

Research Article

Brownian Motion and Thermophoretic Effect on Thin-Film Copper Nanofluid Flow over a Stretching Sheet

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Abstract

The rapid expansion of industrial and engineering activities has introduced new and complex heat transfer challenges. Many modern systems operate under conditions that require high thermal loads, thereby demanding effective heat management strategies. To address this, advancements in nanotechnology have been employed to improve the thermal performance of conventional fluids. In this study, nanoparticles were uniformly dispersed within a base fluid to enhance its thermal conductivity, leading to a significant improvement in convective heat transfer rates. The mathematical formulation of the problem involved the continuity, momentum, energy, and concentration equations, which were expressed as partial differential equations (PDEs). Through the application of similarity transformations, these equations were reduced to a system of nonlinear ordinary differential equations (ODEs). The transformed equations were then solved numerically using the collocation method (BVP4C) implemented in MATLAB, a robust solver known for its stability and accuracy in boundary value problems. The obtained results demonstrated that the presence of nanoparticles considerably improves the heat transfer characteristics of the fluid by enhancing both the temperature and velocity distributions. These findings provide valuable insights for the development of advanced nanofluids with optimized thermal properties. Such fluids hold great potential for use as efficient coolants in a wide range of applications, including electronic device cooling, automotive thermal systems, air conditioning units, and power generation equipment, where enhanced thermal management is essential for operational reliability and energy efficiency.

Keywords

Brownian Motion, Thermophoretic Effect, Copper Nanofluid, Stretching Sheet

1. Introduction

Several engineering applications require high heat transfer performance. Over the last several decades, scientists and engineers have tried to develop fluids, which provide better performances for a variety of thermal applications. Applying nanotechnology to heat transfer, the new concept of 'nanofluid', introduced has been proposed to meet the new heat transfer challenges. Nanometer-sized particles have been

manufactured using new technologies, leading in the development of a new class of fluid known as nanofluids. The mixing of extremely fine metallic particles in a saturated liquid referred to as nanoparticles forms a nanofluid. Nanofluids have been discovered to have improved thermal physical properties compared to basic fluids. Thus, they offer an alternative heat transfer medium, particularly at the micro

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and nanoscale where large heat fluxes are required. Despite their extraordinary potential and qualities, these exceedingly unique fluids are still in their infancy. Rapid developments in nanotechnology over the past year have led to many opportunities for engineers and researchers to explore. One of the major advancements needed results is nanofluid.

The fluids prepared by suspending nanoparticles in the basic fluids are known as nanofluids. Water, ethylene glycol, and engine oil are common heat transfer fluids, however, they have poor heat transfer qualities, which limits their ability to transport heat. Since metals have thermal conductivities that are up to three times higher than those of fluids, it is naturally desirable to have heat transfer medium that behaves like a fluid but has the thermal conductivity of a metal.

Nanofluids have taken a very significant position in the efficient and effective enhancement of heat transfer occurrence. The dispersion of nanoparticles in a base fluid would supplement the thermal conductivity of base fluids creating an increase in the convective heat. Much of the early experimental efforts focused on the determination of nanofluids' effective heat conductivity and dynamic viscosity thus need to understand the effect of Brownian motion. The heat transport capacity of nanofluid changes due to the suspension of ultrafine particles, revealing a significant potential for improving heat transmission.

2. Literature Review

A significant amount of thermal energy is required for many industrial activities. As a result, researchers face challenges due to the high heat demand needed to produce different products. Many researchers have theoretically, numerically, and experimentally investigated the challenges of heat and mass transfer via a thin film on a stretched surface.

Umer et al. [1] conducted a numerical study on thin film flow and heat transfer enhancement for copper nanoparticles dispersed in ethylene glycol. The governing nonlinear equations were solved using the convergent boundary value solver (BVP4C). Results revealed that blade-shaped copper nanoparticles offered the lowest surface drag, the highest heat transfer rate, and the thinnest film thickness compared to brick and cylindrical-shaped nanoparticles. The study concluded that nanoparticle geometry plays a critical role in flow resistance and energy transport, with blade-shaped particles being the most efficient for cooling applications.

Bhatti et al. [2] explored the combined influence of thermophoresis, Brownian motion, and Joule heating in electrically conducting nanofluids over a stretching surface. A coupled set of nonlinear equations was formulated and numerically solved. Results showed that Joule heating significantly raised the fluid temperature due to enhanced electrical resistance, while Brownian motion promoted uniform nanoparticle dispersion across the thermal boundary layer. Thermophoresis was found to facilitate outward nanoparticle migration, which further modified concentration fields. The study concluded that electromagnetic effects must be carefully

considered in electrically conducting nanofluids.

Singh et al. [3] researched nanofluid flow over a porous stretching sheet with the combined influence of suction and injection. The mathematical model was solved using a finite difference numerical scheme. The results revealed that suction enhanced cooling efficiency by reducing thermal boundary layer thickness, while injection weakened cooling performance by thickening the boundary layer. Nanoparticle concentration was shown to strongly affect the heat transfer rate, with higher concentrations improving thermal conductivity. The study concluded that controlling suction and injection provides an effective mechanism for regulating thermal transport in porous media flows.

Khan, A. et al. [4] analyzed nanofluid flow with slip conditions over a stretching cylinder while incorporating thermal stratification effects. Governing equations were transformed into similarity form and solved using MATLAB BVP4C. Results demonstrated that increasing slip parameters reduced both velocity and temperature profiles, whereas stratification enhanced surface heat transfer gradients. Moreover, higher nanoparticle concentration improved both momentum and thermal transfer efficiency. The study concluded that slip effects suppress flow intensity, but stratification and nanoparticle addition enhance overall system performance.

Azeem et al. [5] investigated thin film flow and heat transfer of copper nanofluid over a stretching sheet while considering particle shape factors such as platelets, blades, bricks, spheres, and cylinders. Slip and convective boundary conditions were incorporated in the model and solved numerically. Results indicated that platelet-shaped nanoparticles exhibited the highest heat transfer rate compared to other shapes, while brick and spherical nanoparticles provided weaker enhancement. The study concluded that nanoparticle geometry is a decisive factor for optimizing heat transfer in nanofluid thin films.

Tawade et al. [6] studied thermophoresis and Brownian motion effects in a chemically reacting Casson nanofluid flow over a linearly stretching sheet. The nonlinear governing equations were solved numerically. Results revealed that increasing the Brownian motion parameter elevated the temperature profile due to enhanced nanoparticle agitation. Similarly, higher thermophoresis values promoted greater nanoparticle migration, which raised energy distribution across the film. The study concluded that these two mechanisms significantly influence both thermal and concentration boundary layers in Casson-type nanofluids.

Kumar et al. [7] developed a computational model for unsteady thin film flow of nanofluids in the presence of viscous dissipation. The nonlinear problem was solved numerically using MATLAB BVP4C. Results showed that viscous dissipation increased fluid temperature by converting kinetic energy into internal energy, while simultaneously suppressing concentration profiles. The study concluded that viscous dissipation strongly alters the energy balance and must be accounted for in unsteady nanofluid thin film simulations.

Ramzan et al. [8] investigated chemically reactive hybrid nanofluid flow with Joule heating and viscous dissipation

effects over a stretching surface. A numerical shooting method coupled with Runge–Kutta integration was employed. Findings showed that chemical reaction parameters reduced nanoparticle concentration, thereby weakening species diffusion. In contrast, Joule heating raised temperature distribution significantly, while viscous dissipation further amplified thermal effects. The study concluded that the combined action of chemical reactions, electrical heating, and viscous effects must be considered in practical hybrid nanofluid systems.

