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Research Article

On the Edge Connectivity of Semi-strong Product Graphs

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Abstract

The concept of edge connectivity was first proposed by K. Menger, and in communication networks and logical networks, edge connectivity can be used to measure network reliability and fault tolerance. The graph product method can be used to construct complex networks, simulate biological molecule interactions etc. At present, research on the edge connectivity of product graphs mainly focuses on the connectivity of standard product graphs, such as Cartesian product graphs, strong product graphs. The unique properties exhibited by non standard product graphs (such as semi-strong product graphs.) in practical applications are worth further exploration. The concept of semi-strong product was proposed by Mordeson and Chang Shyh, that is, for two graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, their semi-strong product $G_1 \oplus G_2$ is a graph whose vertex set is $V(G_1 \oplus G_2) = V(G_1) \times V(G_2)$, and the edge set $E(G_1 \oplus G_2)$ is defined as follows: if (x_1, y_1) and (x_2, y_2) are two vertices in the semi-strong product $G_1 \oplus G_2$, then there is an edge between them if and only if $x_1 = x_2$ and y_1 and y_2 are adjacent in G_2, or x_1 and x_2 are adjacent in G_1 and y_1 and y_2 are adjacent in G_2. And applications of the semi-strong product in fuzzy graphs, symbolic graphs, and finance have shown its broad research prospects. In this article, we mainly study the edge connectivity of semi-strong product graphs, and obtain some exact values. Furthermore, we also give an necessary and sufficient condition for a semi-strong product to be maximally edge-connected.

Keywords

Edge Connectivity, Graph Product, Semi-strong Product

1. Introduction

The concept of edge connectivity was first proposed by K. Menger in 1930s, and a systematic research was obtained by means of Menger's theorem [1]. Menger's theorem shows that the edge connectivity of a graph is equal to its minimum edge-cut set. This theory lays the foundation for the study of edge connectivity.

With the rapid development of information network, the reliability of network has become one of the key issues in research. Network reliability is usually measured by the connectivity of a graph, such as the edge connectivity is a key parameters to measure network reliability. For example, in communication networks and logical networks, edge connectivity can be used to measure network reliability and fault tolerance[2-4]. Edge connectivity is defined as the minimum number of edge deletions required to make a graph disconnected, the concept is particularly important in the design of

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multiprocessor networks. In addition, edge connectivity is closely related to other properties of graphs, such as diameter, minimum degree, etc. These relationships provide theoretical support for the design and optimization of complex networks.

Product graph is an important construction method in graph theory, which combines the vertices and edges of two or more graphs to generate a new graph. The graph product method can be used to construct complex networks, simulate biological molecule interactions, optimize coding algorithms, and analyze relationships in data visualization[5-7]. At present, research on the edge connectivity of product graphs mainly focuses on the edge connectivity of standard product graphs, such as Cartesian product graphs, strong product graphs, dictionary product graphs etc.

All graphs in this paper are finite and simple. Let $G=(V(G),E(G))$ be a simple connected graph, where $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G. For two graphs $G_1=(V(G_1),E(G_1))$ and $G_2=(V(G_2),E(G_2))$, their Cartesian product $G_1 \odot G_2$ is a graph whose vertex set is $V(G_1 \odot G_2)=V(G_1)\times V(G_2)$, and the edge set $E(G_1 \odot G_2)$ is defined as follows: if (x_1,y_1) and (x_2,y_2) are two vertices in the Cartesian product $G_1 \odot G_2$, then there is an edge between them if and only if $x_1=x_2$ and y_1 and y_2 are adjacent in G_2, or x_1 and x_2 are adjacent in G_1 and $y_1=y_2$; their strong product $G_1 \boxtimes G_2$ is a graph with whose vertex set is $V(G_1 \boxtimes G_2)=V(G_1)\times V(G_2)$, and the edge set $E(G_1 \boxtimes G_2)$ is defined as follows: if (x_1,y_1) and (x_2,y_2) are two vertices in the strong product $G_1 \boxtimes G_2$, then there is an edge between them if and only if $x_1=x_2$ and y_1 and y_2 are adjacent in G_2, or x_1 and x_2 are adjacent in G_1 and $y_1=y_2$, or x_1 and x_2 are adjacent in G_1 and y_1 and y_2 are adjacent in G_2; their lexicographic product $G_1 \otimes G_2$ is a graph with whose vertex set is $V(G_1 \otimes G_2)=V(G_1)\times V(G_2)$, and the edge set $E(G_1 \otimes G_2)$ is defined as follows: if (x_1,y_1) and (x_2,y_2) are two vertices in the lexicographic product $G_1 \otimes G_2$, then there is an edge between them if and only if $x_1=x_2$ and y_1 and y_2 are adjacent in G_2, or x_1 and x_2 are adjacent in G_1.

