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Exact Travelling Wave Solutions for the Space-time Fractional Benjamin-Ono Equation with Three Types of Fractional Operators

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Abstract: The space-time fractional Benjamin-Ono equation (STFBOE) is of fundamental importance in ocean science, particularly for modeling wave propagation in deep water. This study investigates the STFBOE employing three distinct fractional operators: the conformable derivative, the beta derivative, and the M-truncated derivative. By applying a fractional traveling wave transformation, the original nonlinear fractional partial differential equation is reduced to an ordinary differential equation. We then utilize three analytical techniques-the fractional functional variable method, the modified Kudryashov method, and the improved F-expansion method to derive novel exact traveling wave solutions. To the best of our knowledge, the exact solutions for the fractional Benjamin-Ono equation in the form considered here have been scarcely studied. A key contribution of this work is the introduction and tailored application of these methods to systematically construct solutions under each fractional derivative definition. Accordingly, we establish specific traveling wave variables corresponding to the conformable, beta, and M-truncated derivatives. A significant advantage of the proposed framework is its flexibility and effectiveness, enabling a straightforward derivation of solutions for both time-fractional and space-fractional versions of the Benjamin-Ono equation. Finally, we present a comparative graphical analysis of the obtained solutions using two- and three-dimensional plots, illustrating the spatio-temporal dynamics under selected parameter values. All the derived solutions are entirely new and extend the current understanding of nonlinear wave phenomena described by the STFBOE.

Keywords: Space-time Fractional Benjamin-Ono Equation, Beta Derivative, M-truncated Derivative, Fractional Functional Variable Method, Modified Kudryashov Method, Improved F-expansion Method

1. Introduction

Fractional nonlinear evolution equations (FNEEs) are the generalizations of classical nonlinear evolution equations. Due to its ability to better describe natural phenomena and the complex behavior of dynamic systems, FNEEs serve an essential role in fields such as mathematical physics, fluid dynamics, virus propagation, ocean engineering and so on [1–6].

Finding analytical solutions to the FNEEs can aid researchers in further understanding the nature of the problems, the behavior patterns and interrelationships of the systems. With the development of the FNEEs, a growing number of researchers have devoted themselves to the analytical solutions, which results in a series of effective methods being proposed. Among the more emblematic ones include Lie symmetry analysis [7, 8], F-expansion method [9, 10], Fractal variational method [11–13], Exponential rational function method [14, 15], Fractional sub-equation method [16, 17] and the others [18–22].

Along with the development of fractional calculus in recent years, researchers have developed many fractional derivatives that perform well in scientific and engineering applications, such as Caputo fractional derivative [23], Yang-Abdel-

Aty-Cattani fractional derivative [24] conformable fractional derivative [25], beta derivative [26], M-truncated derivative [27], etc.

In this paper, we generalize the Benjamin-Ono equation to the STFBOEs with conformable fractional, beta and M-truncated derivative, respectively. Fractional functional variable method, modified Kudryashov method and improved F-expansion method in three different fractional senses are implemented to obtain new types of analytical solutions of the STFBOEs.

The classical (1+1)-dimensional Benjamin-Ono equation is presented as

$$D_t^2 u + \alpha D_x^2(u^2) + \beta D_x^4 u = 0, \tag{1}$$

where α and β denote the parameters of the nonlinear and dispersion terms, respectively. The Benjamin-Ono equation is an important nonlinear partial differential equations introduced by Benjamin [28] and Ono [29], which is modeled to describe the internal waves in a stratified fluid. Based on its importance, researchers have extensively studied various types of analytic solutions of the Benjamin-Ono equation [30–35].

The (1+1)-dimensional STFBOE is stated as below

$$D_t^{2\mu} u + \alpha D_x^{2\nu}(u^2) + \beta D_x^{4\nu} u = 0, \tag{2}$$

where μ, $\nu \in (0, 1]$. Here, $D_t^{2\mu}$, $D_x^{2\nu}$ and $D_x^{4\nu}$ are fractional derivative operators. When $\mu = 1$ and $\nu = 1$, 2 can be degenerated to 1. Particularly, if $\nu = 1$, then 2 is transformed into the governing equation in [40], which is employed to model the propagation of surface waves over a thin laminar structure.

To the best of our knowledge, the exact solutions of the fractional Benjamin-Ono equation shaped as 2 have been little studied. In this paper, the main interesting fact is that we introduce the fractional functional variable method, modified Kudryashov method and improved F-expansion method to separately derive exact solutions of the STFBOEs with three types of fractional derivatives. According to the different definitions of conformable, beta and M-truncated derivatives, we establish the traveling wave variables under each of the three senses. An attractive feature of the proposed method is its flexibility and effectiveness, mainly in the sense that we can obtain the solutions of the time-fractional and space-fractional Benjamin-Ono equations in a straightforward manner. Lastly, the spatio-temporal dynamics of these exact travelling wave solutions are illustrated with some exquisite 2D and 3D graphs by choosing suitable parameter values. All the resulting solutions are completely novel.

The paper is organized as follows: In Section 2, we review the definitions and properties of the three fractional derivative operators. Section 3 depicts the fractional functional variable method, modified Kudryashov method and improved F-expansion method based on different definitions of fractional operators, respectively. In Section 4, we obtain the exact travelling wave solutions of the STFBOEs by the proposed methods, respectively. 2D and 3D graphs of the resulting solutions plotted and compared. In the last section, we summarize the paper.