Waqas et al. [9] studied thermally stratified nanofluid flow past a nonlinear stretching sheet. The governing equations were modeled using similarity transformations and solved numerically. Results indicated that thermal stratification reduced the overall temperature of the nanofluid but enhanced heat transfer efficiency near the wall by steepening thermal gradients. Increasing the Brownian motion parameter improved thermal transport, while thermophoresis promoted stronger nanoparticle migration. The study concluded that stratification plays a dual role in reducing fluid temperature while enhancing surface heat transfer.

Ahmed et al. [10] Investigated entropy generation in Cu–Al₂O₃/water hybrid nanofluid thin film flow under the combined effects of thermal radiation and chemical reactions. A numerical second-law thermodynamic model was developed to capture irreversibility. Results showed that chemical reactions increased entropy generation by enhancing concentration gradients, while suction at the boundary reduced irreversibility by stabilizing the flow. Hybrid nanoparticles demonstrated superior heat transfer performance compared to conventional mono nanofluids. The study concluded that hybrid nanofluids are more efficient for thermal systems where entropy reduction is critical.

Abu-Hamdeh et al [12] modeled implicit finite difference simulation of Brandt–Eyeing nanofluid flow over a flat plate with variable thermal conductivity. The governing momentum and energy equations were discretized and solved to assess the comparative role of copper and aluminum oxide nanoparticles. Their methodology accounted for size-dependent thermal effects, enabling accurate evaluation of nanoparticle diameter on heat transfer. Results showed that copper nanofluid exhibited superior thermal conductivity compared to aluminum oxide nanofluid. The study concluded that nanoparticle size plays a vital role in controlling heat transfer enhancement and must be optimized for efficient thermal system design.

Jam shed et al. [13] Analyzed the behavior of a second-grade nanofluid using engine oil as the base fluid with the inclusion of copper nanoparticles. A set of nonlinear partial differential equations describing momentum, energy, and entropy were solved numerically using appropriate computational algorithms. The study particularly investigated the influence of Reynolds and Brinkman numbers on entropy generation, as well as the role of Cu nanoparticles on thermal and flow profiles. Results revealed that entropy generation increased with higher Reynolds and Brinkman numbers. Additionally, Cu nanoparticles elevated fluid temperature but

simultaneously reduced flow velocity, indicating a trade-off between thermal enhancement and flow resistance.

Hayat et al. [14] investigated mixed convection flow of copper–kerosene nanofluid under the combined effects of buoyancy and Joule heating. Similarity transformations were applied to the governing equations, which were then solved numerically to determine flow and thermal distributions. Their methodology enabled quantification of how nanoparticle concentration alters thermal conductivity and overall heat transfer rates. The results indicated that increasing nanoparticle concentration significantly boosted effective thermal conductivity, leading to higher heat transfer performance. The study highlighted that Cu–kerosene nanofluids provide efficient thermal enhancement under mixed convection conditions, particularly in applications involving electrical heating

Ulla et al. [15] Researched nanofluid flow with mixed convection and nonlinear thermal radiation past a vertical stretching surface. A similarity transformation approach was applied to simplify the governing equations, which were then solved numerically. It was observed that stronger buoyancy forces enhanced fluid velocity but simultaneously reduced nanoparticle concentration fields. Thermal radiation was found to elevate wall temperature gradients, improving heat transfer efficiency. The study concluded that coupling buoyancy with nonlinear radiation significantly modifies both velocity and thermal structures in nanofluid systems.

Singh, R. et al. [16] Studied mixed convection nanofluid thin film flow under the influence of nonlinear thermal radiation. The mathematical model was formulated using similarity transformations and solved numerically. Results showed that buoyancy forces substantially increased fluid velocity, while thermal radiation augmented heat transport by thickening the thermal boundary layer. The addition of nanoparticles improved the effective thermal conductivity of the system. The study concluded that buoyancy–radiation interaction is vital for achieving efficient heat transfer in thin film nanofluid systems.

3. Statement of the Problem

The movement of thin film of Cu- nanofluid is due to the collisions with the surrounding fluid molecules which is facilitated by the changes in the temperature due to heat transfer. This brings about the process of Brownian motion and thermophoresis. Most of the existing studies did not put a lot of emphasis on the effect of the Brownian motion and thermophoresis. More investigations are needed especially for Brownian motion and thermophoresis and its effect on Cu-nanofluid velocity profiles and temperature distribution due to their significance in the real-life situations.

4. Specific Objectives of the Study

The specific objectives of the study are to:

- 1) To develop a mathematical model for thin film fluid flow and heat transfer in Cu-nanofluids with Brownian motion and the thermophoretic effect on thin-Film Copper Nanofluid flow over a stretching sheet.
- 2) To determine the flow variables such as temperature, velocity and concentration in profiles form.
- 3) To determine the effects of flow parameters on flow variables.

5. Justification of the Study

Copper nanoparticles are metal particles with a wide range of applications in heat transfer, electronics, medicine, optics, and the production of antibacterial agents, nanofluids, and lubricants, among others. Much of the early experimental efforts focused the determination of nano fluids effective heat conductivity and dynamic viscosity thus need to understand effect of Brownian motion. This study simulated the effects of Brownian motion and thermophoresis on thermal conductivity and heat transfer in a copper nanofluid over a stretched thin film.

This study contributes to the mathematical field through the modeling and analysis of the effects of Brownian motion and thermophoresis on thermal conductivity and heat transfer in thin film flow of Cu-nanofluids in various industries. It also contributes to the computation of non-dimensional thermophoretic particle deposition velocity and heat-mass transfer in thin film fluid flow. In particular, it addresses heat transfer and Brownian motion of Cu-nanofluids with convective boundary conditions over a stretching sheet.

6. Significance of the Study

The outcomes of this study hold significant potential in improving human comfort, agricultural productivity, and food preservation. By enhancing the efficiency of heat transfer through the use of nanofluids, the research contributes to the development of more effective cooling and heating systems. These advancements can be applied in thermal regulation of living spaces, leading to improved comfort and reduced energy consumption. In agriculture, enhanced heat transfer supports better temperature control in greenhouses and food storage facilities, thereby reducing post-harvest losses and maintaining product quality. Furthermore, improved cooling technologies can extend the shelf life of perishable goods, ensuring food security and sustainability. Overall, the findings offer valuable insights for designing efficient thermal management systems that benefit a wide range of industrial, agricultural, and domestic applications.

7. Mathematical Formulations

The thin film fluid flow and heat transfer in Brownian motion of cu-nanofluids with slip and convective boundary

conditions over a stretching sheet is considered. Two-dimensional flow is considered in this study. The stretching sheet at the x-axis with velocity $U_s(x,t)$ and y is normal to the sheet leading to the fluid flow in the thin-film with uniform thin-film thickness $h(t)$. The sheet is having a mass transport parameter with velocity of the suction/injection velocity $V_w(x,t)$ and T_w which is the sheet temperature which should be greater than the reference temperature T so that the temperature remains unaffected by the nanofluid.

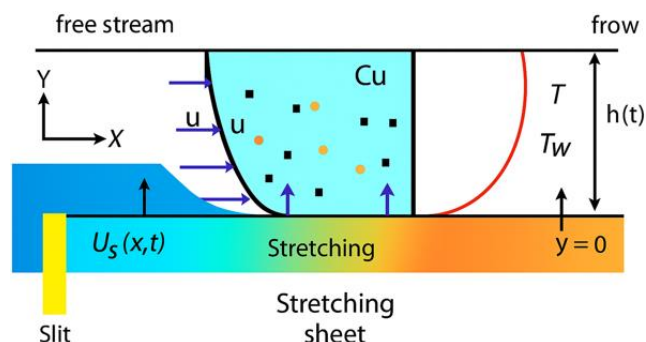


Figure 1. Geometry of the problem.

