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Modeling and Analysis of Flow Problems in Aquifer Systems Involving Nonlinear Source-terms: Application of Schauder's Fixed Point Theorem

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Abstract: This work deals with two mathematical aspects of subsurface flow problems within aquifer systems, namely Mathematical Modeling and Theoretical Analysis. Concerning the Mathematical Modeling, the classical challenges for this class of problems are a rigorous description of diverse interactions that may take place between different involved aquifers. Recall that the main challenge at this stage is a realistic description of the flows from one aquifer to another passing necessarily through an aquitard which is a porous layer with small permeability coefficient and small thickness (compared with the mean thickness of involved aquifers. In the same way as most of flow phenomena, the governing equations of subsurface flows are based upon conservation laws.) To address the mathematical modeling of water exchange between different aquifers separated by aquitards we expose a mathematical approach based upon the Taylor expansion. Introducing the concept of observers located inside the aquitard and the neighboring aquifers, the mass conservation law has been applied and has led to one mass balance equation for each aquifer. Thanks to this original approach we have recovered the well-known mass balance equations exposed in the literature for flow problems in aquifer systems. Due to the assumptions of small thickness and homogeneity of the absolute permeability of aquitards for our framework the water flow is supposed vertical in aquitards and so we deal with one-dimensional flows there. This is the reason why the Taylor expansion deployed there concerns only the vertical space variable. The flux continuity has been applied to get the coupling of flow equations in the two aquifers. Since the flows in aquifers are supposed horizontal it is clear that the interface aquitard/aquifer flux acts as an additional source-term for each aquifer (and not a boundary term). Concerning the Theoretical Analysis of the global system of elliptic equations (as the flow is supposed to be submitted to a steady state) the Schauder Fixed Point Theorem has been applied for facing the nonlinearity of the right-hand sides of the system. This is the way we have got the existence of a solution to the system, but not the uniqueness. Thanks to a monotonicy assumption on the right-hand side vector-function we get the uniqueness of the solution. Finally the stability of that solution has been established under appropriate conditions.

Keywords: 3-Layer Aquifer System, Mathematical Modeling and Analysis, Schauder's Fixed Point Theorem, Stability Result

1. Introduction

The mathematical modeling and analysis of flows in aquifer systems remains a challenging topic for diverse reasons such that: (i) Complexity of interactions between different parts of an aquifer system; (ii) Possibility that source/sink terms depend on hydraulic pressures in the aquifers and so give rise to nonlinearities in the mathematical model; etc. Mathematical models for aquifer systems are well established. One of the main references on this topic is a book due to Bear and Bachmat [1].

In our work we consider an aquifer system made up of two aquifers separated with an aquitard as illustrated in figure 1 below. Our aim is to expose another way to recover the well-known governing equations for this aquifer system (under some simplifying assumptions) and to theoretically analyze them in the case where the source-terms are depending on the hydraulic pressures.

The rest of the work is subdivided into three parts. The next section is the first part of the work devoted to the derivation of the mathematical model under some assumptions on the flows. We are led to a system of two semi-linear elliptic equations as we suppose that source-terms may depend on hydraulic pressures in the aquifers. In the third section, the second part of the work consists to theoretically investigate the existence, the uniqueness and the stability of the solution to the mathematical model under consideration. For that purpose the Schauder's fixed point theorem is applied to prove the existence of (at least) one solution. The uniqueness of the solution follows from a monotony type assumption made on the source-terms. The third section ends with a stability type inequality result. The last section is devoted to the conclusion of the work and its perspectives.

2. Mathematical Modeling of the Physical Problem

Before going through the exposition of the mathematical derivation of the governing equations for aquifer systems we should start with giving some elementary definitions associated with the physical model under consideration in this work.

2.1. Physical Model

2.1.1. Some Definitions and Basic Notions for Flow in Porous Media

In the context of subsurface flows the *hydraulic load* is the pressure of the groundwater in pore spaces or fractures. This point of view remains valid for any hydraulic or porous systems. The hydraulic load is typically expressed in units of pressure such as pascals (Pa) or bars (bar).

In the field of hydrology, a *porous system* refers to a geological or geomorphological medium consisting of interconnected pore spaces that allow for the storage and movement of groundwater.

The *porosity* ϕ_M of a porous medium M is the ratio defined as "volume of pore spaces inside M" over "total volume of M". It is then clear that: $0 \leq \phi_M \leq 1$. The porosity is a macroscopic parameter that may be varying from one macro-point to another (see Figure 1 below for an illustration of the notion of macroscopic point or macro-point in the context of porous media). In the sequel points from the space $\mathbf{R}^3$ are implicitly considered as macro-points since the space $\mathbf{R}^3$ is associated with a system of macroscopic space variable $\mathbf{x} = (x_1, x_2, x_3)$ in this work (unless otherwise mentions).