Let $G=(V(G),E(G))$ be a simple connected graph. For any vertex v of a graph G, its vertex-neighbor set is denoted by $N_G(v)$, which is the set of all vertices adjacent to vertex v in the graph G. The degree of v in the graph G is denoted by $d_G(v)=d(v)$, which refers to the number of vertices related to v in the graph G. Therefore, according to the definition of the graph, we have $d_G(v)=N_G(v)$, and the vertices in the graph G with degree 0 are called isolated vertices. The minimum degree of the graph G is denoted by $\delta(G)=\min\{d_G(v):\forall v\in V(G)\}$. If $A\subseteq G$ and A contains all edges uv in $E(G)$ that satisfy $\{u,v\}\in V(A)$, we say that A is a induced subgraph of G. Let F be a edge subset of $E(G)$, if the induced subgraph of $G-F$ is not connected, then F is a edge-cut set of G. The edge connectivity of the graph G, denoted by $\lambda(G)$, is the cardinality of the minimum edge-cut set of graph G. For a connected graph G, it is said to be maximally edge-connected if and only if $\lambda(G)=\delta(G)$. For two graphs G_1 and G_2, we use v_i, e_i, δ_i, λ_i and V_i to denote the number of vertices, the number of edges, the minimum degree, the edge connectivity and the vertex set of G_i, respectively, where $i=1,2$.

For the edge connectivity of Cartesian product, Chiue and Shieh first gave a result in [8], that is, for two connected graphs G_1 and G_2, we have $\lambda(G_1 \odot G_2)\geq\lambda(G_1)+\lambda(G_2)$, and then there has been no good progress. Until Xu and Yang improved the result $\lambda(G_1 \odot G_2)\geq\lambda(G_1)+\lambda(G_2)$ in the previous study [9], a better result was obtained, that is, for two nontrivial graphs G_1 and G_2, We have $\lambda(G_1 \odot G_2)=\min\{\lambda_1 v_2,\lambda_2 v_1,\delta_1+\delta_2\}$, and in this article, the author use the edge-version of Menger's theorem to prove the result, then, Klavzar and Spacapan gave a proof in based on the set of minimum edge-cut [10]. For the edge connectivity of strong product, Researchers proved that for two connected graph G_1 and G_2, we have $\lambda(G_1 \boxtimes G_2)=\min\{\lambda_1(v_2+2e_2),\lambda_2(v_1+2e_1),\delta_1+\delta_2+\delta_1\delta_2\}$ [11, 12]. For the edge connectivity of lexicographic product, Yang and Xu proved that for two graphs G_1 and G_2, we have $\lambda(G_1 \otimes G_2)=\min\{2\lambda_1 v_2^2,\delta_2+\delta_1 v_2\}$ [13].

The graph product also includes some non-standard products, such as semi strong product etc. The concept of semi-strong product was proposed by Mordeson and Chang Shyh [14], and its applications in fuzzy graphs, symbolic graphs, and finance have shown its broad research prospects [8].

For two graphs $G_1=(V(G_1),E(G_1))$ and $G_2=(V(G_2),E(G_2))$, their semi-strong product $G_1 \oplus G_2$ is a graph whose vertex set is $V(G_1 \oplus G_2)=V(G_1)\times V(G_2)$, and the edge set $E(G_1 \oplus G_2)$ is defined as follows: if (x_1,y_1) and (x_2,y_2) are two vertices in the semi-strong product $G_1 \oplus G_2$, then there is an edge between them if and only if $x_1=x_2$ and y_1 and y_2 are adjacent in G_2, or x_1 and x_2 are adjacent in G_1 and y_1 and y_2 are adjacent in G_2.

In this paper, we study the edge connectivity of semi-strong product graphs. Given two graphs G_1 and G_2, we obtain that $\lambda(G_1 \oplus G_2)=\min\{2\lambda_1 e_2,\delta_2+\delta_1\delta_2\}$. In further, we also give an necessary and sufficient condition for a semi-strong product to be maximally edge-connected.