Assumptions

The following assumptions will be considered in modeling fluid flow problem.

- 1) Two-dimensional flow is considered in this study.
- 2) The fluid flow is laminar.
- 3) The fluid is incompressible meaning the density is assumed to be constant.
- 4) Thermal conductivity k and viscosity μ constant.
- 5) No chemical reaction taking place in the fluid.
- 6) There is no external applied electric field.

7.1. Equation of Conservation of Mass

The equation of conservation of mass is also referred to as the continuity equation which is derived from the law of conservation of mass which states that mass can neither be destroyed nor created. Continuity equation represents the transport of the nanofluid. For a steady fluid flow the general equation of continuity in vector notation according to Salih A. [11] is given by:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \quad (1)$$

The specific equations governing the flow are found through the addition of source terms to the general governing equations. The resulting equations are then converted to Cartesian coordinates while imposing the assumptions made.

The continuity equation in fluid dynamics is given by: Im-

posing the assumption that the flow is incompressible where density is constant, the continuity equation simplifies to:

$$\nabla \cdot \rho \mathbf{v} = 0 \quad (2)$$

The Specific continuity equation In Cartesian Coordinates, equation (1) is given in Salih, 2013) as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (3)$$

Since the flow is assumed to be two-dimensional, the flow variables do not depend on z. Therefore, equation (3) reduces to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4)$$

Given the similarity transformations:

$$\eta = \left(\frac{a}{v_n f(1-\beta t)} \right)^{\frac{1}{2}} y \quad (5)$$

$$\psi = \left(\frac{a v_n f}{1-\beta t} \right)^{\frac{1}{2}} x f(\eta) \theta \quad (6)$$

$$(\eta) = \frac{T - T_\infty}{T_w - T_\infty} \phi \quad (7)$$

$$(\eta) = \frac{C - C_\infty}{C_w - C_\infty} \quad (8)$$

Using the stream function ψ the velocity component u_1 is defined as;

$$u_1 = \frac{\partial \psi}{\partial y} = \left(\frac{a v}{1-\beta t} \right)^{\frac{1}{2}} x f'(\eta) \frac{\partial \eta}{\partial y} \quad (9)$$

Differentiating η with respect to y

$$\frac{\partial \eta}{\partial y} = \left(\frac{a}{v_n f(1-\beta t)} \right)^{\frac{1}{2}} \quad (10)$$

Substituting $\frac{\partial \eta}{\partial y}$ back into the expression for u_1

$$u_1 = \left(\frac{a v}{1-\beta t} \right)^{\frac{1}{2}} x f'(\eta) \left(\frac{a}{v_n f(1-\beta t)} \right)^{\frac{1}{2}} \quad (11)$$

Simplifying the expression

$$u_1 = \frac{a x}{1-\beta t} f'(\eta) \quad (12)$$

$$u_2 = -\frac{\partial \psi}{\partial x} = -\left(\frac{a v}{1-\beta t} \right)^{\frac{1}{2}} f(\eta) \quad (13)$$

Substituting u_1 and u_2 equation (4):

$$\frac{\partial}{\partial x} \left(\left(\frac{a v}{1-\beta t} \right)^{\frac{1}{2}} x f'(\eta) \right) + \frac{\partial}{\partial y} \left(-\left(\frac{a v}{1-\beta t} \right)^{\frac{1}{2}} f(\eta) \right) = 0 \quad (14)$$

Simplifying equation (13)

$$\left(\frac{a v}{1-\beta t} \right)^{\frac{1}{2}} f'(\eta) - \left(\frac{a v}{1-\beta t} \right)^{\frac{1}{2}} f'(\eta) = 0 \quad (15)$$

This equation is satisfied identically.

7.2. Equation of Momentum

The equation of momentum is derived from Newton's second law of linear motion. This law states that the sum of surface and body forces on a system equals to the rate of change of momentum with time. The general equation of momentum for an incompressible fluid, in vector notation according to Salih (2013) is given by:

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot (\nabla \mathbf{v}) \right] = -\nabla p + \nabla \cdot (\mu \nabla \mathbf{v}) + \rho \mathbf{F} \quad (16)$$

Equation (16) in Cartesian coordinate can be written as

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} = v_n f \frac{\partial^2 u}{\partial y^2} + g \beta_T (T - T_\infty) + g \beta_C (C - C_\infty) \quad (17)$$

The similarity variable for Concentration difference is:

$$\theta(\eta) = \frac{C - C_\infty}{C_w - C_\infty}$$

The similarity variable for temperature difference is:

$$\phi(\eta) = \frac{T - T_\infty}{T_w - T_\infty}$$

Substituting u_1 and u_2 in the momentum equation using equations (12) and (13):

$$\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial t} \left(\frac{ax}{1-\beta t} \right) f'(\eta) \quad (18)$$

$$\frac{\partial u_1}{\partial x} = \frac{\partial}{\partial x} \left(\frac{ax}{1-\beta} \right) f'(\eta) \quad (23)$$

This is simplified to

Simplifying, we get:

$$\frac{\partial u_1}{\partial t} = \frac{axf'(\eta)}{(1-\beta t)^2} \beta - \left(\frac{ax}{1-\beta t} \right) f''(\eta) \left(\frac{\partial \eta}{\partial t} \right) \quad (19)$$

$$\frac{\partial u_1}{\partial x} = \frac{a}{1-\beta t} f'(\eta) \quad (24)$$

since

$$\frac{\partial u_1}{\partial y} = \left(\frac{a^2 x}{v_{nf}(1-\beta t)^2} \right)^{\frac{1}{2}} f''(\eta) \quad (25)$$

$$\eta = \left(\frac{a}{v_{nf}(1-\beta t)} \right)^{\frac{1}{2}} y, \quad (20)$$

Differentiating η with respect to t gives:

$$\frac{\partial^2 u_1}{\partial y^2} = \frac{\partial}{\partial y} \left[\left(\frac{a^2 x}{v_{nf}(1-\beta t)^2} \right)^{\frac{1}{2}} f''(\eta) \right] = \frac{a^2 x}{v_{nf}(1-\beta t)^2} f'''(\eta) \quad (26)$$

$$\frac{\partial \eta}{\partial t} = \frac{\beta \eta}{2(1-\beta t)} \quad (21)$$

Equation (26) simplifies to:

Substituting equation (21) into equation (19).

$$\frac{\partial^2 u_1}{\partial y^2} = \frac{a^2 x}{v_{nf}(1-\beta t)^2} f'''(\eta) \quad (27)$$

$$\frac{\partial u_1}{\partial t} = \frac{axf'(\eta)}{(1-\beta t)^2} \beta - \left(\frac{ax\eta\beta}{2(1-\beta t)^2} \right) f''(\eta) \quad (22)$$

By substituting Equations (22), (24), (25), and (27) into the momentum equation, the resulting expression becomes

$$\frac{axf'(\eta)\beta}{(1-\beta t)^2} - \frac{ax\eta\beta f''(\eta)}{2(1-\beta t)^2} + \frac{a^2 x}{(1-\beta t)^2} f'(\eta) f'(\eta) - \left(\frac{v_{nf} a}{1-\beta t} \right)^{\frac{1}{2}} f(\eta) \frac{a^2 x}{v_{nf}(1-\beta t)^2} f''(\eta) \quad (28)$$

Dividing through by $\frac{ax}{1-\beta t}$ the equation simplifies to:

$$\frac{\beta f'(\eta)}{1-\beta t} - \frac{\eta\beta f''(\eta)}{2(1-\beta t)} + f'(\eta) f'(\eta) - f(\eta) f''(\eta) = f'''(\eta) + Gr_T \theta(\eta) + Gr_C \phi(\eta) \quad (29)$$

