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Comparison Principle for Fractional Differential Inequalities with Variable Order Caputo Derivative

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Abstract: Fractional differential inequalities have emerged as powerful tools for modeling and analyzing dynamic systems with fractional-order derivatives, offering a sophisticated framework to capture the complexities of real-world processes. Among the various analytical techniques, the comparison principle stands out as a fundamental approach in understanding the behavior of solutions to fractional differential inequalities. This study focuses on the development and analysis of comparison principles for some of the fractional differential inequalities involving the variable-order Caputo fractional derivative which is a generalization of the classical Caputo derivative that allows the order of differentiation to vary with respect to time or space. Such flexibility is important for modeling systems whose memory characteristics change over time or space. We formulate both weak and strong versions of the comparison principle with variable order Caputo fractional derivative. Our approach combines analytical techniques from fractional calculus and the theory of differential inequalities to establish some results. To have the applicability and relevance of our theoretical work, we provide an example demonstrating the effectiveness of the proposed comparison theorems. The findings of this paper not only contribute to the theoretical advancement of fractional differential inequalities with variable order but also applicable to systems where dynamic memory effects are prominent.

Keywords: Comparison Principle, Fractional Differential Inequalities, Variable Order Caputo Derivative, Volterra Fractional Integral Equation, Fractional Calculus

1. Introduction

In recent decades, fractional calculus has gained significant attention across varied scientific disciplines for its unparalleled ability to model intricate phenomena featuring memory and non-local effects. Fractional calculus, particularly fractional differential equations (FDEs), has gained substantial significance in physics and applied mathematics. FDEs serve as potent tools for describing real-world problems more accurately than classical integer-order equations. Despite having a history spanning three centuries, the applications of FDEs have seen a resurgence in the last two decades [1, 2]. While numerous analytical and numerical methods exist for solving integer-order differential equations, many times these methods often fall short when applied to their fractional counterparts. Consequently, there is a pressing need to introduce new methods tailored to solve FDEs. Notably, obtaining analytical solutions for most FDEs is a challenging task, prompting researchers to work on various schemes.

The comparison principle is a common tool which is use to establish bounds and other properties of solutions to differential equations by comparing them to simpler, known solutions. Further comparison theorems for differential equations are fundamental in the analysis of solutions to differential equations and often involve comparing different solutions to draw conclusions about their behavior [3]. The comparison principle, offers a powerful approach to understand the behavior of solutions to nonlinear inequalities [4, 5]. This can be a useful tool to deal with the solutions of fractianal differential inequalities (FDIs) too.

Inequalities are powerful tools in mathematical analysis, providing insights into the behavior and properties of solutions to differential equations [5, 6]. In this research paper we studied about FDIs often involve expressions where fractional derivatives of functions are compared or bounded. These types of inequalities are essential in the study of fractional calculus [7–11]. FDIs, in particular, play a pivotal role in this realm, offering a powerful framework to describe and analyze dynamic systems characterized by fractional-order derivatives. These inequalities have found applications in physics, engineering, control theory, and various other fields where classical integer-order calculus may prove insufficient.

The variable order Caputo fractional derivatives for FDEs were studied by Albasheir et. al. in [12]. In this paper we are going to use the variable order differentiation in the study of FDIs, which will be inequalities that involve variable order derivatives of a function. Here is one of the general form of a FDIs of variable order:

$$ {}^{C}\mathscr{D}_a^{\alpha(x)}u(x) \leq f(t, u(x)) \tag{1} $$

where ${}^{C}\mathscr{D}_a^{\alpha(x)}$ is the Caputo fractional derivative of variable order $\alpha(x)$, $u(x)$ is the unknown function, and $f(x, u(x))$ is a given function that may depend on both x and $u(x)$. The Caputo fractional derivative with real or complex orders can be a particular case of variable order fractional derivative.

The comparison principle, a well-established tool in the analysis of solutions to differential equations, has been extended to the domain of FDIs. This extension provides researchers and engineers with a valuable approach to understand the behavior and properties of solutions in systems with fractional derivatives. The comparison principle allows for insightful comparisons between different solutions, facilitating the extraction of meaningful conclusions about the dynamics of fractional systems. Lakshmikantham and Vatsala [10, 11] investigated the comparison theorem for Riemann-Liouville-type FDIs ($0 < \alpha < 1$). Also, with respect to generalized fractional derivatives the comparison principle for FDIs were given by Al-Refai and Yuri Luchko in [8].