2. Preliminaries

In the present section, we briefly review the definitions and basic properties of three types of fractional operators.

2.1. Conformable Fractional Derivative

Definition 2.1. [25] Let $u : [0, +\infty) \to \mathbb{R}$. The conformable fractional derivative of u of order μ is denoted by

$${}^{C}D_x^{\mu} u(x) = \lim_{\varepsilon \to 0} \frac{u(x + \varepsilon x^{1-\mu}) - u(x)}{\varepsilon}, \quad \mu \in (0, 1), \ x > 0. \tag{3}$$

Theorem 2.1. [25] Let $\mu \in (0, 1)$. If u is differentiable and μ-differentiable at $x>0$, then

$${}^{C}D_x^{\mu} u(x) = x^{1-\mu} D_x u(x). \tag{4}$$

Theorem 2.2. [41] Assume that for $x>0$, u is k-order differentiable and $k\mu$-differentiable, then

$${}^{C}D_x^{k\mu} u(x) = \underbrace{{}^{C}D_x^{\mu}\, {}^{C}D_x^{\mu} \cdots {}^{C}D_x^{\mu}}_{k\ times} u(x), \tag{5}$$

where k is a positive integer.

Further, more detailed mathematical descriptions of the conformable derivative can be found in [42].

2.2. Beta Derivative

Definition 2.2. [26] Let a function $u : [0, +\infty) \to \mathbb{R}$. The μ-order beta fractional derivative of u is defined as

$${}^{B}D_x^{\mu} u(x) = \lim_{\varepsilon \to 0} \frac{u\left(x + \varepsilon\left(x + \frac{1}{\Gamma(\mu)}\right)^{1-\mu}\right) - u(x)}{\varepsilon}, \tag{6}$$
$$\mu \in (0, 1), \quad x > 0.$$

Theorem 2.3. [26] Let $\mu \in (0, 1)$. If u is differentiable and μ-order differentiable, then

$${}^{B}D_x^{\mu} u(x) = \left(x + \frac{1}{\Gamma(\mu)}\right)^{1-\mu} D_x u(x). \tag{7}$$

Based on the definition and related properties of the beta derivative, Theorem 2.2 also holds for beta derivative. For more details, see [43].

2.3. M-truncated Derivative

Definition 2.3. [27] Assume that $u : [0, +\infty) \to \mathbb{R}$. The M-truncated fractional derivative of u of order μ is stated, as

$${}^{M}D_x^{\mu} u(x) = \lim_{\varepsilon \to 0} \frac{u(x\ {}_i\mathbb{E}_{\delta}(\varepsilon x^{-\mu})) - u(x)}{\varepsilon}, \quad \mu \in (0, 1), \ x > 0, \tag{8}$$

where ${}_{i}\mathbb{E}_{\delta}(z) = \sum_{k=0}^{i} \frac{z^k}{\Gamma(\delta k+1)}$, $\delta > 0$, $z \in \mathbb{C}$ denotes a truncated Mittag-Leffler function of one parameter.

Theorem 2.4. [27] Let $\mu \in (0,1)$. If u is differentiable, then

$$ {}^{M}D_{x}^{\mu}u(x) = \frac{x^{1-\mu}}{\Gamma(\delta+1)}D_{x}u(x), \ \ \delta > 0. \tag{9}$$

Similarly to the beta derivative, the M-truncated derivative also satisfies Theorem 2.2. In addition, the mathematical properties of M-truncated derivative are discussed in [44].

3. Methodology

To better depict the methods for seeking exact travelling wave solutions, we first consider the generalized form of the STFBOE as follows

$$F(D_t^{2\mu}, D_x^{2\nu}, D_x^{4\nu}, u, u^2) = 0. \tag{10}$$

For different definitions of fractional derivatives, we provide three new variables for the traveling wave transform $u = U(\xi)$.
(a) Conformable fractional derivative

For the purpose of better simplifying 10, we employ the following traveling wave variable according to Theorem 2.1:

$$\xi^{C} = \frac{A}{\mu}t^{\mu} + \frac{B}{\nu}x^{\nu}, \tag{11}$$

where A and B are real parameters that affect the width and soliton velocity of traveling wave.
(b) Beta derivative

Considering Theorem 2.3 for the beta derivative, the traveling wave variable is chosen as follows

$$\xi^{B} = \frac{A}{\mu}\left(t + \frac{1}{\Gamma(\mu)}\right)^{\mu} + \frac{B}{\nu}\left(x + \frac{1}{\Gamma(\nu)}\right)^{\nu}, \tag{12}$$

where A and B have the same meanings as above.
(c) M-truncated derivative

For the M-truncated derivative possessing TTheorem 2.3, the traveling wave variable is considered, as

$$\xi^{M} = \frac{A\Gamma(\delta+1)}{\mu}t^{\mu} + \frac{B\Gamma(\delta+1)}{\nu}x^{\nu}, \tag{13}$$

which leads to ${}^{M}D_t^{2\mu}u = A^2 D_{\xi^M}^2 U$ by means of Theorem 2.2. For the sake of convenience, later in the paper, ξ^C, ξ^B and ξ^M are abbreviated as ξ, due to their similar properties.