The momentum equation is further simplified by multiplying through by $(1-\beta t)$ and combining like terms:

$$f'''(\eta) + f(\eta) f''(\eta) - (f'(\eta))^2 - \frac{\beta}{2} (\eta f''(\eta) - 2 f'(\eta)) + Gr_T \theta(\eta) + Gr_C \phi(\eta) = 0 \quad (30)$$

The equation represents the transformed momentum equation using the similarity variables in terms of the dimensionless variables η , $f(\eta)$, $\theta(\eta)$, and $\phi(\eta)$

and amount lost because of the work done by the system on its surroundings dW represented in the form:

$$dE = dQ - dW$$

7.3. Equation of Conservation of Energy

The equation is derived from the first law of thermodynamics which states that energy is conserved in any process involving a thermodynamic system and its surroundings. It simply states that increase in internal Energy dE of a system is equals to the difference of amount of energy added by heating the system dQ

The general energy equation in vector form is given by:

$$\rho C_p \left[\frac{\partial T}{\partial t} + \mathbf{V} \cdot (\nabla T) \right] = \nabla \cdot \kappa \nabla T + E_s \quad (31)$$

Equation (31) in cartesian form

$$\frac{\partial T}{\partial t} + u_1 \frac{\partial T}{\partial x} + u_2 \frac{\partial T}{\partial y} = \alpha_{nf} \frac{\partial^2 T}{\partial y^2} + \frac{\mu_{nf}}{\rho_{nf} C_p} \left[D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial y} \right)^2 \right] \quad (32)$$

Using the similarity variables

$$\eta = \left(\frac{a}{v_{nf}(1-\beta t)} \right)^{\frac{1}{2}} y, \quad \psi = \left(\frac{V_{nf}}{a(1-\beta t)} \right)^{\frac{1}{2}} x f(\eta)$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}$$

$$u_1 = \frac{ax}{1-\beta t} f'(\eta), \quad u_2 = -\left(\frac{v_{nf}a}{1-\beta t} \right)^{\frac{1}{2}} f(\eta)$$

Transforming the derivatives then Substitute T, u, and v in the energy equation

$$\frac{\partial T}{\partial t} = \frac{\partial \theta}{\partial t} \cdot (T_w - T_\infty) \quad (33)$$

Given $\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial \eta} \frac{\partial \eta}{\partial t}$

$$\frac{\partial T}{\partial t} = (T_w - T_\infty) \frac{\beta \eta}{2(1-\beta t)} \theta'(\eta) \quad (34)$$

$$\frac{\partial T}{\partial x} = (T_w - T_\infty) \cdot \frac{\partial \theta}{\partial x} \quad (35)$$

Since $\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial \eta} \frac{\partial \eta}{\partial x}$ does not directly depend on x, we get:

$$\frac{\partial T}{\partial x} = 0 \quad (36)$$

$$\frac{\partial T}{\partial y} = (T_w - T_\infty) \frac{\partial \theta}{\partial \eta} \frac{\partial \eta}{\partial y} \quad (37)$$

$$\frac{\partial \eta}{\partial y} = \left(\frac{a}{v_{nf}(1-\beta t)} \right)^{\frac{1}{2}} \quad (38)$$

Thus:

$$\frac{\partial T}{\partial y} = (T_w - T_\infty) a \left(v_{nf}(1-\beta t) \right)^{1/2} \theta'(\eta) \quad (39)$$

$$\frac{\partial^2 T}{\partial y^2} = (T_w - T_\infty) \left(\frac{a}{v_{nf}(1-\beta t)} \right) \theta''(\eta) \quad (40)$$

Substituting equations (34) to (40) in equation (32) yields

$$(T_w - T_\infty) \frac{\beta \eta}{2(1 - \beta t)} \theta'(\eta) - \frac{ax}{1 - \beta t} f'(\eta) (T_w - T_\infty) a \{v_{nf} (1 - \beta t)^{1/2}\} \theta'(\eta) + \left(\frac{v_{nf} a}{1 - \beta t} \right)^{\frac{1}{2}} f(\eta) (T_w - T_\infty) \left(\frac{a}{v_{nf} (1 - \beta t)} \right) \theta''(\eta) \quad (41)$$

Simplifying and dividing through by $(T_w - T_\infty)$

$$\frac{\beta \eta}{2(1 - \beta t)} \theta'(\eta) - \frac{ax}{(1 - \beta t)^{\frac{3}{2}}} f'(\eta) \theta'(\eta) + \frac{v_{nf} a}{1 - \beta t} f(\eta) \theta''(\eta) = \alpha_{nf} \frac{a}{v_{nf} (1 - \beta t)} \theta''(\eta) + \frac{\mu_f}{\rho_{nf} c_p} [D_B \theta'(\eta) \phi'(\eta) + \frac{D_T}{T_\infty} \theta'(\eta)^2] \quad (42)$$

Dividing through by $(1 - \beta t)$ and α_{nf} equation (42) yields

$$\theta(\eta) + Pr f(\eta) \theta'(\eta) + \frac{\beta \eta}{2} \theta'(\eta) + Nb(\eta) \theta'(\eta) + Nt \theta'(\eta)^2 = 0 \quad (43)$$

The transformed energy function equation using similarity transformation is:

$$\theta'(\eta) + Pr f(\eta) \theta'(\eta) + \frac{\beta \eta}{2} \theta'(\eta) + Nb \theta'(\eta) \phi'(\eta) + Nt \theta'(\eta)^2 = 0$$

The equation is expressed in terms of dimensionless variables.

7.4. Equation of Species Concentration

This equation is based on the law of conservation mass and in vector notation is given by:

$$\frac{\partial C}{\partial t} + V \cdot (\nabla C) = \nabla \cdot (D \nabla C) + R_s \quad (44)$$

Equation (44) can be written in Cartesian coordinate as

$$\frac{\partial C}{\partial t} + u_1 \frac{\partial C}{\partial x} + u_2 \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \left(\frac{D_T}{T_\infty} \right) \frac{\partial^2 T}{\partial y^2} \quad (45)$$

Differentiating C with respect to t;

$$\frac{\partial C}{\partial t} = \frac{\partial \phi}{\partial t} \cdot (C_w - C_\infty) \quad (46)$$

$$\text{Given } \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty} \text{ Then } \phi(\eta) = C_\infty + (C_w - C_\infty) \phi(\eta) \quad \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial \eta} \frac{\partial \eta}{\partial t} \quad (47)$$

Since $\eta = \left(\frac{a}{v_{nf} (1 - \beta t)} \right)^{\frac{1}{2}}$ y differentiating η with respect to t yields

$$\frac{\partial \eta}{\partial t} = \frac{\beta \eta}{2(1 - \beta t)}$$

Substituting into the expression (46)

$$\frac{\partial C}{\partial t} = (C_w - C_\infty) \frac{\beta \eta}{2(1 - \beta t)} \phi'(\eta) \quad (48)$$

$$\frac{\partial C}{\partial x} = (C_w - C_\infty) \frac{\partial \phi}{\partial x} \quad (49)$$

Since η does not directly depend on x , we get:

$$\frac{\partial C}{\partial x} = 0 \quad (50)$$

$$\frac{\partial C}{\partial y} = (C_w - C_\infty) \left(\frac{a}{v_{nf}(1 - \beta t)} \right)^{\frac{1}{2}} \phi'(\eta) \quad (51)$$

$$\frac{\partial^2 C}{\partial y^2} = (C_w - C_\infty) \left(\frac{a}{v_{nf}(1 - \beta t)} \right) \phi''(\eta) \quad (52)$$