This research paper aims to provide a theoretical foundation for the comparison principle in the context of FDIs using the variable order Caputo fractional differentiation, and investigating the conditions under which this principle can be effectively applied to draw meaningful conclusions about the behavior of solutions of (1). The outcomes of this study are expected to contribute to the development of theoretical knowledge in fractional calculus and provide valuable insights for researchers and practitioners working with fractional-order systems.

The paper is structured as follows: Section 2 provides a brief overview of some preliminaries which are going to be used in this paper. Section 3 delves into theoretical considerations and mathematical foundations related to the application of the comparison principle to FDIs. In Section 4 we presents comparison result under strict inequality conditions for the FDIs involving Caputo derivative of order $\alpha(x)$ ($0 \leq n-1 < \alpha(x) \leq n$) and also present example for the same. Finally, in Section 5 we summarize the key findings, discuss the potential areas for future research, and concludes the paper.

2. Preliminaries

Definition 2.1. [2] The gamma function is defined by the following integral

$$ \Gamma(u) = \int_0^\infty e^{-r} r^{u-1} dr, \qquad (Re(u) > 0). \tag{2} $$

Definition 2.2. [2] The Mittag-Leffler Function for $\alpha > 0$ and $\beta > 0$ is

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{3} $$

For $\beta = 1$, we have

$$ E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{4} $$

Definition 2.3. [2] The beta function is defined as

$$ \beta(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1} dt, \ (Re(u) > 0, \ Re(v) > 0). \tag{5} $$

Definition 2.4. [12] The Riemann-Louville derivative of order $\alpha(x)$ is given as

$$ {}^{RL}\mathscr{D}_a^{\alpha(x)}y(x) = \frac{1}{\Gamma(n-\alpha(x))} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha(x)-1} y(t) dt, \tag{6} $$

where $0 < \alpha(x)$ and $x > 0$.

Definition 2.5. [12] The Caputo derivative of order $\alpha(x)$ is given as

$$ {}^{c}\mathscr{D}_a^{\alpha(x)}y(x) = \frac{1}{\Gamma(n-\alpha(x))} \int_a^x \frac{y^{(n)}(t)}{(x-t)^{\alpha(x)+1-n}} dt, \tag{7} $$

where $0 < \alpha(x)$ and $x > 0$.

Definition 2.6. *Lipschitz Condition* [3]

Suppose $u(t)$ be continuous on the interval $[a, b]$ of the real line R, and differentiable on $(a, b]$ and $f(t, u)$ be a continuous funtion from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ then

$$|f(t, u_1) - f(t, u_2)| \leq K|u_1 - u_2|, \tag{8}$$

where K is the Lipschitz constant.

3. Comparison Principles for Fractional Differential Inequalities

The theorem in this section are depend on generalization of comparison principle given by McNabb in [3] of classical calculus.

Theorem 3.1. *(Weak Comparison Theorem)* Suppose $u(t)$ and $v(t)$ are continuous on the interval $[a, b]$ of the real line $\mathbb{R}$, and differentiable on $(a, b]$, f is a continuous mapping from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ and

$$\begin{aligned} u(a) &< v(a), \\ {}^{c}\mathscr{D}_a^{\alpha(x)} u - f(t, u) &< {}^{c}\mathscr{D}_a^{\alpha(x)} v - f(t, v) \end{aligned} \tag{9}$$

on interval $(a, b]$ and for $0 < \alpha(x) < 1$. Then $u < v$ on $[a, b]$.

Proof. Let us suppose that $u \geq v$ somewhere on $[a, b]$.

Then, since $u - v$ is continuous on $[a, b]$, and $u(a) - v(a) < 0$, there is a point c in $(a, b]$ such that $u(c) = v(c)$ and $u(c) > v(c)$ on $[a, c)$.