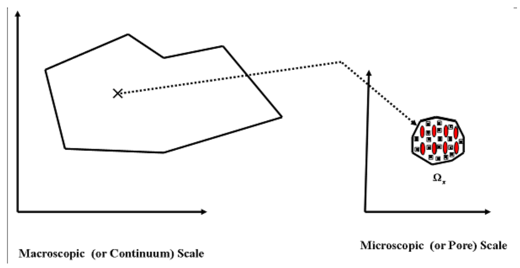

Figure 1. *Representation of a point* $\mathbf{x}$ *from macroscopic (or continuum) scale by a small porous medium* $\Omega_{\mathbf{x}}$ *at pore (or microscopic) scale.*

The *Hydrology* is a scientific area that studies the dynamics and temporal/spatial distribution of water, as well as its mechanical and physical properties, in various components of the hydrological cycle including porous systems like aquifer systems (see G. de Marsily [4]). Recall that an aquifer is a geological formation, typically a porous rock or sedimentary layer, that gets the ability to store and transmit water in significant quantities. For some range of macroscopic scales (in opposition to the pore scale) that ability is quantifiable and is called the *absolute permeability*. It is currently expressed in *darcy* for reference to the french hydraulic engineer Henry Darcy (1803-1858) who was the first to discover in 1856 the Law governing macroscopically the filtration of an incompressible fluid through a porous medium. We will be coming back to the Darcy's Law further. So the absolute permeability is a macroscopic hydrogeological parameter. It is well known that the water flow in a porous structure (in general not only in aquifer systems) is rigorously described at the macroscopic scale by the continuity (or mass balance) equation that expresses the conservation of the mass of water (see for instance C.M. Marle [6], J. Bear and Y. Bachmat [1], A. Njifenjou [7])

$$\frac{\partial[\phi\rho]}{\partial t} + div[\rho\mathbf{V}] = f \qquad in\, a\, porous\, medium\, \mathcal{M} \quad (1)$$

where ϕ is the porosity function (it is generally assumed that ϕ does not depend on time variable, physically it means that the porous medium $\mathcal{M}$ is almost non-deformable), ρ is the water macroscopic density, f models diverse sources (continuously or discretely distributed in the porous medium under consideration) while $\mathbf{V}$ is the Darcy's velocity of the water flowing inside $\mathcal{M}$. It is well known that this velocity is given by the so-called Dary's law stating that :

$$V = -\frac{K(\mathbf{x})}{\mu} grad\,[P + \rho g x_3] \quad (2)$$

where $\mathbf{V}$ the Darcy's velocity and P the macroscopic water pressure are two unknown functions, and where $K(.)$ is the absolute permeability that may be a full symmetric tensor (it is the case for anisotropic porous media) spatially varying (it is the case for non-homogeneous porous media), μ is the water viscosity supposed constant and $g = |\mathbf{g}|$, with $\mathbf{g}$ the gravitational acceleration. Note that the vertical axis (O, x_3) is oriented from down to top and that $|.|$ is the Euclidean norm in $\mathbb{R}^3$.

Aquifers are characterized by their high permeability, which allows water to flow relatively freely through interconnected pore spaces. They serve as important underground reservoirs of water that could be exploited in case of necessity for human or industrial needs.

An aquitard is a thin layer made up of a geological formation separating two aquifers (at least) and getting a lower permeability compared to those of associated aquifers. Such association of aquifers and aquitards is called Aquifer System in this work. It is worth mentioning that the thickness of an aquitard is too small in front of that of an aquifer. It is the reason why we will be neglecting terms of the form $O(e^2)$ in what follows, where e stands for the thickness of aquitards under consideration in what follows. Note that an aquitard may also contain water within its pore spaces but in smaller quantities. Roughly speaking an aquitard restricts the flow of water and acts as a barrier or confining layer, impeding the vertical or horizontal movement of groundwater. Aquitards are often made up of compacted clay, shale, or low-permeability rock formations.

2.2. *Mathematical Modeling of the Flow within Aquifer Systems: Case of a 3-layer System*

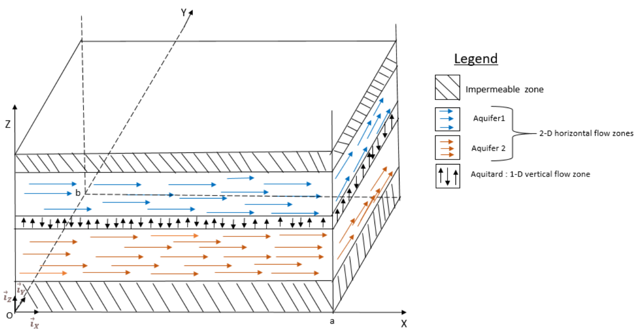

Figure 2. *A system of 2 aquifers interconnected through an aquitard.*

Physical Assumptions

(A1) -Water displacements in aquifers 1 and 2 are two-dimensional displacements parallel to the reference-plane (O, i_X, i_Y) where is embedded the reference-domain Ω defined as: $\Omega =]0, a[\times]0, b[$ (see Figure 2 above);
(A2) -Water displacements in aquitard are parallel to the vector i_Z (i.e. the flow in the aquitard is vertical);
(A3) -The thickness e and the absolute permeability k of the aquitard are strictly positive constant parameters;
(A4) -The overall flow is incompressible i.e. the water density is a constant;
(A5) -There is no source term.