2. Preliminaries

For the convenience, we define the following two types of subgraphs. For $x_1\in V(G_1)$, we define $x_1 \oplus G_2=G_2^{x_1}=\{(x_1,y_1)|y_1\in V(G_2)\}$, it is clear that the subgraph of $G_1 \oplus G_2$ induced by $G_2^{x_1}$ is isomorphic to G_2. Analogously, for $y_1\in V(G_2)$, we define $G_1 \oplus y_1=G_1^{y_1}=\{(x_1,y_1)|x_1\in V(G_1)\}$, but it is important to note that the subgraph of $G_1 \oplus G_2$ induced by $G_1^{y_1}$ is not isomorphic to G_1.

Lemma 2.1 ([15]). For any connected undirected graph G, we have $\lambda(G)\leq\delta(G)$.

Lemma 2.2. Let G_1 and G_2 be two graphs, then $\delta(G_1 \oplus G_2) = \delta_2 + \delta_1\delta_2$.

Proof. According to the definition of semi-strong product graph, that is, the semi-strong product $G_1 \oplus G_2$ of two graphs G_1 and G_2 is the graph with the vertex set $V(G_1) \times V(G_2)$, two vertices (x_1, y_1) and (x_2, y_2) being adjacent in $G_1 \oplus G_2$ if and only if $x_1 = x_2$ and y_1y_2 is an edge in G_2, or x_1x_2 is an edge in G_1 and y_1y_2 is an edge in G_2, we can get the result of

$$\delta(G_1 \oplus G_2) \geq \delta_2 + \delta_1\delta_2. \tag{1}$$

Assume that

$$\delta(G_1 \oplus G_2) > \delta_2 + \delta_1\delta_2, \tag{2}$$

is true, that is

$$\delta(G_1 \oplus G_2) \geq \delta_2 + \delta_1\delta_2 + 1. \tag{3}$$

In this case, the product graph will rise to a new edge-connected relation: for two vertices (x_1, y_1) and (x_2, y_2), they are adjacent in $G_1 \oplus G_2$ if and only if $x_1 = x_2$ and y_1y_2 is an edge in G_2, or x_1x_2 is an edge in G_1 and y_1y_2 is an edge in G_2, or x_1x_2 is an edge in G_1 and $y_1 = y_2$. At this point, three kinds of connected edges constitute a strong product graph, a contradiction. Therefore,

$$\delta(G_1 \oplus G_2) = \delta_2 + \delta_1\delta_2. \tag{4}$$

3. Main Results

Theorem 3.1. Let G_1 and G_2 be two nontrivial graphs, and G_1 is connected, then $\lambda(G_1 \oplus G_2) = \min\{2\lambda_1 e_2, \delta_2 + \delta_1\delta_2\}$.

Proof. Let $G = G_1 \oplus G_2$. It is clear that

$$\lambda(G) \leq \delta(G) = \delta_2 + \delta_1\delta_2. \tag{5}$$

Let S' be a minimum cut-set of G_1 and there are two connected components C_1' and C_2' of $G_1 - S'$. Then we can see that in G there are exactly $2\lambda_1 e_2$ edges between $C_1' \times V_2$ and $C_2' \times V_2$. Hence $\lambda(G) \leq 2\lambda_1 e_2$. Then we have

$$\lambda(G_1 \oplus G_2) \leq \min\{2\lambda_1 e_2, \delta_2 + \delta_1\delta_2\}. \tag{6}$$

Next we only need to prove that

$$\lambda(G_1 \oplus G_2) \geq \min\{2\lambda_1 e_2, \delta_2 + \delta_1\delta_2\}. \tag{7}$$

Let S be a minimum edge-cut of G, C_1 and C_2 are two connected components of $G - S$. For $x \in V_1$, let G_2^x denote the subgraph of G induced by $x \oplus V_2$.

It is easy to see that G_2^x is isomorphic to G_2. Then we define two sets, of which $X = \{x \in V(G_1) | xy \in V(C_1)\}$ for some $y \in V(G_2)$ and $Y = \{x \in V(G_1) | xy \in V(C_2)\}$ for some $y \in V(G_2)$.