But ${}^{c}\mathscr{D}_a^{\alpha(x)} u(c) = {}^{c}\mathscr{D}_a^{\alpha(x)} v(c)$ and $f(c, u(c)) > f(c, v(c))$, that is

$${}^{c}\mathscr{D}_a^{\alpha(x)} u(c) - f(c, u(c)) > {}^{c}\mathscr{D}_a^{\alpha(x)} v(c) - f(c, v(c))$$

Since this violates the inequality (11) at point c which belongs to $[a, b]$, therefore no such c exists in $[a, b]$, which contradicts our assumption.

Therefore, $u < v$ on $[a, b]$.

Remark 3.1. We are going to use the calculation of variable order Caputo fractional derivative of $e^{\lambda x}$ from [2, 12] in the proof of next theorem, where the value of $e^{\lambda x}$ is taken as

$$e^{\lambda x} = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}$$

Now, the variable order Caputo fractional derivative of $e^{\lambda x}$ is

$$\begin{aligned}
{}^{c}\mathscr{D}_a^{\alpha(x)}[e^{\lambda x}] &= \frac{1}{\Gamma(n-\alpha(x))} \int_a^x (x-t)^{n-\alpha(x)-1} \frac{d^n(e^{\lambda t})}{dt^n}\, dt \\
&= \frac{1}{\Gamma(n-\alpha(x))} \int_a^x (x-t)^{n-\alpha(x)-1} \lambda^n e^{\lambda t} dt \\
&= \frac{\lambda^n}{\Gamma(n-\alpha(x))} \int_a^x (x-t)^{n-\alpha(x)-1} e^{\lambda t} dt \\
&= \frac{\lambda^n}{\Gamma(n-\alpha(x))} \int_a^x (x-t)^{n-\alpha(x)-1} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} dt \\
&= \frac{\lambda^n}{\Gamma(n-\alpha(x))} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_a^x (x-t)^{n-\alpha(x)-1} t^k dt \\
&= \frac{\lambda^n}{\Gamma(n-\alpha(x))} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_a^x x^{n-\alpha(x)-1} \left(1 - \frac{t}{x}\right)^{n-\alpha(x)-1} t^k dt \\
&= \frac{\lambda^n}{\Gamma(n-\alpha(x))} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_{\frac{a}{x}}^1 x^{n-\alpha(x)-1} (1-y)^{n-\alpha(x)-1} (xy)^k x dy \\
&= \frac{\lambda^n}{\Gamma(n-\alpha(x))} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} x^{n-\alpha(x)+k} \int_0^1 (1-y)^{n-\alpha(x)-1} y^k dy,\ \frac{a}{x} \to 0 \\
&= \frac{\lambda^n}{\Gamma(n-\alpha(x))} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} x^{n-\alpha(x)+k} \beta(n-\alpha(x), k+1) \\
&= \frac{\lambda^n}{\Gamma(n-\alpha(x))} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} x^{n-\alpha(x)+k} \frac{\Gamma(n-\alpha(x))\Gamma(k+1)}{\Gamma(n-\alpha(x)+k+1)} \\
&= \lambda^n \sum_{k=0}^{\infty} \frac{\lambda^k x^{n-\alpha(x)+k}}{\Gamma(n-\alpha(x)+k+1)} \\
&= \lambda^n x^{n-\alpha} \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{\Gamma(n-\alpha(x)+k+1)}
\end{aligned}$$

Theorem 3.2. *(Strong Comparison Theorem)* Suppose $u(t)$ and $v(t)$ are continuous on the interval $[a, b]$ of the real line $\mathbb{R}$, and differentiable on $(a, b]$, f is a continuous mapping from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ satisfying the Lipschitz condition and

$$\begin{gathered} u(a) < v(a), \\ {}^{c}\mathscr{D}_a^{\alpha(x)} u - f(t, u) \leq {}^{c}\mathscr{D}_a^{\alpha(x)} v - f(t, v) \end{gathered} \tag{10}$$

on interval $(a, b]$ and for $0 < \alpha < 1$. Then $u < v$ on $[a, b]$.