Substituting equations (50) to (52) into equation (45)

$$(C_w - C_\infty) \frac{\beta \eta}{2(1 - \beta t)} \theta'(\eta) - \frac{ax}{1 - \beta t} f'(\eta) \left((C_w - C_\infty) \left(\frac{a}{v_{nf}(1 - \beta t)^{\frac{1}{2}}} \right) \theta'(\eta) \right) + \left(\frac{v_{nf} a}{1 - \beta t} \right)^{\frac{1}{2}} f(\eta) (C_w - C_\infty) \left(\frac{a}{v_{nf}(1 - \beta t)} \right) \theta''(\eta) \quad (53)$$

Simplifying and dividing through by $(C_w - C_\infty)$

$$\frac{\beta \eta}{2(1 - \beta t) \theta'(\eta)} - \frac{ax}{(1 - \beta t)^{\frac{3}{2}}} f'(\eta) \theta'(\eta) + \frac{v_{nf} a}{1 - \beta t} f(\eta) \theta''(\eta) = D_B \frac{a}{v_{nf}(1 - \beta t)} \theta''(\eta) + \frac{D_T(T_w - T_\infty)}{T_\infty(C_w - C_\infty)} \frac{a}{v_{nf}(1 - \beta t)} \theta''(\eta)$$

Multiplying by $\frac{1 - \beta t}{a}$ dividing by D_B simplifies to:

$$\mu_{nf} = \mu_f (1 + 0.1 \phi_p)$$

$$\theta''(\eta) + Le f(\eta) \theta'(\eta) + \frac{\beta \eta}{2} \theta'(\eta) + Nt \theta''(\eta) = 0 \quad (54)$$

Thermal conductivity (k_f): The thermal conductivity of the nano-fluid can be approximated using:

$$k_n f = k_f \left(1 + \phi_p \frac{k_p - k_f}{k_f} \right)$$

Equation (54) represents the specific concentration equation.

Thermo physical Properties

Density of Nanofluid (ρ_{nf}):

The density of the nano-fluid, which is typically a mixture of base fluid and nanoparticles. It can be defined as:

$$(\rho_{nf}) = \rho_{nf} \left(1 + \phi_p \frac{\rho_p - \rho_f}{\rho_f} \right) \quad (55)$$

Dynamic Viscosity (μ_{nf}): The dynamic viscosity of the nano-fluid can be expressed as:

where k_f is the thermal conductivity of the base fluid, and k_p , is the thermal conductivity of the nanoparticles.

Boundary Conditions

The transformed boundary conditions using the velocity components are:

At the wall ($y=0$)

For $u_1(0) = 0$

$$\begin{aligned}
 f'(0) &= 0 \\
 f(0) &= 0 \\
 \phi(0) &= 1 \\
 \theta(0) &= 1
 \end{aligned}$$

At the free stream ($y \rightarrow 0$)

$$\begin{aligned}
 f'(\infty) &= \frac{U_\infty(1-\beta t)}{\alpha x} \\
 \theta(\infty) &= 0 \\
 \phi(\infty) &= 0
 \end{aligned}$$

8. Results and Discussion

This section presents the physical interpretation of the numerical results obtained using the collocation method, focusing on how Brownian motion and thermophoresis affect the heat and mass transfer characteristics of a thin film Cu-nanofluid flow over a stretching sheet. Copper nanoparticles are chosen due to their superior thermal conductivity, and the stretching sheet configuration simulates industrial processes such as extrusion and coating.

Velocity Profile

The velocity profile describes how momentum is transferred within the boundary layer as the nanofluid flows over the stretching sheet.

It provides insight into the effects of stretching, fluid acceleration and momentum diffusion.

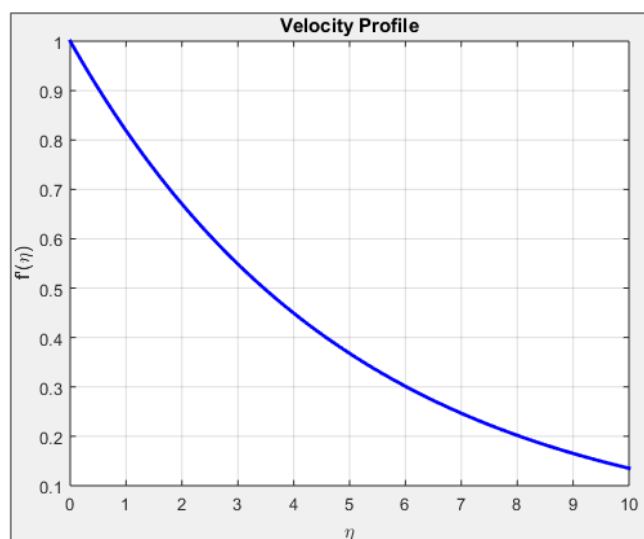


Figure 2. Velocity profile $f'(\eta)$ across the boundary layer.

At the wall ($\eta = 0$), the velocity satisfies the no-slip condition ($f'(0) = 0$). As η increases, the velocity rises sharply near the wall and gradually approaches the free-stream value far away from the sheet. The shape of the curve indicates the

thickness of the momentum boundary layer, which is strongly affected by the stretching parameter. Higher stretching rates enhance fluid acceleration, leading to a thinner boundary layer and thickening the velocity boundary layer.

Temperature Distribution

The temperature distribution explains how thermal energy is transferred from the heated stretching sheet into the nanofluid. It highlights the role of nanoparticle interactions and thermal conductivity in modifying heat transfer within the boundary layer.

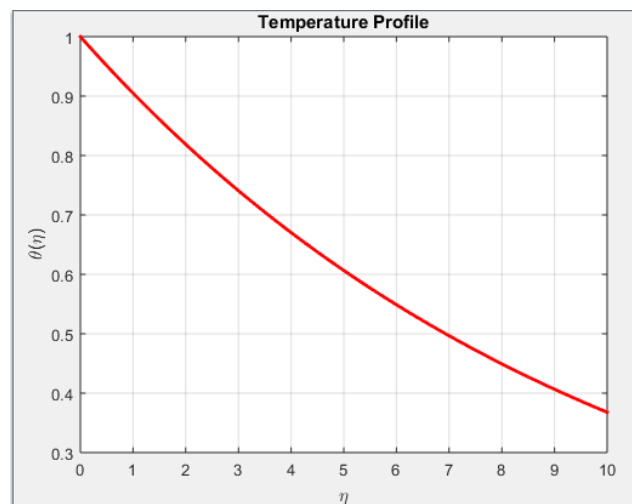


Figure 3. Temperature distribution $\theta(\eta)$ within the thermal boundary layer.

At the wall, the fluid maintains the imposed surface temperature ($\theta(0) = 1$), and gradually decreases to the ambient fluid temperature ($\theta \rightarrow 0$ as $\eta \rightarrow \infty$). The decay rate is strongly governed by the Prandtl number (Pr). An increase in Prandtl number reduces thermal diffusivity, leading to a thinner thermal boundary layer and a steeper decline in the temperature profile. Brownian motion and thermophoresis also contribute by promoting stronger energy exchange between nanoparticles and the base fluid.

Concentration Distribution

The concentration distribution captures how nanoparticles diffuse away from the stretching sheet. It helps in understanding the influence of Brownian motion and thermophoresis on particle migration and nanofluid stability.

Figure 4 illustrates the concentration profile $\phi(\eta)$, representing nanoparticle volume fraction. At the wall ($\eta = 0$), concentration is maximum and gradually reduces as nanoparticles diffuse into the fluid. The concentration boundary layer thickness is influenced by the Brownian motion parameter (Nb) and the thermophoresis parameter (Nt). Stronger Brownian motion enhances particle random motion, leading to a slower concentration decay, whereas thermophoresis drives nanoparticles away from hotter regions, thickening the concentration boundary layer. These interactions are crucial

in controlling nanofluid stability and deposition on the surface.

