensional setting. Additionally, the isogeometric framework will be employed to model various physical phenomena.

Abbreviations

CFL	Courant-Friedrichs-Lewy
FEM	Finite Element Method
FVM	Finite Volume Method
CAD	Computer Aided Design
IGA	IsoGeometric Analysis

Conflicts of Interest

The authors declare no conflicts of interest.

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