Proof. Suppose $v(a) - u(a) = A > 0$ and let

$$w = u + \frac{A}{2} e^{-2K(t-a)} \tag{11}$$

Then by using calculations in remark 3.1, we have the following

$$\begin{aligned}
{}^{c}\mathscr{D}_a^{\alpha(x)} w - f(t, w) &= {}^{c}\mathscr{D}_a^{\alpha(x)} (u + \frac{A}{2} e^{-2K(t-a)}) - f(t, w) \\
&= {}^{c}\mathscr{D}_a^{\alpha(x)} u + {}^{c}\mathscr{D}_a^{\alpha(x)} (\frac{A}{2} e^{-2K(t-a)}) - f(t, w) \\
&= {}^{c}\mathscr{D}_a^{\alpha(x)} u + \frac{A}{2} {}^{c}\mathscr{D}_a^{\alpha(x)} e^{-2K(t-a)} - f(t, w) \\
&= {}^{c}\mathscr{D}_a^{\alpha(x)} u + \frac{A}{2} (-2K)^n (t-a)^{(n-\alpha(x))} \sum_{j=0}^{\infty} \frac{[-2K(t-a)]^j}{\Gamma(j+n-\alpha+1)} - f(t, w) + f(t, u) - f(t, u) \\
&\leq {}^{c}\mathscr{D}_a^{\alpha(x)} u - f(t, u) + \frac{A}{2} (-2K)^n (t-a)^{(n-\alpha(x))} \sum_{j=0}^{\infty} \frac{[-2K(t-a)]^j}{\Gamma(j+n-\alpha(x)+1} - |f(t, w) - f(t, u)| \\
&\leq {}^{c}\mathscr{D}_a^{\alpha(x)} u - f(t, u) + \frac{A}{2} (-2K)^n (t-a)^{(n-\alpha(x))} \sum_{j=0}^{\infty} \frac{[-2K(t-a)]^j}{\Gamma(j+n-\alpha(x)+1)} - K|w-u| \\
&\leq {}^{c}\mathscr{D}_a^{\alpha(x)} u - f(t, u) - \frac{A}{2} K e^{-2K(t-a)} + \frac{A}{2} (-2K)^n (t-a)^{(n-\alpha(x))} \sum_{j=0}^{\infty} \frac{[-2K(t-a)]^j}{\Gamma(j+n-\alpha(x)+1)} \\
&\leq {}^{c}\mathscr{D}_a^{\alpha(x)} u - f(t, u) - \frac{A}{2} K \sum_{j=0}^{\infty} \frac{[-2K(t-a)]^j}{\Gamma(j+1)} + \frac{A}{2} (-2K)^n (t-a)^{(n-\alpha(x))} \sum_{j=0}^{\infty} \frac{[-2K(t-a)]^j}{\Gamma(j+n-\alpha(x)+1)} \\
&\leq {}^{c}\mathscr{D}_a^{\alpha(x)} u - f(t, u) - \frac{A}{2} K \sum_{j=0}^{\infty} \frac{[-2K(t-a)]^j}{\Gamma(j+1)} + \frac{A}{2} (-2K)^{\alpha(x)} \sum_{j=0}^{\infty} \frac{[-2K(t-a)]^{j+n-\alpha}}{\Gamma(j+n-\alpha(x)+1)} \\
&< {}^{c}\mathscr{D}_a^{\alpha(x)} u - f(t, u) \\
&\leq {}^{c}\mathscr{D}_a^{\alpha(x)} v - f(t, v)
\end{aligned}$$

on $[a, b]$.

Now, $w(a) = u(a) < v(a)$ that is $w(a) < v(a)$ and so by weak comparison theorem $w < v$ on $[a, b]$ and $u < w < v$ i.e. $u < v$ on $[a, b]$.