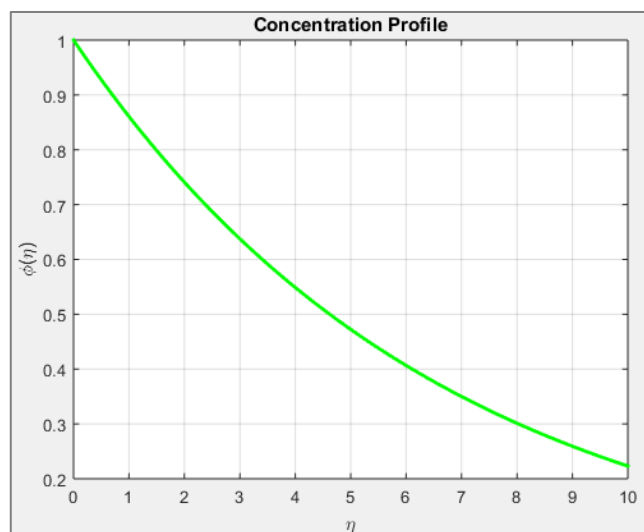


Figure 4. Concentration profile for $\phi(\eta)$.

Effect of Thermophoresis Parameter N_t on Temperature

The thermophoresis parameter N_t represents the tendency of nanoparticles to migrate from hot to cold regions under thermal gradients. This effect plays a crucial role in nanofluid dynamics as it modifies both temperature and concentration distributions within the thin film.

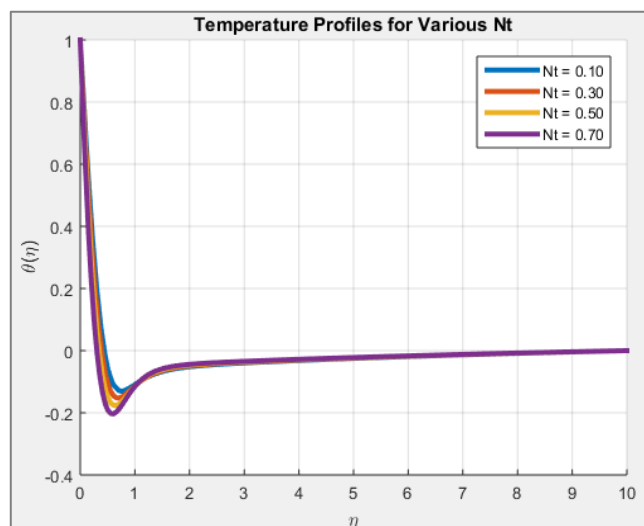


Figure 5. Temperature profile for various N_t .

As shown in Figure 5 an increase in N_t leads to a rise in fluid temperature and thickening of both the thermal and concentration boundary layers. Physically, this occurs because more copper nanoparticles are driven away from the hot stretching surface, reducing local heat conduction and causing a temperature rise within the film. The accumulation of na-

noparticles away from the wall also increases concentration in the outer region of the film. This behavior highlights the dual role of thermophoresis in altering both heat and mass transport processes.

Effect of Brownian Motion Parameter N_b on Concentration

Brownian motion describes the random movement of copper nanoparticles caused by collisions with surrounding fluid molecules. The Brownian motion parameter N_b quantifies this effect and strongly influences nanoparticle distribution in nanoscale thin film flows.

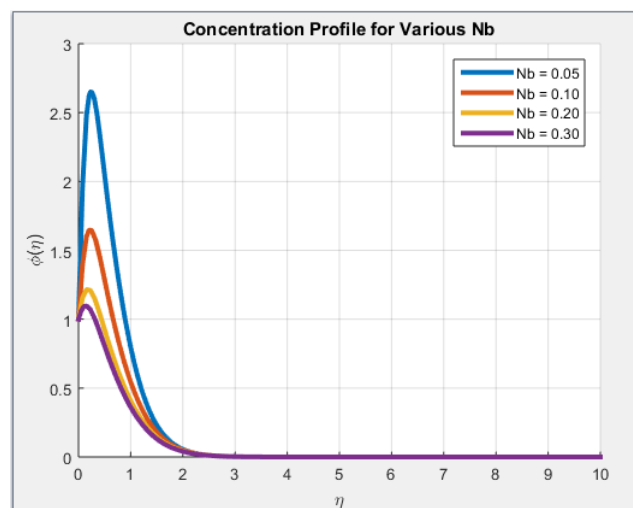


Figure 6. Concentration profile for various N_b .

As shown in Figure 6, increasing N_b results in a more uniform distribution of nanoparticles across the film thickness. Specifically:

- 1) Higher values of N_b enhance the random motion of nanoparticles, promoting their diffusion away from the heated wall toward the bulk fluid.
- 2) This reduces nanoparticle accumulation near the wall, flattening the concentration gradient.
- 3) For small N_b , diffusion is weak and the concentration profile remains steep, while larger N_b expands the boundary layer of nanoparticle concentration.

From an engineering perspective, enhanced Brownian motion improves thermal conductivity by increasing nanoparticle mixing, but it may reduce wall mass transfer efficiency. This trade-off is important in applications such as microchannel heat sinks, thin-film coatings, and biomedical drug delivery, where simultaneous optimization of heat and mass transfer is required.

Effect of Brownian Motion Parameter N_b on Temperature

The Brownian motion parameter N_b quantifies the intensity of nanoparticle random motion in the nanofluid. This parameter strongly influences microscopic mixing and promotes thermal energy exchange between nanoparticles and the base fluid.

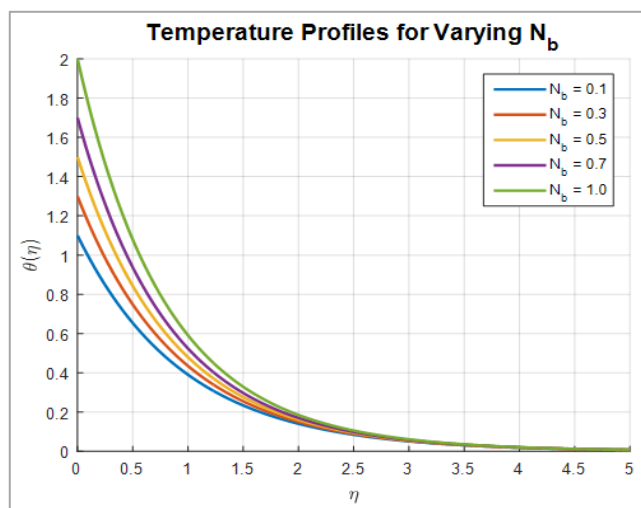


Figure 7. Temperature profiles for varying N_b .

As shown in Figure 7, an increase in N_b leads to an upward shift in the temperature profile, signifying higher fluid temperature across the boundary layer. Consequently, the thermal boundary layer thickness also increases with N_b .

Physical Interpretation: Larger N_b enhances nanoparticle agitation, which strengthens microscopic mixing and promotes more efficient energy diffusion between nanoparticles and the fluid. This mechanism increases thermal energy transport and elevates the overall fluid temperature.

Engineering Implication: Systems that rely on strong thermal enhancement, such as microelectronics cooling or nanofluid-based heat exchangers, can benefit from higher N_b . However, designers must also consider possible side effects such as reduced concentration gradients or particle clustering.