Derivation of the Governing Equations for flows in a 3-layer system

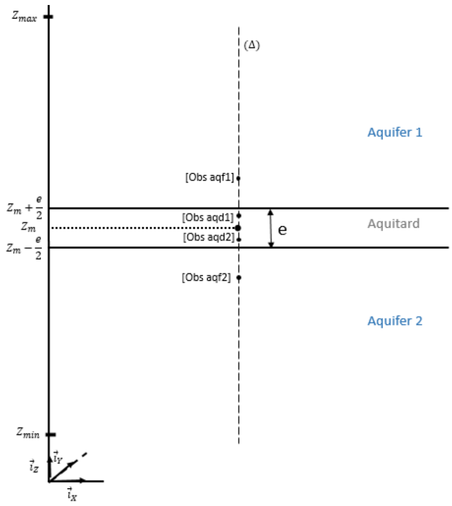

Figure 3. *Water flow exchange between the two aquifers, passing through the aquitard, with focal points in each zone materialized with observers positioned along the line-segment* (Δ) *.*

Recall that the governing equations for one-phase flows in a 3-layer system are mathematical expressions of mass conservation principle. Roughly speaking the flows in aquifers 1 and 2 involve diffusion terms $-div\left[\Lambda^k \nabla u^k\right]$ and source terms f_k, for k=1,2. In addition there is an exchange of water between aquifers 1 and 2 through the aquitard. In view to get the governing equations for the flows in Aquifer 1 and Aquifer 2 let us investigate the mathematical expression of the mass exchange terms. Denoting by $[ExchangeTerm]_1$ and $[ExchangeTerm]_2$ the mass exchange terms respectively in Aquifer 1 and Aquifer 2, we have necessarily:

$$[ExchangeTerm]_1 + [ExchangeTerm]_2 = 0$$

as a consequence of the assumptions *(A4)* and *(A5)*. In other words the aquitard serves as a transit zone only. The structure of the mass balance equation in these two aquifers is:

$$-div[\Lambda^i \nabla u^i] + [ExchangeTerm]_i = f^i(x, u^1, u^2)$$

$$in \quad \Omega, \quad for\, i = 1, 2.$$

Denote by u the overall hydraulic head in the system. We then have:

$$\forall (X, Y) \in \overline{\Omega} \qquad \forall Z_{min} \le Z \le Z_{max}$$

$$u(X,Y,Z) = \begin{cases} u^1(X,Y) \;\; if \;\; Z_m + \dfrac{e}{2} \le Z \le Z_{max} \\ \qquad\qquad\qquad \text{(Aquifer 1)} \\ u_{aqd}(Z) \;\; if \;\; Z_m - \dfrac{e}{2} \le Z \le Z_m + \dfrac{e}{2} \\ \qquad\qquad\qquad \text{(Aquitard)} \\ u^2(X,Y) \;\; if \;\; Z_{min} \le Z \le Z_m - \dfrac{e}{2} \\ \qquad\qquad\qquad \text{(Aquifer 2).} \end{cases}$$

Let us now concentrate on the quantification of the mass of water exchanged between aquifers 1 and 2 per unit of surface (for a given period of time). If we assume that the overall hydraulic head belongs to the function space $C^3(\overline{\Omega} \times [Z_{min}, Z_{max}])$, then the continuity of the hydraulic head gradient is ensured at any point (X, Y, Z) of $\overline{\Omega} \times [Z_{min}, Z_{max}])$. In particular the hydraulic head gradient in the Z-direction is continuous at the point (x, y, Z_m). Therefore we have:

$$u'_{aqd}(Z_m^-) = u'_{aqd}(Z_m^+) = u'_{aqd}(Z_m). \tag{3}$$

Applying the Taylor expansion to u_{aqd} (assumed to be from the class of $C^3 - functions$) around the point Z_m from the axis (Δ) leads to what follows:

$$\begin{aligned} u_{aqd}(Z_m + \frac{e}{2}) &= u_{aqd}(Z_m) + \frac{1}{1!}u'_{aqd}(Z_m)\frac{e}{2} + \\ &+ \frac{1}{2!}u''_{aqd}(Z_m)\left(\frac{e}{2}\right)^2 + O(e^3) \qquad (4) \\ u_{aqd}(Z_m - \frac{e}{2}) &= u_{aqd}(Z_m) + \frac{1}{1!}u'_{aqd}(Z_m)\left(-\frac{e}{2}\right) + \\ &+ \frac{1}{2!}u''_{aqd}(Z_m)\left(-\frac{e}{2}\right)^2 + O(e^3) \qquad (5) \end{aligned}$$

By subtracting side by side (5) from (4), we obtain:

$$e\, u'_{aqd}(Z_m) = u_{aqd}(Z_m + \frac{e}{2}) - u_{aqd}(Z_m - \frac{e}{2}) + O(e^3) \tag{6}$$

Let us multiply the two sides of the preceding equation by $-k/e$. Then we get the flux Φ of water at the interface $[aquifer\,1/aquitard]$ in the direction i_Z as seen by the observer [Obs aqd**1**] (when the term $O(e^2)$ is negligible):

$$\Phi\left[\left(Z_m + \frac{e}{2}\right)^-\right] = \frac{-k}{e}\left[u_{aqd}(Z_m + \frac{e}{2}) - u_{aqd}(Z_m - \frac{e}{2})\right]. \tag{7}$$

By continuity of the hydraulic head u we get what follows:

$$u_{aqd}(Z_m + \frac{e}{2}) = u^1(x, y) \quad and \quad u_{aqd}(Z_m - \frac{e}{2}) = u^2(x, y) \tag{8}$$

Inserting (8) in (7) yields the following relation:

$$\Phi\left[\left(Z_m + \frac{e}{2}\right)^-\right] = \frac{-k}{e}\left[u^1(x, y) - u^2(x, y)\right]. \tag{9}$$

A similar reasoning leads to what follows:

$$\Phi\left[\left(Z_m - \frac{e}{2}\right)^+\right] = \frac{-k}{e}\left[u^2(x, y) - u^1(x, y)\right]. \tag{10}$$