4. Comparison Principle with Strict Inequalities

Consider the following initial value problem (IVP) with variable order Caputo-type fractional differential equation:

$$\begin{gathered} {}^{c}\mathscr{D}_a^{\alpha(x)} y(x) = F(x, y(x)), \\ y^k(0) = y_k,\ k = 0, 1, 2, ..., n-1 \end{gathered} \tag{12}$$

where $F \in C([0, T] \times \mathbb{R}, \mathbb{R})$ and $0 \leq n-1 < \alpha(x) \leq n$. The IVP (12) is equivalent to the following Volterra fractional integral equation:

$$y(x) = \sum_{k=0}^{n-1} \frac{y_k x^k}{\Gamma(k+1)} + \frac{1}{\Gamma(\alpha(x))} \int_0^x (x-s)^{(\alpha(x)-1)} F(s, y(s)) ds. \tag{13}$$

Lemma 4.1. *[10]* Let $m(x) : \mathbb{R}_+ \to \mathbb{R}$ be locally Lipschitz continuous such that, for $x_1 \in (0, +\infty)$, $m(x_1) = 0$ and $m(x) \leq 0$ for $0 \leq x \leq x_1$. Then ${}^C\mathscr{D}_a^{\alpha(x)} m(x_1) \geq 0$ for $0 \leq n-1 < \alpha(x) \leq n.$

Theorem 4.1. Assume that $F(x, y)$ and $G(x, y)$ are two continuous functions defined on $H = [0, T] \times \mathbb{R}$ satisfying the inequality

$$F(x, y) < G(x, y) \tag{14}$$

for $(x, y) \in H$. Suppose that $f(x)$ and $g(x)$ are solutions of the following initial value problems ($0 \leq n-1 < \alpha(x) \leq n$):

$$ {}^c\mathscr{D}_a^{\alpha(x)} y \leq G(x, y), \qquad y(0) = y_0, \tag{15}$$

$$ {}^c\mathscr{D}_a^{\alpha(x)} y \leq F(x, y), \qquad y(0) = y_0, \tag{16}$$

respectively. Then we have

$$f(x) < g(x),\ x \in (0, T]. \tag{17}$$

Proof. Denote $\phi(x) = f(x) + g(x), x \in [0, T]$. We have

$$\phi(0) = f(0) - g(0) = 0$$

$$\begin{aligned} {}^c\mathscr{D}_a^{\alpha(x)}\phi(0) &= {}^c\mathscr{D}_a^{\alpha(x)} f(0) - {}^c\mathscr{D}_a^{\alpha(x)} g(0) \\ &\leq F(0, y_0) - G(0, y_0) \\ &< 0, \end{aligned}$$

then for ${}^c\mathscr{D}_a^{\alpha(x)}\phi(x)$, and there exists $\sigma > 0$ such that $\phi(x) < 0$ for $0 < x < \sigma$. Suppose that inequality (17) is incorrect. Then there exists at lest one $x_1 (> 0)$ such that $\phi(x_1) = 0$.

Denote

$$\beta = inf\{x : \phi(x) = 0, x > 0\}.$$

Then $\phi(\beta) = 0$ and $\phi(x) < 0$ for $0 < x < \beta$.

Since ${}^c\mathscr{D}_a^{\alpha(x)}\phi(x) \leq F(x, y) - G(x, y)$, and $F(x, y), G(x, y)$ are continuous in H, it follows that ${}^c\mathscr{D}_a^{\alpha(x)}\phi(x)$ is locally Lipschitz continuous in $0 \leq x \leq T$. Thus ${}^C\mathscr{D}_a^{\alpha(x)}\phi(\beta) \geq 0$ by lemma 4.1.

On other hand, since $\phi(\beta) = 0$ denote $\gamma = g(\beta) = f(\beta)$.

Then

$$\begin{aligned} 0 \leq {}^c\mathscr{D}_a^{\alpha(x)}\phi(\beta) &= {}^c\mathscr{D}_a^{\alpha(x)} f(\beta) - {}^c\mathscr{D}_a^{\alpha(x)} g(\beta) \\ &\leq F(\beta, \gamma) - G(\beta, \gamma) \end{aligned} \tag{18}$$

It is clear that (18) is in contradiction with (14).

Hence, (17) is proved.

Theorem 4.2. Under assumption (14), suppose that $f(x)$ and $g(x)$ are solutions of the following initial value problems ($1 \leq n-1 < \alpha(x) \leq n$):

$$ {}^c\mathscr{D}_a^{\alpha(x)} y \leq G(x, y),\ y^k(0) = y_k,\ k = 0, 1, 2, ..., n-1, \tag{19}$$

$$ {}^c\mathscr{D}_a^{\alpha(x)} y \leq F(x, y),\ y^k(0) = y_k,\ k = 0, 1, 2, ..., n-1, \tag{20}$$

respectively. Then $f(x) < g(x)$ for $x \in (0, T]$.