Then, 10 is transformed into the following ordinary differential form

$$F(D_\xi^2, D_\xi^4, U, U^2) = 0. \tag{14}$$

3.1. Fractional Functional Variable Method

Step 1: We introduce a new function as the derivative of the unknown function U, defined as

$$D_\xi U = P(U). \tag{15}$$

It is convenient to obtain that

$$\begin{cases} D_\xi^2 U = \dfrac{1}{2}D_\xi(P^2), \\ D_\xi^3 U = \dfrac{1}{2}D_\xi^2(P^2)\sqrt{P^2}, \\ D_\xi^4 U = \dfrac{1}{2}[D_\xi^3(P^2)P^2 + D_\xi^2(P^2)D_\xi(P^2)]. \end{cases} \tag{16}$$

Step 2: An appropriate number of integration operations are performed on 14 to minimize the orders of derivations of U. With the help of Eqs. (15) and (16), it follows that

$$F\left(U, U^2, U^3, \frac{1}{2}D_\xi(P^2), \cdots\right) = 0. \tag{17}$$

Step 3: Integrating 17 with respect to ξ, we obtain an expression for $P(U)$. With the help of 15, we can give an ordinary differential equation with independent variable U. By solving this equation, we can obtain the exact traveling wave solutions of 2.

3.2. Modified Kudryashov Method

Step 1: We assume that the solution of 14 can be expressed as a polynomial in $\varphi(\xi)$ as

$$U(\xi) = \sum_{n=0}^{N} a_n\varphi(\xi), \tag{18}$$

where $a_n, n = 0, 1, 2, \cdots, N$ are real constants with $a_N \neq 0$ (N is a positive integer obtained through homogeneous balance method). Here, $\varphi(\xi)$ is the solution of the linear ordinary differential equation used to assist in solving:

$$D_\xi U(\xi) = U(\xi)(U(\xi) - 1)\, ln(a), \tag{19}$$

where a is a positive constant and $a \neq 1$. Besides, 19 provide the following form of solution

$$\varphi(\xi) = \frac{1}{1 + da^{\xi}}, \tag{20}$$

where $d \in \mathbb{R}$.

Step 2: Substituting 18 and 19 into 14 and merging like terms based on the powers of $\varphi(\xi)$. Then we set each coefficient of the obtained new polynomial of $\varphi(\xi)$ to zero. With the help of Maple, we can solve the algebraic system regarding $a_n's, A, B, a$, and d mentioned above.

Step 3: Substituting the results together with the solutions of 18, we will obtain the exact traveling wave solutions of the space-time fractional Benjamin-Ono 2.

3.3. Improved F-expansion Method

Step 1: Assume that the exact solution of 14 can be represented by the following polynomial in $Q(\xi)$:

$$U(\xi)=\sum_{n=0}^{N}a_n(h+Q(\xi))^n+\sum_{n=1}^{N}b_n(h+Q(\xi))^{-n}, \quad (21)$$

where $a_n, b_n, n = 0,1,2,\cdots,N$ are real constants with $a_N, b_N \neq 0$. Moreover, $Q(\xi)$ satisfies the Riccati equation

$$D_\xi Q(\xi)=l+Q^2(\xi), \quad (22)$$

with the following three cases:

(1)When $l<0$, the general solutions are

$$Q_1=-\sqrt{-l}\,tanh(\sqrt{-l}\xi), \quad (23a)$$
$$Q_2=-\sqrt{-l}\,coth(\sqrt{-l}\xi). \quad (23b)$$

(2)When $l>0$, the general solutions are

$$Q_3=\sqrt{l}\,tan(\sqrt{l}\xi), \quad (24a)$$
$$Q_4=-\sqrt{l}\,cot(\sqrt{l}\xi). \quad (24b)$$

(3)When $l=0$, the general solution is

$$Q_5=-\frac{1}{\xi}. \quad (25)$$

Step 2: Substituting Eqs. (21) and (22) into 14, we obtain a new polynomial. We then collect the coefficients of all terms of the same power of $Q(\xi)$ and make them equal to zero. A system of algebraic equations for $a_n's, b_n's, h, A, B$ and l is given. By using the symbolic computation function of the Maple software, we can get the sets of solutions for these parameters.

Step 3: Substituting the set of solutions above into 21, we will obtain the exact traveling wave solutions of the STFBOE.

4. STFBOE and Its Exact Solutions

The (1+1)-dimensional STFBOE is stated, as

$${}^{C}D_t^{2\mu}u+\alpha\,{}^{C}D_x^{2\nu}(u^2)+\beta\,{}^{C}D_x^{4\nu}u=0. \quad (26)$$

Under the sense of conformable fractional derivative, using fractional traveling wave transform

$$\begin{cases} u=U(\xi^C), \\ \xi^C=\dfrac{a^\mu}{\mu}t^\mu+\dfrac{b^\nu}{\nu}x^\nu+e, \end{cases} \quad (27)$$

then 26 is converted to

$$A^2D_\xi^2U+2\alpha B^2((D_\xi U)^2+UD_\xi^2U)+\beta B^4D_\xi^4U=0. \quad (28)$$

Integrating the above equation, we get

$$A^2D_\xi U+2\alpha B^2UD_\xi U+\beta B^4D_\xi^3U=0. \quad (29)$$

Integrating once more, we have

$$A^2U+\alpha B^2U^2+\beta B^4D_\xi^2U=0. \quad (30)$$

4.1. Appling the Fractional Functional Variable Method

Substituting 16 into 30 and integrating it, 30 is transformed to

$$\frac{1}{2}A^2U^2+\frac{1}{3}\alpha B^2U^3+\frac{1}{2}\beta B^4(P(U))^2=0, \quad (31)$$

namely

$$P(U)=\pm\sqrt{-\frac{3A^2U^2+2\alpha B^2U^3}{3\beta B^4}}. \quad (32)$$

In the view of 15, there is the following relationship

$$D_\xi U=\pm\sqrt{-\frac{3A^2U^2+2\alpha B^2U^3}{3\beta B^4}}. \quad (33)$$

Further, we solve 33 and derive the fractal soliton wave solutions of (1+1)-dimensional STFBOE.