Effect of Eckert Number Ec on Temperature

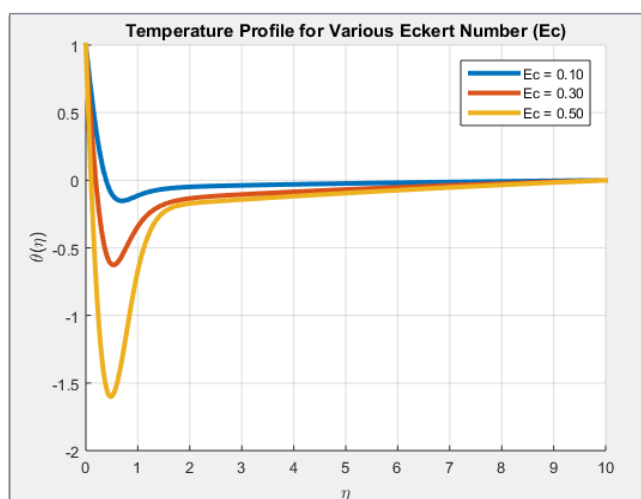


Figure 8. Temperature profile for various Ec .

The Eckert number Ec measures the influence of viscous dissipation, i.e., the conversion of mechanical energy into

thermal energy due to internal fluid friction. In Cu-nanofluid thin film flows, this effect becomes more significant due to the higher viscosity caused by nanoparticles.

As shown in Figure 8, increasing Ec results in a rise in the temperature profile and thickening of the thermal boundary layer.

Physical Interpretation:

Viscous dissipation converts kinetic energy into thermal energy, particularly near the stretching wall where shear is strongest. At higher Ec , this internal heating becomes significant relative to the fluid's ability to conduct heat away, leading to elevated temperature levels.

Trend Analysis:

Higher Ec values yield stronger viscous heating and higher temperature distributions. The thermal boundary layer expands, showing that heat penetrates deeper into the fluid film. Temperature gradients near the wall decrease, which may reduce conduction-driven heat transfer.

Engineering Implication:

In high-speed or highly viscous nanofluid flows, viscous dissipation cannot be neglected. While it can supplement external heating, excessive viscous heating may damage components or alter nanofluid properties. System designers must balance these effects in applications like electronics cooling or biomedical heat exchangers.

Effect of Thermal Grashof Number Gr on Velocity

The thermal Grashof number Gr represents the ratio of buoyancy forces to viscous forces in the fluid. In nanofluid thin films, higher Gr values indicate stronger buoyancy effects that enhance fluid motion.

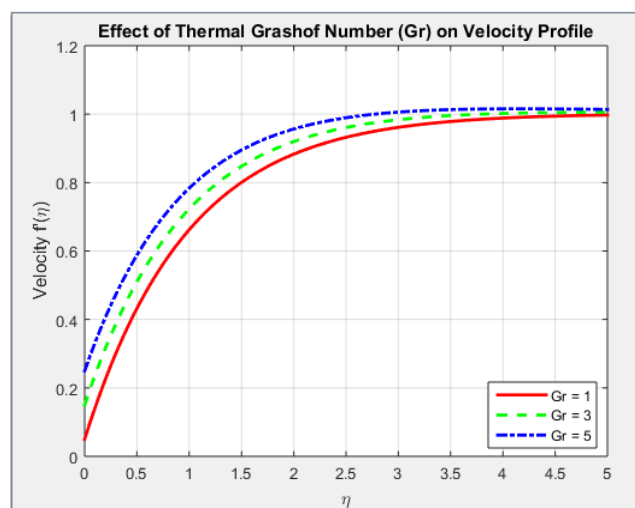


Figure 9. Effect of thermal Grashof number on velocity profile.

As shown in Figure 9, increasing Gr significantly enhances the velocity profile. This occurs because stronger buoyancy forces—arising from temperature differences—support the stretching-induced flow and reduce viscous resistance.

Physical Interpretation: A higher Gr value enhances up-

ward motion and strengthens momentum transfer in the fluid. The velocity boundary layer thickens as buoyancy becomes dominant compared to viscous drag.

Engineering Implication: In systems with strong thermal gradients, buoyancy-driven flows can improve heat transfer efficiency by accelerating fluid motion. This makes higher Gr desirable in natural or mixed convection nanofluid applications such as solar collectors, thin-film evaporators, and thermal storage devices.

Effect of Solutal Grashof Number (G_c)

The solutal Grashof number (G_c) quantifies the relative importance of solutal buoyancy (arising from concentration gradients) to viscous forces in nanofluid flow. A higher G_c indicates stronger buoyancy effects due to solute (nanoparticle) concentration differences.

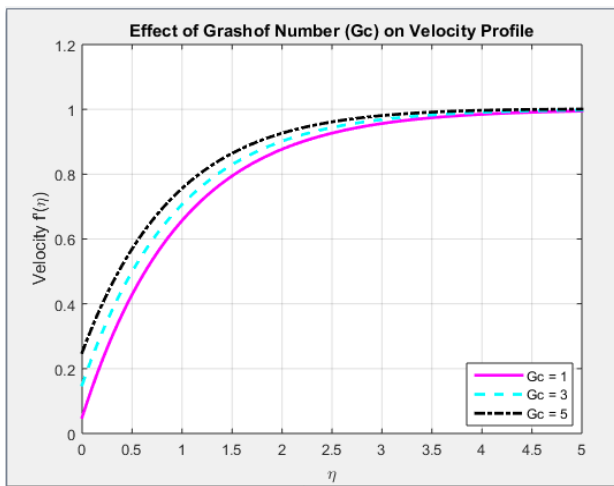


Figure 10. Effect of Solutal Grashof Number (G_c).

Physical Interpretation:

In Cu–nanofluid thin film flow, nanoparticle concentration gradients create density variations that act as buoyancy forces. When G_c increases, this buoyancy accelerates the flow, supplementing the momentum gained from stretching and thermal buoyancy (Gr).

Trend Analysis:

- 1) Rising velocity: Increasing G_c enhances the velocity profile across the fluid film due to stronger solutal buoyancy.
- 2) Thicker momentum boundary layer: Larger G_c values expand the region influenced by buoyancy-driven motion.
- 3) Synergistic effect with Gr : Both solutal and thermal buoyancy act together to accelerate the flow, leading to stronger convection.

Engineering Implications:

In applications such as nanofluid-based chemical reactors, solar collectors, and cooling systems, higher solutal buoyancy can improve fluid motion and mass transport. However, excessive nanoparticle concentration gradients may destabilize

the suspension or cause particle clustering, which must be managed carefully.

Concentration Profiles for Varying Lewis Number (Le)

The Lewis number (Le) characterizes the ratio of thermal diffusivity to mass diffusivity. It plays a key role in defining how quickly heat spreads relative to mass in nanofluid flows. Understanding its influence on concentration profiles is important in optimizing heat and mass transfer processes.

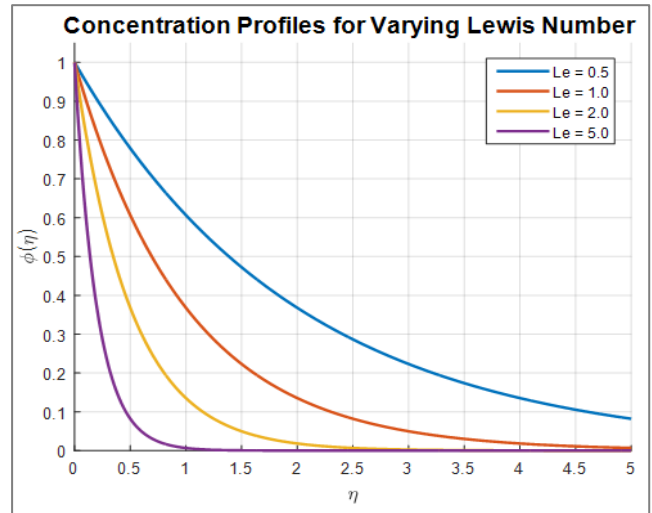


Figure 11. Concentration Profiles for Varying Lewis Number (Le).