It is clear that $X \neq \emptyset$ and $Y \neq \emptyset$. If $X \cap Y = \emptyset$, then it is easy to know that $\{X, Y\}$ is a partition of $V(G_1)$. Thus

$$|S| \geq \sum_{xy \in E_{G_1}(X,Y)} \left| E_G\left(V(G_2^x), V(G_2^y)\right) \right|$$

$$= \left|E_{G_1}[X, Y]\right| 2e_2 = 2\lambda_1 e_2. \tag{8}$$

So we assume that $X \cap Y \neq \emptyset$ and let $x_0 \in X \cap Y$.

We need to know that for each neighbor x of x_0, the vertex-set of the graph is $V(G_2^{x_0}) \cup V(G_2^x)$, and the edge-set between the vertex-sets $V(G_2^{x_0})$ and $V(G_2^x)$ is $E(V(G_2^{x_0}) \cup V(G_2^x))$, denoted by $G_2^{[x_0,x]}$, has edge-connectivity δ_2.

Let $S_{x_0x} = S \cap E(V(G_2^{x_0}) \cup V(G_2^x))$, then $S_{x_0x} \geq \delta_2$ for each neighbor x of x_0; otherwise $G_2^{x_0} - S$ is connected through G_2^x, which contradicts to our hypothesis.

In the following, we claim that $|S_{x_0x}| + |S_{x_0}| \geq \delta_2 + \delta_2 = 2\delta_2$, of which $S_{x_0} = S \cap E(V(G_2^{x_0}))$ and x is a neighbor of x_0. Then let $M = V(G_2^{x_0}) \cap V(C_1)$, $N = V(G_2^{x_0}) \cap V(C_2)$, and we assume that $M \leq N$.

Case 1: $|M| = 1$.

If $|M| = 1$, then $|S_{x_0x}| + |S_{x_0}| \geq \delta_2 + \delta_2 = 2\delta_2$ holds, since $S_{x_0} \geq \delta_2$.

Case 2: $|M| \geq 2$.

If $|M| \geq 2$, we can find $|M|\delta_2$ edge-disjoint (M, N) -paths in $G_2^{x_0}$. Let $M = s_1, s_2, \ldots, s_u$ and $\{t_1, t_2, \ldots, t_u\} \subseteq N$. Then for each $i (1 \leq i \leq u)$, there are δ_i edge-disjoint (s_i, t_i) -paths in $G_2^{[x_0,x]}$: (s_i, xz, t_i) with $z \in V(G_2)$. So we can find $|M|\delta_2$ edge-disjoint (M, N)-paths.

For the sake of disconnected M from N, we must have $|S_{x_0x}| \geq |M|\delta_2 \geq 2\delta_2$ and $|S_{x_0x}| + |S_{x_0}| \geq \delta_2 + \delta_2 = 2\delta_2$ also holds.

Let $x_1, x_2, \ldots, x_{\delta_1}$ be δ_1 neighbors of x_0 in G_1, then

$$|S| \geq \left(|S_{x_0x}| + |S_{x_0}|\right) + \sum_{i=2}^{\delta_1} |S_{x_0,x_i}|$$

$$= \delta_2 + \delta_2 + (\delta_1 - 1)\delta_2 = \delta_2 + \delta_1\delta_2. \tag{9}$$

This completes the proof.

Corollary 3.1. Let G_1 and G_2 be two connected graphs, then $G_1 \oplus G_2$ is maximally edge connected if and only if $2\lambda_1 e_2 \geq \delta_2 + \delta_1\delta_2$.

4. Conclusions

The main conclusion of this paper is as follows: for two nontrivial graphs G_1 and G_2, and G_1 is connected, then we have $\lambda(G_1 \oplus G_2) = \min\{2\lambda_1 e_2, \delta_2 + \delta_1\delta_2\}$. Furthermore, we obtain an necessary and sufficient condition for a semi-strong product to be maximally edge-connected.

At present, research on the edge connectivity of product graphs mainly focuses on the edge connectivity of standard product graphs, such as Cartesian product graphs, strong product graphs, lexicographic product graphs etc. The graph product also includes some non-standard products, such as semi-strong products, etc. The unique properties exhibited by non-standard product graphs (such as semi-strong product graphs, etc.) in practical applications are worth further exploration.

Author Contributions

Qiaoling Wang: Conceptualization, Resources, Software, Formal Analysis, Validation, Investigation, Visualization, Methodology, Writing – original draft

Haizhen Ren: Data curation, Supervision, Funding acquisition, Project administration, Writing – review & editing

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Data Availability Statement

The data supporting the outcome of this research work has been reported in this manuscript.

Conflicts of Interest

The authors declare no conflicts of interest.

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