Case 1: $\beta<0$, we obtain
Family-1:

$$u_1(t,x)=-\frac{A^2}{2\alpha B^2}\mathrm{sech}^2\left(\frac{A}{2B^2\sqrt{-\beta}}\left(\frac{A}{\mu}t^\mu+\frac{B}{\nu}x^\nu\right)\right), \quad (34a)$$
$$u_2(t,x)=\frac{A^2}{2\alpha B^2}\mathrm{csch}^2\left(\frac{A}{2B^2\sqrt{-\beta}}\left(\frac{A}{\mu}t^\mu+\frac{B}{\nu}x^\nu\right)\right). \quad (34b)$$

Case 2: $\beta>0$, we have
Family-2:

$$u_3(t,x)=-\frac{A^2}{2\alpha B^2}\sec^2\left(\frac{A}{2B^2\sqrt{\beta}}\left(\frac{A}{\mu}t^\mu+\frac{B}{\nu}x^\nu\right)\right), \quad (35a)$$
$$u_4(t,x)=-\frac{A^2}{2\alpha B^2}\csc^2\left(\frac{A}{2B^2\sqrt{\beta}}\left(\frac{A}{\mu}t^\mu+\frac{B}{\nu}x^\nu\right)\right). \quad (35b)$$

Remark 4.1. For the STFBOEs in the sense of beta and M-truncated, we just need to replace ξ^C with ξ^B and ξ^M, i.e., 34a is rewritten as

$$u_1^B(t,x)=-\frac{A^2}{2\alpha B^2}\mathrm{sech}^2\left(\frac{A}{2B^2\sqrt{-\beta}}\left(\frac{A}{\mu}\left(t+\frac{1}{\Gamma(\mu)}\right)^\mu+\frac{B}{\nu}\left(x+\frac{1}{\Gamma(\nu)}\right)^\nu\right)\right), \quad (36)$$

and

$$u_1^M(t,x) = -\frac{A^2}{2\alpha B^2}\operatorname{sech}^2\left(\frac{A}{2B^2\sqrt{-\beta}}\left(\frac{A\Gamma(\delta+1)}{\mu}t^\mu + \frac{B\Gamma(\delta+1)}{\nu}x^\nu\right)\right), \tag{37}$$

respectively. The same is true for 34b, 35a, and 35b. This fully reflects the flexibility and universality of the proposed method.

4.2. Appling the Modified Kudryashov Method

Balancing U^2 and D_ξ^2 U according to 30 gives $N = 2$. We seek solutions of the form

$$U(\xi) = a_0 + a_1\varphi(\xi) + a_2\varphi^2(\xi), \tag{38}$$

Inserting 19 and 38 into 14 and setting the coefficients of the like powers of $\varphi(\xi)$ equal to 0, we obtain the following algebraic system for $a_n's, A, B, a$, and d:

$$\begin{aligned}
&B^2a_0^2\alpha + A^2a_0 = 0,\\
&B^4\beta\left(-3a_1(\ln a)^2 + 4a_2(\ln a)^2\right)\\
&\quad + B^2\alpha\left(2a_0a_2 + a_1^2\right) + A^2a_2 = 0,\\
&B^4\beta a_1(\ln a)^2 + 2\alpha B^2a_0a_1 + A^2a_1 = 0,\\
&6(\ln a)^2B^4a_2\beta + B^2a_2^2\alpha = 0,\\
&B^4\beta\left(2a_1(\ln a)^2 - 10a_2(\ln a)^2\right) + 2\alpha B^2a_1a_2 = 0.
\end{aligned} \tag{39}$$

To solve the above system using maple, we give

Case 1: $\beta > 0,\ A = \pm(\ln a)B^2\sqrt{\beta},\ a_0 = -\frac{B^2(\ln a)^2\beta}{\alpha},\ a_1 = \frac{6B^2(\ln a)^2\beta}{\alpha},\ a_2 = -\frac{6B^2(\ln a)^2\beta}{\alpha}$.

Family-3:

$$u_5(t,x) = \frac{B^2(\ln a)^2\beta}{\alpha} + \frac{6B^2(\ln a)^2\beta}{\alpha\left(1 + d\,a^{\frac{(\ln a)B^2\sqrt{\beta}t^\mu}{\mu} + \frac{B\,x^\nu}{\nu}}\right)} - \frac{6B^2(\ln a)^2\beta}{\alpha\left(1 + d\,a^{\frac{(\ln a)B^2\sqrt{\beta}t^\mu}{\mu} + \frac{B\,x^\nu}{\nu}}\right)^2}, \tag{40}$$

$$u_6(t,x) = \frac{B^2(\ln a)^2\beta}{\alpha} + \frac{6B^2(\ln a)^2\beta}{\alpha\left(1 + d\,a^{-\frac{(\ln a)B^2\sqrt{\beta}t^\mu}{\mu} + \frac{B\,x^\nu}{\nu}}\right)} - \frac{6B^2(\ln a)^2\beta}{\alpha\left(1 + d\,a^{-\frac{(\ln a)B^2\sqrt{\beta}t^\mu}{\mu} + \frac{B\,x^\nu}{\nu}}\right)^2}. \tag{41}$$

Case 2: $\beta < 0,\ A = \pm(\ln a)B^2\sqrt{-\beta}\,,\ a_0 = 0,\ a_1 = \frac{6B^2(\ln a)^2\beta}{\alpha},\ a_2 = -\frac{6B^2(\ln a)^2\beta}{\alpha}$.