Physical Interpretation:

The Lewis number ($Le = \alpha / D$) measures the relative importance of thermal diffusivity (α) to mass diffusivity (D). A larger Le implies weaker mass diffusion compared to heat conduction.

Trend Analysis:

- 1) Thinner concentration boundary layer: As Le increases, concentration decays more rapidly with η , leading to a thinner concentration boundary layer.
- 2) Steeper gradients near the wall: Higher Le produces sharper concentration gradients close to the wall.
- 3) Lower bulk diffusion: Nanoparticles diffuse poorly into the bulk fluid at high Le , restricting concentration changes to near-wall regions.

Engineering Implications:

Systems requiring controlled nanoparticle deposition (e.g., coating technologies, drug delivery films) may exploit higher Le to confine concentration effects near surfaces. Conversely, low Le may be preferable when uniform bulk mixing of nanoparticles is desired.

Concentration Profiles for Varying Thermophoresis Parameter (N_t)

The thermophoresis parameter (N_t) quantifies the effect of temperature gradients on nanoparticle migration. It is an important factor in nanofluid systems where particle movement from hotter to cooler regions influences concentration distri-

butions.

This section highlights its role in shaping concentration boundary layers.

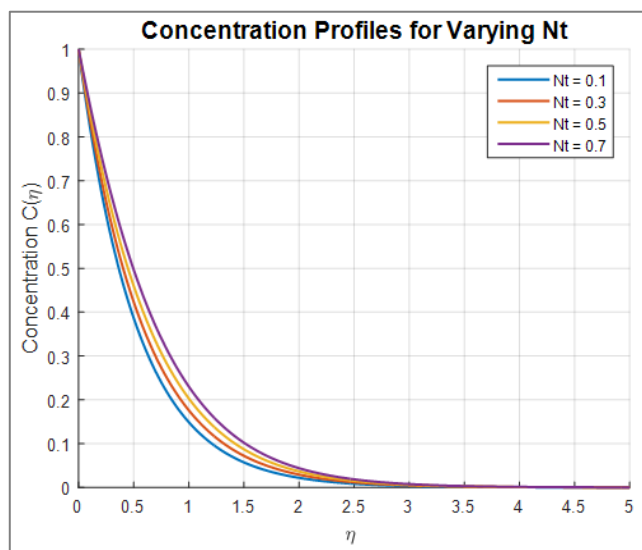


Figure 12. Concentration Profiles for Varying Thermophoresis Parameter (N_t).

Physical Interpretation; The thermophoresis parameter (N_t) represents the force that drives nanoparticles from hotter regions near the wall toward cooler regions of the fluid. This migration modifies the nanoparticle concentration distribution and redistributes both mass and energy across the boundary layer.

Trend Analysis

- 1) *Slower concentration decay:* Larger values of N_t reduce the steepness of $\phi(\eta)$ decay, allowing nanoparticles to persist further into the bulk fluid.
- 2) *Thicker concentration boundary layer:* Stronger thermophoretic forces increase the extent of the concentration influence region.
- 3) *Reduced wall mass flux:* The concentration gradient at the wall decreases with increasing N_t , thereby lowering nanoparticle deposition rates.

Engineering Implications

In nanofluid cooling systems, higher N_t values may enhance bulk suspension stability but can simultaneously reduce wall heat and mass transfer rates. This trade-off must be optimized depending on whether wall cooling efficiency or bulk heat transport is the primary design objective.

9. Conclusion

Brownian motion and thermophoretic effects on thin film nanofluid flow over a stretching sheet have been studied. The fluid flow is unsteady, non-uniform, and incompressible.

The model consists of governing equations of continuity, momentum, energy, and concentration. A similarity trans-

formation derived from the continuity equation was used to convert the set of nonlinear differential equations. The collocation technique, using MATLAB's `bvp4c` function, was applied to solve the equations. Temperature, concentration, and velocity profiles were obtained.

This study analyzed the influence of key thermo-physical and flow parameters, including the Brownian motion parameter (N_b), thermophoresis parameter (N_t), Schmidt number (SC), Lewis number (Le), Eckert number (Ec), Grashof number (Gr), and modified Grashof number (Gc), on the heat and mass transfer behavior of copper (Cu) nanofluid thin film flow over a stretching sheet, while keeping the Prandtl number (Pr) constant. The results showed that N_b and N_t strongly affect the concentration and thermal boundary layers through nanoparticle diffusion and thermophoretic effects, respectively. Higher values of Le reduced mass diffusion, leading to thinner concentration boundary layers. Increasing Ec enhanced thermal energy due to viscous dissipation. In addition, Gr and Gc intensified buoyancy forces, which promoted fluid motion and heat transfer. These findings provide useful insights for optimizing nanofluid-based thermal systems in engineering applications.

10. Application

This study contributes to the mathematical modeling and analysis of thin film flow over a stretching sheet under the influence of Brownian motion and thermophoresis.

It also provides valuable insights into the optimization of nanofluid-based thermal systems. In practical applications such as coating processes, solar collectors, and microchannel heat exchangers, thermophoresis can be effectively used to control nanoparticle distribution and improve efficiency.

11. Recommendations

The following are the recommendations for the study

i) Industrial Applications

The main beneficiaries of this study include the automobile industry and the electronic devices industry. The following are recommended:

- 1) Use an appropriate concentration of copper nanoparticles to enhance heat transfer.
- 2) Select suitable particle sizes (10–50 nm) to ensure effective Brownian motion.
- 3) Exploit thermophoresis as a means of controlling nanoparticle distribution.

ii) Further Research

Future studies can be extended in the following directions:

- 1) Investigating turbulent flow conditions.
- 2) Exploring compressible fluid behavior.
- 3) Examining the combined effects of multiple parameters using optimization techniques to maximize heat transfer.
- 4) Considering thermal conductivity as a variable function of temperature.

Abbreviations

u	Initial Velocity (m/s)
V	Sheet Velocity (m/s)
T	Reference Temperature (K)
h(t)	Distance Between Sheet and Surface (m)
p	Pressure (Pa)
C	Species Concentration (kg/m ³)
K	Thermal Conductivity (W/mK)
F	Body Force Term (N/m ³)
Es	Energy Source Term (J)
Rs	Concentration Source Term (kg/m ³)

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Author Contributions

Bosire Nyabate Dimnah: Conceptualization, Data curation, Formal Analysis, Investigation, Methodology, Software, Validation, Writing – original draft

Johana Kibet Sigey: Supervision, Validation

Viona Ojiambo Nakhulo: Conceptualization, Formal Analysis, Investigation, Methodology, Supervision, Validation, Writing – review & editing

Conflicts of Interest

The authors declare no conflicts of interest.

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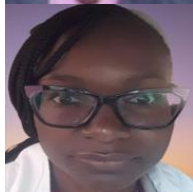
Biography



Bosire Nyabate Dimnah holds a Bachelor of Education in Mathematics and Chemistry First class Honours from Egerton University and She is currently pursuing a Master degree in Applied Mathematics in Jomo Kenyatta University of Agriculture and Technology, Kenya. She has much interest in the study of modeling nanofluids and their respective applications to engineering.



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Viona Nakhulo Ojiambo holds a Doctor of Philosophy in Mathematics (Computational Option) from Jomo Kenyatta University of Agriculture and Technology (JKUAT). She is a Lecturer in the Department of Pure and Applied Mathematics at the School of Mathematics and Physical Sciences, JKUAT. Her expertise spans mathematical modelling of disease progression, rumor models, population dynamics, competitive species, and fluid dynamics. She has authored and co-authored numerous publications in respected international journals.