It follows from the continuity of the flux at the points $Z_m + \frac{e}{2}$ and $Z_m - \frac{e}{2}$ that :

$$\begin{cases} \Phi\left[\left(Z_m + \frac{e}{2}\right)^-\right] + \Phi\left[\left(Z_m + \frac{e}{2}\right)^+\right] = 0 \\ \Phi\left[\left(Z_m - \frac{e}{2}\right)^-\right] + \Phi\left[\left(Z_m - \frac{e}{2}\right)^+\right] = 0. \end{cases} \tag{11}$$

Hence we get the following flux equations:

$$\begin{cases} \Phi\left[\left(Z_m + \frac{e}{2}\right)^+\right] = \dfrac{k}{e}\left[u^1(x, y) - u^2(x, y)\right] \\ \Phi\left[\left(Z_m - \frac{e}{2}\right)^-\right] = \dfrac{k}{e}\left[u^2(x, y) - u^1(x, y)\right]. \end{cases} \tag{12}$$

The preceding equations describe flux exchanges seen respectively by Observer aqf1 and Observer aqf2 placed respectively in aquifer 1 and aquifer 2. We can deduce that the water-flow mass balance in the aquifer 1 is given by the following equation:

$$\overbrace{-div\left[\Lambda^1 \nabla u^1\right]}^{diffusion\ term} + \overbrace{\gamma.(u^1 - u^2)}^{exchange\ term} = \overbrace{f^1(x, u^1, u^2)}^{source\ term} \ \ in\ \Omega$$

where we have set:

$$\gamma = \frac{k}{e}$$

with the following *Dirichlet* boundary condition:

$$u^1 = g^1 \qquad on\ \partial\Omega.$$

Similarly the water-flow mass balance in the aquifer 2 is given by:

$$\overbrace{-div\left[\Lambda^2 \nabla u^2\right]}^{diffusion\ term} + \overbrace{\gamma.(u^2 - u^1)}^{exchange\ term} = \overbrace{f^2(x, u^1, u^2)}^{source\ term} \ \ in\ \Omega. \tag{13}$$

with the associated *Dirichlet* boundary conditions set as follows :

$$u^2 = g^2 \qquad on\ \partial\Omega.$$

We have to solve (in a relevant function space) the problem consisting to find a pair of functions $\underline{u} = \{u^1, u^2\}$ such that (s.t. for short in the sequel) :

$$\begin{cases} -div\left[\Lambda^1 \nabla u^1\right] + \gamma.(u^1 - u^2) = f^1(x, \underline{u}) & in\ \Omega \\ \qquad \text{(Mass balance in Aquifer 1)} & \\ -div\left[\Lambda^2 \nabla u^2\right] + \gamma.(u^2 - u^1) = f^2(x, \underline{u}) & in\ \Omega \\ \qquad \text{(Mass balance in Aquifer 2)} & \\ u^1 = g^1 & on\ \partial\Omega \\ u^2 = g^2 & on\ \partial\Omega. \end{cases} \tag{14}$$

Discussion : As mentioned in our introduction the previous system of balance equations that we are led to is well known. But the technique used here to derive those governing balance equations is not conventional as the one based on the so-called Groundwater Control-Volume exposed by several authors in the literature (see for instance Bear and Bachmat [1] and the references therein). Notice that one-dimensional vertical flow in the aquitard is a common assumption in the context of multi-layers flows. In this context the number of unknown scalar functions (pressure head in each aquifer) grows with the number of involved aquifers and the interactions between them lead to more complex system of partial differential equations.

This work is a starting point of our research project that concerns mathematical modeling, analysis and finite volume computation of groundwater flows in geologically complex aquifers.

3. Existence and Uniqueness of Solution to System (14)

In this section we focus on the theoretical analysis of the system of semi-linear elliptic equations (14) obtained in the previous section, accounting with the prescribed (non-homogeneous) Dirichlet boundary conditions. Notice that the conventional Discrete Duality Finite Volume method is one of powerful tools used today for addressing numerical resolution of anisotropic flow problems in porous media (see for instance Njifenjou et al. [8]).

3.1. Required Sobolev Spaces and Assumptions on the Data

Let us start with recalling some fundamental function spaces and associated results that we will be utilizing in this section and the following ones.

For the sake of conformity to conventional notations of the following well-known Sobolev spaces let us set:

$$H^1(\Omega) \overset{def}{=} \{v \in L^2(\Omega) \,/\, \nabla v \in L^2(\Omega) \times L^2(\Omega)\} \qquad (15)$$

where $\nabla v = \frac{\partial v}{\partial x_i}]_{i=1;2}$ denotes the gradient of v, with partial derivatives of v understood in the sense of distributions.

Note that the following bilinear form:

$$(v\,,\,w)_{H^1} \;=\; \int_\Omega v(x)w(x)dx + \int_\Omega \nabla v(x)\cdot\nabla w(x)dx \quad (16)$$

is the standard scalar product of the Sobolev space $H^1(\Omega)$. The associated norm is denoted by $\|\;.\;\|_{H^1}$. Let us recall the following facts:

1) The space $H^1(\Omega)$ is a Hilbert space with respect to its standard scalar product defined by (16) (see for instance: Brezis [2]);

2) The space $C^\infty(\overline{\Omega})$ is a dense subspace of $H^1(\Omega)$ (see: Theorem 6.6-4 in Ciarlet [3]);

3) The linear continuous operator:

$$H^1(\Omega) \supset C^\infty(\overline{\Omega}) \ni v \;\longmapsto\; v_{|_\Gamma} \in C^0(\Gamma) \subset L^2(\Gamma) \qquad (17)$$

can be extended into a linear continuous operator (called trace operator and sometimes denoted by γ_0) from $H^1(\Omega)$ to $L^2(\Gamma)$, where Γ denotes the boundary of the domain Ω.