Family-4:

$$u_7(t,x) = \frac{6\beta B^2(\ln a)^2}{\alpha\left(1 + d\,a^{\frac{\sqrt{-\beta}\,(\ln a)B^2t^\mu}{\mu} + \frac{B\,x^\nu}{\nu}}\right)} - \frac{6\beta B^2(\ln a)^2}{\alpha\left(1 + d\,a^{\frac{\sqrt{-\beta}\,(\ln a)B^2t^\mu}{\mu} + \frac{B\,x^\nu}{\nu}}\right)^2}, \tag{42}$$

$$u_8(t,x) = \frac{6\beta B^2(\ln a)^2}{\alpha\left(1 + d\,a^{-\frac{\sqrt{-\beta}\,(\ln a)B^2t^\mu}{\mu} + \frac{B\,x^\nu}{\nu}}\right)} - \frac{6\beta B^2(\ln a)^2}{\alpha\left(1 + d\,a^{-\frac{\sqrt{-\beta}\,(\ln a)B^2t^\mu}{\mu} + \frac{B\,x^\nu}{\nu}}\right)^2}. \tag{43}$$

Remark 4.2. To obtain results in the beta and M-truncated sense, we replace ξ^C with ξ^B and ξ^M, i.e., 40 is rewritten as

$$u_5^B(t,x) = \frac{B^2(\ln a)^2\beta}{\alpha} + \frac{6B^2(\ln a)^2\beta}{\alpha\left(1 + d\,a^{\frac{(\ln a)B^2\sqrt{\beta}t_B}{\mu} + \frac{B\,x_B}{\nu}}\right)} - \frac{6B^2(\ln a)^2\beta}{\alpha\left(1 + d\,a^{\frac{(\ln a)B^2\sqrt{\beta}t_B}{\mu} + \frac{B\,x_B}{\nu}}\right)^2}, \tag{44}$$

where $t_B = (t + \frac{1}{\Gamma(\mu)})^\mu$ and $x_B = (x + \frac{1}{\Gamma(\nu)})^\nu$, and

$$u_5^M(t,x) = \frac{B^2(\ln a)^2\beta}{\alpha} + \frac{6B^2(\ln a)^2\beta}{\alpha\left(1 + d\,a^{\frac{(\ln a)B^2\sqrt{\beta}t_M}{\mu} + \frac{B\,x_M}{\nu}}\right)} - \frac{6B^2(\ln a)^2\beta}{\alpha\left(1 + d\,a^{\frac{(\ln a)B^2\sqrt{\beta}t_M}{\mu} + \frac{B\,x_M}{\nu}}\right)^2}, \tag{45}$$

where $t_M = \Gamma(\delta+1)t^\mu$ and $x_M = \Gamma(\delta+1)x^\nu$. The same applies to Eqs. (41) to (43).

4.3. Appling the Improved F-expansion Method

According to the homogeneous balance method, we can derive $N = 2$. Thus, 21 can be expressed as follows

$$U(\xi) = a_0 + a_1(h+Q(\xi)) + a_2(h+Q(\xi))^2 + b_1(h+Q(\xi))^{-1} + b_2(h+Q(\xi))^{-2}. \tag{46}$$

Substituting 46 into 14 and holding the coefficients of each power of $Q(\xi)$ to zero. We will then gain a system of algebraic equations with respect to $a_n's, b_n's, h, A, B$ and l. By means of Maple, we can solve this system and give the following four solution cases:

Case 1: $l = -20,\ a_0 = 10a_2,\ a_1 = -10a_2,\ \alpha = \frac{A^2}{20B^2a_2},\ b_1 = -50a_2,\ b_2 = 25a_2,\ \beta = -\frac{A^2}{120B^4},\ h = 5.$

Family-5:

$$u_9(t,x) = -\frac{400a_2\left(\tanh(R_1)+1\right)\left(\tanh(R_1)-1\right)}{\left(2\sqrt{5}\,\tanh(R_1)-5\right)^2} \times \left(\sqrt{5}\,\tanh(R_1) - \tanh^2(R_1) - \frac{3}{2}\right), \tag{47}$$

$$u_{10}(t,x) = -\frac{400a_2\left(\coth(R_1)-1\right)\left(\coth(R_1)+1\right)}{\left(2\sqrt{5}\,\coth(R_1)-5\right)^2} \times \left(\sqrt{5}\,\coth(R_1) - \coth^2(R_1) - \frac{3}{2}\right), \tag{48}$$

where $R_1 = \frac{2\sqrt{5}\,(A\,t^{\mu}\nu + B\,x^{\nu}\mu)}{\mu\nu}$.