As a consequence of the previous third fact the following definition is formulated:

Definition 3.1. (Trace image)
We set:

$$H^{\frac{1}{2}}(\Gamma) \overset{def}{=} \{\, v_{|_\Gamma}; \quad v\ being\ running\ in\ H^1(\Omega)\,\} \quad (18)$$

or, equivalently,

Definition 3.2. (Equivalent definition of the Trace image)
We set:

$$H^{\frac{1}{2}}(\Gamma) \overset{def}{=} \{\, \gamma_0(v); \quad v\ being\ running\ in\ H^1(\Omega)\,\}. \quad (19)$$

An important fact to notice is that:

Proposition 3.1. The Trace image $H^{\frac{1}{2}}(\Gamma)$ is a dense subspace of the Hilbert space $L^2(\Gamma)$ with respect to the norm associated with the standard scalar product of $L^2(\Gamma)$.

A very useful closed subspace of $H^1(\Omega)$ made up of functions v from $H^1(\Omega)$ (of course) with null trace on the boundary of Ω is the so-called $H^1_0(\Omega)$:

$$H^1_0(\Omega) \overset{def}{=} \{v \in H^1(\Omega) \,/\, v_{|_\Gamma} \,=\, 0\,\}. \qquad (20)$$

Note that the following bilinear form:

$$(v\,,\,w)_{H^1_0} \;=\; \int_\Omega \nabla v(x)\cdot\nabla w(x)dx \qquad (21)$$

is the standard scalar product of the Sobolev space $H^1_0(\Omega)$. The associated norm is denoted by $\|\;.\;\|_{H^1_0}$. This norm is equivalent to the one induced in $H^1_0(\Omega)$ by $\|\;.\;\|_{H^1}$. This equivalence is a consequence of the following well known result :

Theorem 3.1. (Poincaré-Friedrichs inequality)
The assumption that $\mathit{\Omega}$ is bounded ensures that:

$$\begin{aligned}\exists\lambda > 0 \quad s.t. \quad &\forall v \in H^1_0(\mathit{\Omega})\\ &\|v\|_{L^2(\mathit{\Omega})} \le \lambda\|\nabla\, v\|_{L^2(\mathit{\Omega})}.\end{aligned} \qquad (22)$$

For $k \in \{1,2\}$ *:*

$\mathit{\Lambda}^k$ is an absolute permeability tensor function (divided by the viscosity supposed to be a constant real number) that satisfies the following properties :

$$\left.\begin{array}{ll}\exists\beta \in \mathbb{R}^*_+\ s.t.\ \forall\xi \in \mathbb{R}^2 & \mathit{\Lambda}^k(x)\xi.\xi \le \beta|\xi|^2\\ \exists\alpha \in \mathbb{R}^*_+\ s.t.\ \forall\xi \in \mathbb{R}^2 & \alpha|\xi|^2 \le \mathit{\Lambda}^k(x)\xi.\xi\end{array}\right\}$$
$$\text{almost everywhere in } \mathit{\Omega} \qquad (23)$$

Remark 3.1. (important for the sequel)

The first inequality of the assumptions (23) ensures that the coefficients $\mathit{\Lambda}^k_{ij}(.)]_{1\le i,j\le 2}$ are $L^\infty(\Omega)-$functions.

For $k \in \{1,2\}$ *:*

$$g^k \in H^{\frac{1}{2}}(\Gamma) \qquad (24)$$

$$f^k : \mathit{\Omega}\times\mathbb{R}^2 \;\to \mathbb{R} \text{ is a Carathéodory function i.e.} \qquad (25)$$
$$x \longmapsto f^k(x,s) \text{ is measurable for all } s \in \mathbb{R}^2$$

and

$$s \longmapsto f^k(x,s) \text{ is continuous for almost every } x \in \Omega.$$

Remark 3.2. (very important for the sequel)

Note that if $v : \Omega \ni x \longmapsto v(x) \in \mathbb{R}^2$ is a measurable function then the function $f^k(., v(.)) : \Omega \ni x \longmapsto f^k(x, v(x))$ is a measurable function.

$\exists \psi_k \in L^2(\Omega)$ such that :

$$\forall s \in \mathbb{R}^2 \qquad \left.\begin{array}{rl} \psi_k(x) \geq 0 & \\ \left|f^k(x,s)\right| \leq & \psi_k(x) \end{array}\right\} \text{a.e. in } \Omega \tag{26}$$

where $|\ .\ |$ is the Euclidean norm on $\mathbb{R}^d$ for $d \in \{1,\ 2,\ 3\}$.

Additional assumption (of Monotony type):

$$\forall \theta \in \mathbb{R}^2 \qquad \forall \tau \in \mathbb{R}^2$$

$$< [f(x,\theta) - f(x,\tau)], (\theta - \tau) > \leq 0 \qquad \text{a.e. in } \Omega. \tag{27}$$

where $< .,. >$ is the standard scalar product defined in $\mathbb{R}^2$ and where f is a vector function with components $(f^1, f^2)^t$, where $(.,.)^t$ stands for the transposition operator.