Case 2: $l = -20,\ a_0 = 30a_2,\ a_1 = -10a_2,\ \alpha = -\frac{A^2}{20B^2a_2},\ b_1 = -50a_2,\ b_2 = 25a_2,\ \beta = \frac{A^2}{120B^4},\ h = 5.$

Family-6:

$$u_{11}(t,x) = -\frac{100a_2}{\left(2\sqrt{5}\,\tanh(R_1)-5\right)^2} \times \left(4\sqrt{5}\,\tanh^3(R_1) - 4\tanh^4(R_1) - 6\tanh^2(R_1) + 1\right), \tag{49}$$

$$u_{12}(t,x) = \frac{100a_2}{\left(2\sqrt{5}\,\coth(R_1)-5\right)^2} \times \left(4\coth^4(R_1) - 4\coth^3(R_1)\sqrt{5} + 6\coth^2(R_1) - 1\right). \tag{50}$$

Case 3: $a_0 = -\frac{A^2}{4\alpha B^2},\ a_1 = 0,\ b_1 = 0,\ b_2 = \frac{9A^4}{64B^4a_2\alpha^2},\ \beta = -\frac{\alpha a_2}{6B^2},\ h = 0,\ l = \frac{3A^2}{8\alpha a_2 B^2} < 0.$

Family-7:

$$u_{13}(t,x) = \frac{A^2}{4\alpha B^2} - \frac{3A^2\tanh^2(R_2)}{8\alpha B^2} - \frac{3A^2}{8\alpha B^2\tanh^2(R_2)}, \tag{51a}$$

$$u_{14}(t,x) = \frac{A^2}{4\alpha B^2} - \frac{3A^2\coth^2(R_2)}{8\alpha B^2} - \frac{3A^2}{8\alpha B^2\coth^2(R_2)}, \tag{51b}$$

where $R_2 = \sqrt{-\frac{3A^2}{8\alpha a_2 B^2}}\,\left(\frac{A\,t^{\mu}}{\mu} + \frac{B\,x^{\nu}}{\nu}\right)$.

Case 4: $a_0 = -\frac{A^2}{4\alpha B^2},\ a_1 = 0,\ b_1 = 0,\ b_2 = \frac{9A^4}{64B^4a_2\alpha^2},\ \beta = -\frac{\alpha a_2}{6B^2},\ h = 0,\ l = \frac{3A^2}{8\alpha a_2 B^2} > 0.$

Family-8:

$$u_{15}(t,x) = \frac{A^2}{4\alpha B^2} + \frac{3A^2\tan^2(R_3)}{8\alpha B^2} + \frac{3A^2}{8B^2\alpha\tan^2(R_3)}, \tag{52a}$$

$$u_{16}(t,x) = \frac{A^2}{4\alpha B^2} + \frac{3A^2\cot^2(R_3)}{8\alpha B^2} + \frac{3A^2}{8B^2\alpha\cot^2(R_3)}. \tag{52b}$$

where $R_3 = \frac{\sqrt{6}\sqrt{\frac{A^2}{\alpha a_2 B^2}}\,\left(\frac{A\,t^{\mu}}{\mu} + \frac{B\,x^{\nu}}{\nu}\right)}{4}$.

Remark 4.3. Here, we replace 11 with 12 and 13, i.e., 47 is rewritten as

$$u_9^B(t,x) = \frac{400a_2\left(\tanh(R_B)-1\right)\left(\tanh(R_B)+1\right)}{\left(2\sqrt{5}\,\tanh(R_B)-5\right)^2} \times \left(\tanh^2(R_B) - \sqrt{5}\,\tanh(R_B) + \frac{3}{2}\right), \tag{53}$$

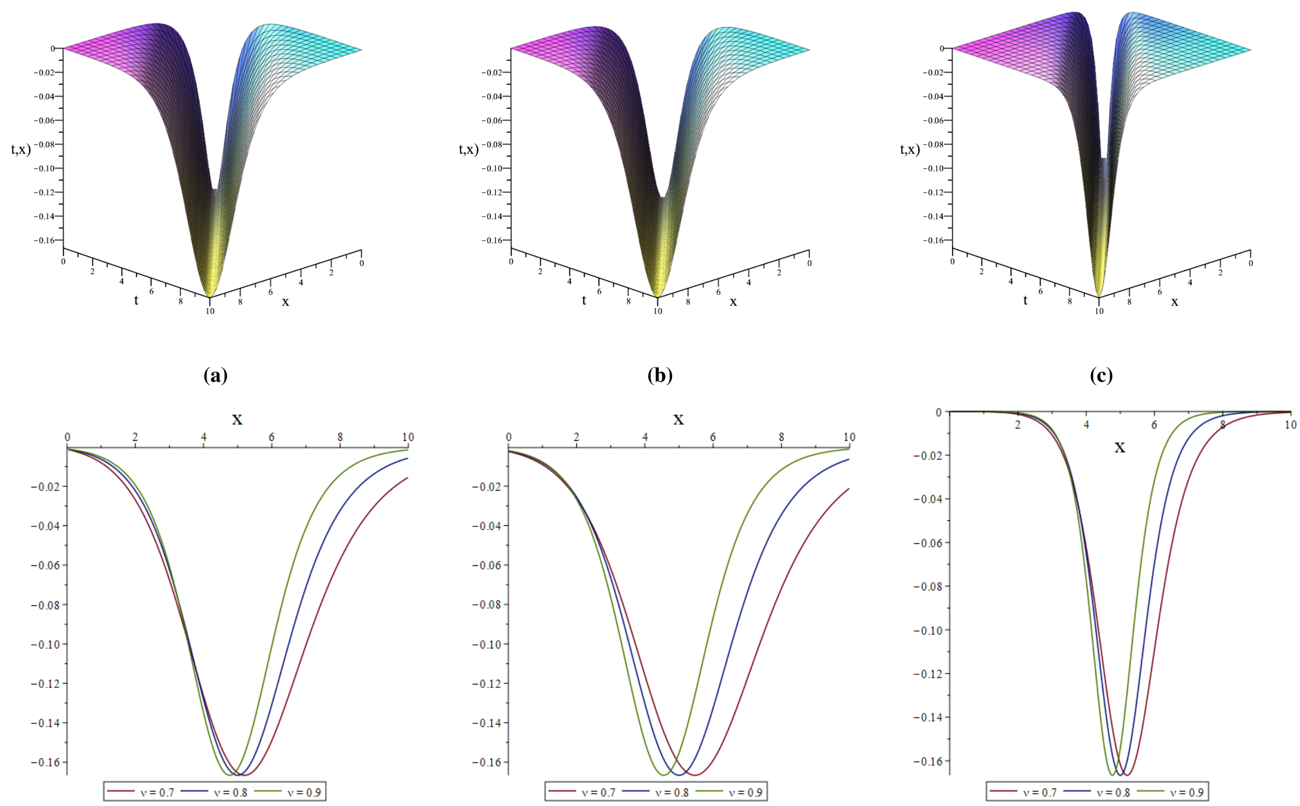
where $R_B = \frac{2\sqrt{5}\left(A\left(\frac{t\Gamma(k_1)+1}{\Gamma(k_1)}\right)^{k_1}k_2+B\left(\frac{x\Gamma(k_2)+1}{\Gamma(k_2)}\right)^{k_2}k_1\right)}{k_1k_2}$, and

$$u_9^M(t,x) = -\frac{400a_2\left(\tanh(R_M)-1\right)\left(\tanh(R_M)+1\right)}{\left(2\sqrt{5}\tanh(R_M)-5\right)^2} \times \left(\sqrt{5}\tanh(R_M) - \tanh^2(R_M) - \frac{3}{2}\right), \tag{54}$$

where $R_M = \frac{2\sqrt{5}\left(A\,t^{k_1}k_2+B\,x^{k_2}k_1\right)\Gamma(\delta+1)}{k_1k_2}$. Other traveling wave solutions behave similarly.

4.4. Comparative Analysis of Some Solutions with Three Fractional Operators

In this subsection, in order to compare some of the solutions obtained in the three fractional senses, we illustrate 2D and 3D graphs using appropriate parameter values.



(a) **(b)** **(c)**

(d) **(e)** **(f)**

Figure 1. *The dark soliton solutions plots. Left column: conformable derivative; Middle column: beta derivative; Right column: M-truncated derivative.*

In Figure 1, we give graphs for 34a, 36 and 37. The parameters are selected as follows: $\alpha = 3, \beta = -0.5, A = 2, B = -2, \mu = 0.8, \nu = 0.8$ and $\delta = 2$. We plot the red line, blue line, green line respectively for $\nu = 0.7, \nu = 0.8$ and $\nu = 0.9$ in Figures 1d, 1e and 1f. Figure 1 represents the dark soliton solutions.

Then 40, 44 and 45 are plotted in Figure 2 when $\alpha = 1, B = 1, a = 0.5, d = 1, \mu = 0.8, \nu = 0.8$ and $\delta = 2$. For different fractional orders, we plot the red line, blue line, green line respectively for $\nu = 0.7, \nu = 0.8$ and $\nu = 0.9$ in Figures 2d, 2e and 2f. Figure 2 represents the multiple bright-dark soliton solutions.

Finally, we set $A = 1, B = -1, a_2 = 1, \mu = 0.8, \nu = 0.8$ and $\delta = 2$. The 3D graphs of 47, 53 and 54 are plotted in Figures 3a, 3b and 3c. In Figures 3d, 3e and 3f, we plot the 2D graphs for $\nu = 0.7, \nu = 0.8$ and $\nu = 0.9$. It can be seen that the obtained solutions utilizing different fractional operators are compared. Figure 3 represents the bright soliton solutions.