3.2. *Variational Formulation of the System* (14) *and its Analysis*

Let us start with setting:

$$\mathcal{L}^2(\Omega) \stackrel{def}{=} L^2(\Omega) \times L^2(\Omega),$$
$$\mathcal{H}^1(\Omega) \stackrel{def}{=} H^1(\Omega) \times H^1(\Omega),$$
$$\mathcal{H}^1_0(\Omega) \stackrel{def}{=} H^1_0(\Omega) \times H^1_0(\Omega),$$

and recalling the following classical results that we will be applying further.

Theorem 3.2. *(Lax-Milgram)*

Let H be a real Hilbert space, $\Phi(.,.)$ a bilinear form on $H \times H$ and L(.) a linear form on H. We assume that:

1. $\Phi(.,.)$ is continuous:

$$\exists \alpha > 0 \ \ s.t. \ \ \forall v, w \in H \qquad |\Phi(v,w)| \leq \alpha \|v\|_H \|w\|_H$$

2. $\Phi(.,.)$ is coercive:

$$\exists \beta > 0 \ \ s.t. \ \ \forall v \in H \qquad \beta \|v\|^2_H \leq \Phi(v,v)$$

3. L(.) is continuous:

$$\exists M > 0 \ \ s.t. \ \ \forall v \in H \qquad |L(v)| \leq M \|v\|_H \tag{28}$$

Under the above assumptions there exists a unique $\widetilde{u}$ in H such that

$$\Phi(\widetilde{u}, v) = L(v) \qquad \forall v \in H\,. \tag{29}$$

Theorem 3.3. (*Schauder fixed point theorem*)

Let $(\mathbb{E}, \|\ .\ \|_{\mathbb{E}})$ be a Banach space and consider $\mathbb{K} \subseteq \mathbb{E}$ such that $\mathbb{K}$ is non-empty, convex and compact. If $\mathbb{T} : \mathbb{K} \longrightarrow \mathbb{K}$ is a continuous mapping (with respect to the norm $\|\ .\ \|_{\mathbb{E}}$) then $\mathbb{T}$ gets at least one fixed point from $\mathbb{K}$ i.e. there exists some individual in $\mathbb{K}$, let us say ξ, such that $\xi = \mathbb{T}(\xi)$.

Theorem 3.4. (*Rellich-Kondrachov theorem*)

Let $\Omega \subseteq \mathbb{R}^N$ be an open, bounded Lipschitz-domain, and let $1 \leq P < \infty$, The space $W^{1,P}(\Omega)$ is compactly imbedded in $L^P(\Omega)$. In particular, $H^1(\Omega)$ is compactly imbedded in $L^2(\Omega)$.

Without harming the generality, we assume in the current section that for $k = 1, 2$:

$$g^k \equiv 0 \qquad \text{on} \quad \Gamma$$

and we introduce the following elliptic problem :

$$\begin{cases} \text{Find } \ \underline{u} \in \mathcal{H}^1_0(\Omega) \text{ such that :} \\ -div\left[\Lambda^1 \nabla\left(u^1\right)\right] + \gamma(u^1 - u^2) = f^1(x, \underline{u}) \quad in\ \Omega \\ -div\left[\Lambda^2 \nabla\left(u^2\right)\right] + \gamma(u^2 - u^1) = f^2(x, \underline{u}) \quad in\ \Omega \end{cases} \tag{30}$$

Definition 3.3. We call a weak solution to the system (30), any pair of functions $\underline{u} = \left(u^1, u^2\right)$ getting ability to solve the following problem:

$$\begin{cases} Find\ \underline{u} = (u^1, u^2) \ \in \mathcal{H}^1_0(\Omega)\ such\ that \\ \mathcal{B}(\underline{u}, \underline{v}) = L(\underline{v}) \qquad\qquad \forall \underline{v} = (v^1, v^2) \in \mathcal{H}^1_0(\Omega), \end{cases} \tag{31}$$

where we have set

$$\mathcal{B}(\underline{u}, \underline{v}) := \sum_{k=1}^{2} \int_\Omega \Lambda^k \nabla u^k . \nabla v^k dx + \\ + \int_\Omega \gamma(u^1 - u^2)(v^1 - v^2) dx, \tag{32}$$

$$L(\underline{v}) := \sum_{k=1}^{2} \int_\Omega f^k(x, \underline{u}) v^k dx. \tag{33}$$

Remark 3.3. Note that the elliptic problem (30) is equivalent to the variational problem (31).

Let us give now an important result of our work. The proof of that result is based on an application of Schauder's fixed point theorem.

Proposition 3.2. Under the assumptions (23)-(27) the semi-linear variational problem (31) gets one and only one solution.

Proof: We shall proceed in two steps.