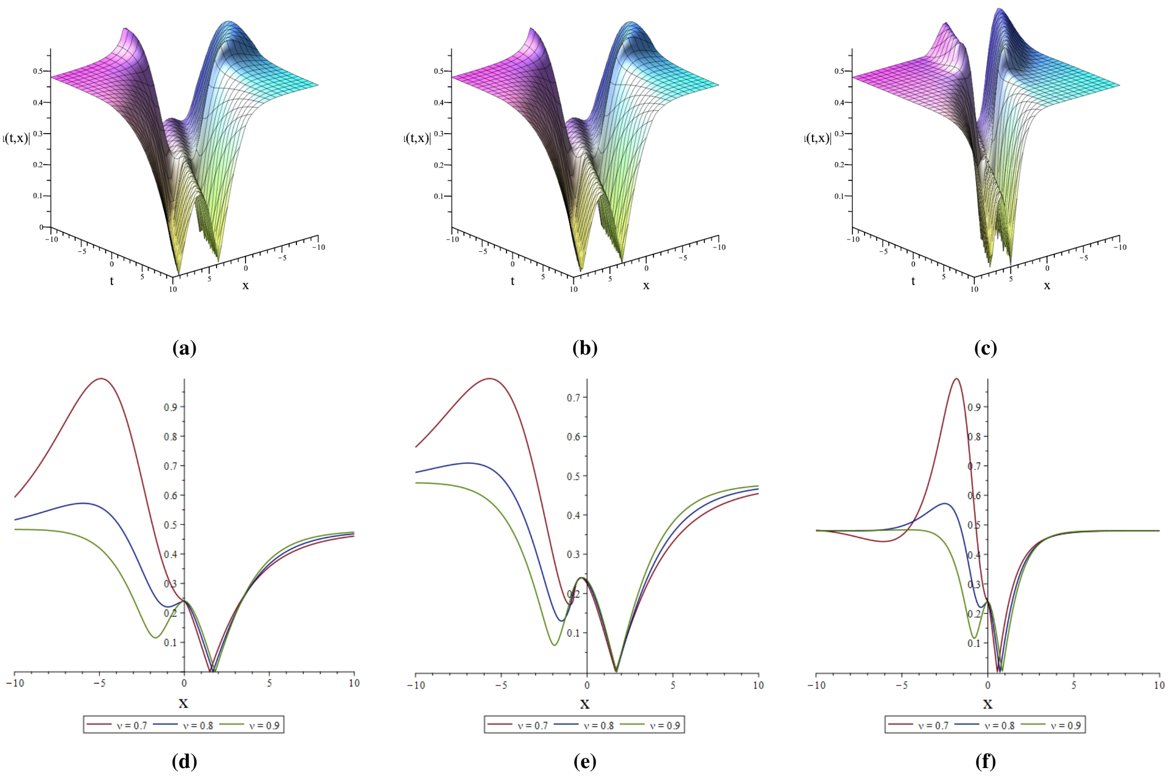


Figure 2. *The multiple bright-dark soliton solutions plots. Left column: conformable derivative; Middle column: beta derivative; Right column: M-truncated derivative.*

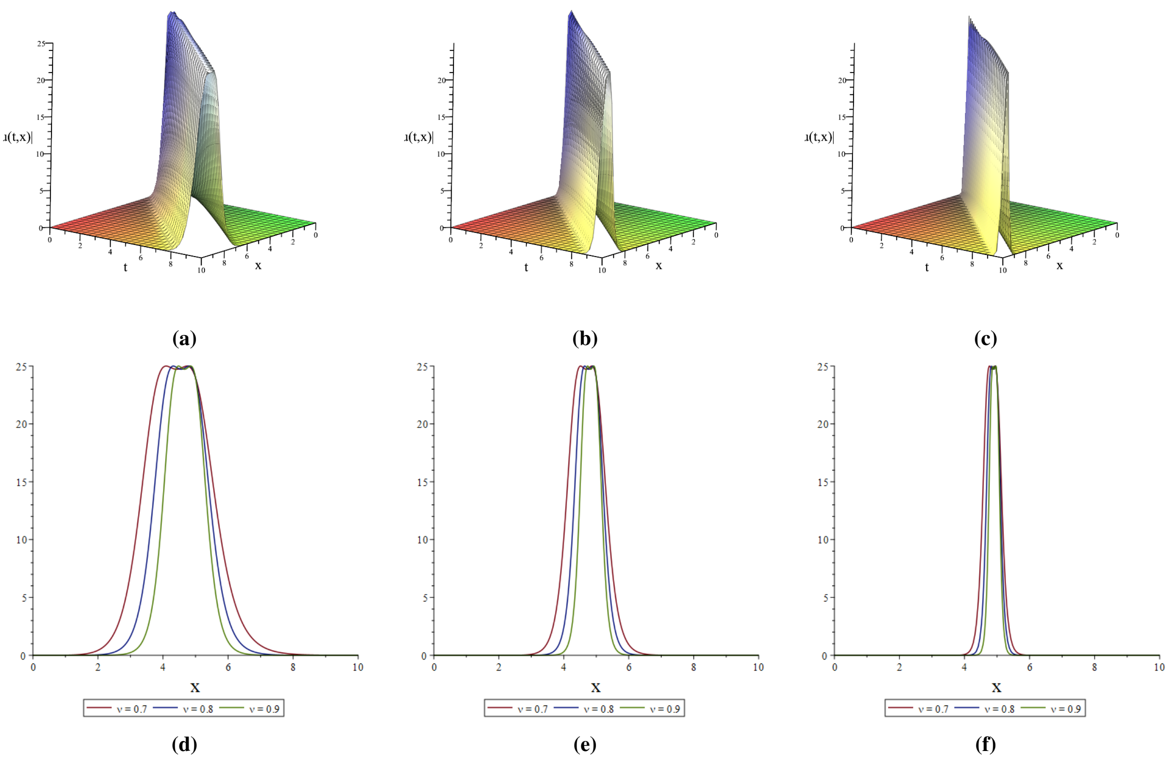


Figure 3. *The bright soliton solutions plots. Left column: conformable derivative; Middle column: beta derivative; Right column: M-truncated derivative.*

5. Conclusion

In this paper, new analytical solutions of the STFBOEs in three different fractional senses are reported to have been successfully obtained by utilizing fractional functional variable method, modified Kudryashov method and improved F-expansion method. Symbolic calculation is performed using Maple software and some results for the conformable, beta and M-truncated derivatives are described so as to demonstrate the behavior of the solution. All results have been verified to be accurate. The above techniques are suitable for solving fractional differential equation models employed in science and engineering.

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Abbreviations

FNEEs	Fractional Nonlinear Evolution Equations
STFBOE	Space-time Fractional Benjamin-Ono Equation
2D	Two-dimensional
3D	Three-dimensional

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Conflicts of Interest

The authors declare no conflicts of interest.

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