First step: Existence of a solution to the semi-linear variational problem (31)

Let us put in place ingredients for proving the existence result concerning a solution to (31). For that purpose let ζ be a given function from $\mathcal{L}^2(\Omega)$ (so ζ is a datum) and consider the following variational problem :

$$\begin{cases} Find\, u_\zeta = (u^1_\zeta, u^2_\zeta) \ \in \mathcal{H}^1_0(\Omega) \text{ such that} \\ \mathcal{B}(u_\zeta, v) = L_\zeta(v) \qquad \forall v = (v^1, v^2) \in \mathcal{H}^1_0(\Omega), \end{cases} \tag{34}$$

where we have set :

$$\mathcal{B}(u_\zeta, v) := \sum_{k=1}^{2} \int_\Omega \Lambda^k \nabla u^k . \nabla v^k dx + \\ + \int_\Omega \gamma(u^1 - u^2)(v^1 - v^2) dx, \tag{35}$$

$$L_\zeta(v) := \sum_{k=1}^{2} \int_\Omega f^k(x, \zeta) v^k dx. \tag{36}$$

Under the assumptions (23)-(26) the Lax-Milgram Theorem ensures the existence and uniqueness of the solution of (34). On the other hand we have

$$\lambda^- \|u_\zeta\|^2_{\mathcal{H}^1_0(\Omega)} \le \mathcal{B}(u_\zeta, u_\zeta) = \\ = L(u_\zeta) \le 2\|g\|_{L^2} \|u_\zeta\|_{\mathcal{H}^1_0(\Omega)}$$

which implies that

$$\|u_\zeta\|_{\mathcal{H}^1_0(\Omega)} \le R \qquad \forall \zeta \in \mathcal{L}^2(\Omega),$$

with

$$R = \frac{2\|g\|_{L^2}}{\lambda^-} \quad \text{(notice that } R \text{ is independent of } \zeta). \tag{37}$$

Now let us introduce the following mapping :

$$\mathbb{T} : \mathcal{L}^2(\Omega) \to \mathcal{H}^1_0(\Omega) \subseteq \mathcal{L}^2(\Omega) \\ \zeta \longmapsto u_\zeta = \mathbb{T}(\zeta)$$

where u_ζ is a function that solves the variational problem (34). As we have shown above the existence and uniqueness of u_ζ are guaranteed by Lax-Milgram theorem under assumptions from (23)-(26). So the mapping $\mathbb{T}$ is well defined over the function space $\mathcal{L}^2(\Omega)$.

Let us now exhibit a convex compact subset of $\mathcal{L}^2(\Omega)$ that is globally invariant for $\mathbb{T}$. We start with considering the following set:

$$K = \{u \in \mathcal{H}^1_0(\Omega), \|u\|_{\mathcal{H}^1_0(\Omega)} \le R\}.$$

where R is given by 37.

It is clear that K is a bounded convex subset of $\mathcal{H}^1_0(\Omega)$. So thanks to Rellich-Kondrachov imbedding theorem, K is a relatively compact subset of $\mathcal{L}^2(\Omega)$. Therefore the closure of K in the sense of $\mathcal{L}^2(\Omega)$-norm, denoted by $\overline{K}^{\|\cdot\|_{\mathcal{L}^2}}$, is a convex compact subset of $\mathcal{L}^2(\Omega)$ such that

$$\mathbb{T}(\overline{K}^{\|\cdot\|_{\mathcal{L}^2}}) \subset \overline{K}^{\|\cdot\|_{\mathcal{L}^2}}. \tag{38}$$

Continuity of $\mathbb{T} : \mathcal{L}^2(\Omega) \longrightarrow \mathcal{L}^2(\Omega)$

Let $(\zeta_n)_n$ be a sequence of $\mathcal{L}^2(\Omega)$ and $\zeta \in \mathcal{L}^2(\Omega)$ such that

$$\zeta_n \longrightarrow \zeta \ \text{ in } \ \mathcal{L}^2(\Omega).$$

We set for $n \in \mathbb{N}$:

$$u_n = T(\zeta_n).$$

Therefore we have

$$\{u_n\}_{n\in\mathbb{N}} \subseteq K \subseteq \mathcal{H}^1_0(\Omega)$$

So the sequence $\{u_n\}$ is bounded in $\mathcal{H}^1_0(\Omega)$. Thus we can extract a subsequence from $\{u_n\}$ still denoted by $\{u_n\}$ such that there exists a function u from $\mathcal{H}^1_0(\Omega)$ satisfying what follows:

$$\begin{cases} u_n \to \ u \text{ in } \mathcal{L}^2(\Omega) \quad (strong\, convergence) \\ u_n \rightharpoonup \ u \text{ in } \mathcal{H}^1_0(\Omega) \quad (weak\, convergence) \end{cases}$$

We should first prove that $u = \mathbb{T}(\zeta)$.
For all $v \in \mathcal{H}^1_0(\Omega)$, we have :

$$\mathcal{B}(u_n, v) = L_{\zeta_n}(v) \tag{39}$$

Applying the dominated convergence theorem to the right-hand side of the equation (39) while passing to the limit in the two sides of this equation yields :

$$\mathcal{B}(u, v) = L_\zeta(v) \qquad \forall v \in \mathcal{H}^1_0(\Omega).$$

The uniqueness of the solution of (34) leads to

$$u = \mathbb{T}(\zeta)$$

So it is clear that all the sequence u_n converges strongly to u in the sense of $\mathcal{L}^2(\Omega)$-norm. So the continuity of the mapping $\mathbb{T}$ from $\mathcal{L}^2(\Omega)$ to $\mathcal{L}^2(\Omega)$ is proven. Therefore we get the continuity of $\mathbb{T}$ over $\overline{K}^{\mathcal{L}^2(\Omega)}$.

First-step conclusion: Thanks to the Schauder's fixed-point theorem (see Theorem 3.3 above) the mapping $\mathbb{T}$ gets at least one fixed point. So we have the guarantee that the problem (31) gets (at least) one solution.

Second step : Uniqueness of the solution to (31).

Suppose that u and $\widetilde{u}$ are two solutions to the problem (31). So we have:

$$\forall v \in \mathcal{H}^1_0(\Omega) \qquad \begin{cases} \mathcal{B}(u, v) = L_u(v), \\ \mathcal{B}(\widetilde{u}, v) = L_{\widetilde{u}}(v). \end{cases}$$

We then deduce that

$$\mathcal{B}(u - \widetilde{u}, v) = L_u(v) - L_{\widetilde{u}}(v) \qquad \forall v \in \mathcal{H}^1_0(\Omega)$$

For $v = u - \widetilde{u}$, we get:

$$\mathcal{B}(u - \widetilde{u}, u - \widetilde{u}) = L_u(u - \widetilde{u}) - L_{\widetilde{u}}(u - \widetilde{u}).$$

Thanks to the coerciveness of $\mathcal{B}(.,.)$ and the assumption

(27) we can see that

$$0 \leq \lambda^{-} \|u - \widetilde{u}\|^2_{\mathcal{H}^1_0(\Omega)} \leq \mathcal{B}\,(u - \widetilde{u}, u - \widetilde{u}) =$$

$$= \sum_{k=1}^{2} \int_{\Omega} \langle \left(f^k(x,u) - f^k(x,\widetilde{u})\right), (u^k - \widetilde{u}^k) \rangle dx \leq 0$$

Therefore

$$\lambda^{-} \|u - \widetilde{u}\|^2_{\mathcal{H}^1_0(\Omega)} = 0$$

Since Ω is a convex (and thus a connected) subset of $\mathbb{R}^2$, we have

$$u \,-\, \widetilde{u} \,=\, \theta \qquad \text{a.e. in } \ \Omega.$$

where θ is a given real number. Since

$$(u \,-\, \widetilde{u})_{|\partial\Omega} \,=\, 0$$

we know that θ is necessarily zero. Hence

$$u \,=\, \widetilde{u} \qquad \text{a.e. in } \ \Omega.$$

3.3. An Inequality Result of Stability Type

We are going to show in what follows that the solution to the variational problem (31)-(33) satisfies a stability type inequality. For that purpose let us take $\underline{v} = \underline{u}$ in the variational equation (31). Therefore we get

$$\sum_{k=1}^{2} \int_{\Omega} \Lambda^k \nabla u^k . \nabla u^k dx + \int_{\Omega} \gamma (u^1 - u^2)^2 \, dx \;=\; = \;\sum_{k=1}^{2} \int_{\Omega} f^k(x, \underline{u}) \, u^k \, dx. \qquad (40)$$

So we deduce that

$$\sum_{k=1}^{2} \int_{\Omega} \Lambda^k \nabla u^k . \nabla u^k dx + \int_{\Omega} \gamma (u^1 - u^2)^2 \, dx \;=\; = \;\sum_{k=1}^{2} \int_{\Omega} f^k(x, \underline{u}) \, u^k \, dx. \qquad (41)$$

Let us set

$$LHS \;=\; \sum_{k=1}^{2} \int_{\Omega} \Lambda^k \nabla u^k . \nabla u^k dx + \int_{\Omega} \gamma (u^1 - u^2)^2 \, dx$$

and

$$RHS \;=\; \sum_{k=1}^{2} \int_{\Omega} f^k(x, \underline{u}) \, u^k \, dx.$$

From the coercivity of the bilinear form $\mathcal{B}(.,.)$ we can easily see that

$$\lambda^{-} \|\underline{u}\|^2_{\mathcal{H}^1_0} \;\leq\; LHS\,. \qquad (42)$$

Thanks to the assumption (26), Cauchy-Schawrz inequality and Poincaré-Friedrichs inequality (see Theorem 3.1) we obtain that

$$RHS \;\leq\; C \|\underline{\psi}\|_{\mathcal{L}^2(\Omega)} \|\underline{u}\|_{\mathcal{H}^1_0} \qquad (43)$$

where C is a nonnegative real number not depending on $\underline{u}$ and where we have set $\underline{\psi} = (\psi_1, \psi_2)$. Thus we have

$$\|\underline{u}\|_{\mathcal{H}^1_0} \;\leq\; C \, \|\underline{\psi}\|_{\mathcal{L}^2(\Omega)}.$$

Therefore we can state what follows:

Proposition 3.3. (Result from stability type)
Under the assumption (26) there exists a nonnegative real number not depending on $\underline{u}$ (the weak solution of the problem (30)), such that :

$$\|\underline{u}\|_{\mathcal{H}^1_0} \;\leq\; C \, \|\underline{\psi}\|_{\mathcal{L}^2(\Omega)}$$

where $\underline{\psi} \stackrel{def}{=} (\psi_1, \psi_2)$. $\square$

4. Conclusion and Perspectives

This work has exposed an alternative way to get governing equations for incompressible flows in an aquifer system. Considering Dirichlet boundary conditions and assuming that source-terms depend on hydraulic pressures we have shown (thanks to Schauder's theorem and a monotony condition on the source-terms) that the obtained semi-linear elliptic equations get a unique solution. Our next objective is the finite volume treatment of the mathematical model analyzed in this work. In that perspective the Brouwer's theorem is expected to play the same role as Schauder's theorem for addressing existence of a finite volume solution. We are also interested in comparing the conventional finite volume solution rate convergence (exposed in [9] for an incompressible flow problem) with the one computed with a new finite volume exposed in [10].

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Abbreviations

a.e.	almost everywhere
s.t.	such that

Conflicts of Interests

The authors have no conflicts of interest from any nature to declare.

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