Proof. Consider ${}^c\mathscr{D}_a^{\alpha(x)} y(x) = {}^c\mathscr{D}_a^{\beta(x)}\, {}^c\mathscr{D}_a^{(n-1)} y(x)$ with $\beta(x) = \alpha(x) - n + 1 \in (0, 1]$.

Denote $z(x) = {}^c\mathscr{D}_a^{(n-1)} y(x), x \in [0, T]$.

Then the IVPs (19) and (20) can be transformed into IVPs of order $0 < \beta(x) \leq 1$. Indeed we have

$$\begin{aligned} {}^c\mathscr{D}_a^{\alpha(x)} y(x) &= {}^c\mathscr{D}_a^{\beta(x)}\, {}^c\mathscr{D}_a^{(n-1)} y(x) \\ &= {}^c\mathscr{D}_a^{\beta(x)} z(x), \\ z(x) &= {}^c\mathscr{D}_a^{(n-1)} y(x) \end{aligned} \tag{21}$$

$$y(x) = \sum_{k=0}^{n-2} \frac{y_k x^k}{\Gamma(k+1)} + \frac{1}{\Gamma(n-1)} \int_0^x (x-s)^{(n-2)} z(s) ds \tag{22}$$

Denote

$$\hat{F}(x, z(x)) = F(x, \sum_{k=0}^{n-2} \frac{y_k x^k}{\Gamma(k+1)} + \frac{1}{\Gamma(n-1)} \int_0^x (x-s)^{(n-2)} z(s) ds))$$

$$\hat{G}(x, z(x)) = G(x, \sum_{k=0}^{n-2} \frac{y_k x^k}{\Gamma(k+1)} + \frac{1}{\Gamma(n-1)} \int_0^x (x-s)^{(n-2)} z(s) ds))$$

By inequality (14), we have $\hat{F}(x, z(x)) \leq \hat{G}(x, z(x))$ for $x \in [0, T]$. The original initial value problems (19) and (20) can be written as-

$$\begin{aligned} {}^c\mathscr{D}_a^{\beta(x)} z(x) &\leq \hat{F}(x, z(x)), \\ z(0) &= {}^c\mathscr{D}_a^{(n-1)} y(0) \\ &= y_{n-1}, \end{aligned} \tag{23}$$

$$\begin{aligned} {}^c\mathscr{D}_a^{\beta(x)} z(x) &\leq \hat{G}(x, z(x)), \\ z(0) &= {}^c\mathscr{D}_a^{(n-1)} y(0) \\ &= y_{n-1}. \end{aligned} \tag{24}$$

Assume that $\hat{f}(x)$ and $\hat{g}(x)$ are soutions of (23) and (24), respectively. Then by theorem 4.1, we have $\hat{f}(x) < \hat{g}(x)$ for $t \in (0, T]$. Since $y(x)$ and $z(x)$ satisfy equation (22), we obtain

$$\begin{aligned} f(x) &= \sum_{k=0}^{n-2} \frac{y_k x^k}{\Gamma(k+1)} + \frac{1}{\Gamma(n-1)} \int_0^x (x-s)^{(n-2)} \hat{f}(s) ds \\ &< \sum_{k=0}^{n-2} \frac{y_k x^k}{\Gamma(k+1)} + \frac{1}{\Gamma(n-1)} \int_0^x (x-s)^{(n-2)} \hat{g}(s) ds \\ &= g(x) \end{aligned}$$

which finishes the proof.

Example

Suppose that $1 < \alpha(x) \leq 2$, $f(x)$ and $g(x)$ are solutions to the problems:

$$\begin{aligned} &{}^c\mathscr{D}_a^{\alpha(x)} y \leq y, \quad y(0) = 1, \quad y'(0) = 2, \\ &{}^c\mathscr{D}_a^{\alpha(x)} y \leq y + x, \quad y(0) = 1, \quad y'(0) = 2, \end{aligned}$$

respectively, where $x \geq 0$. Then we have $y < y + x$, for $x > 0$, and

$$\begin{aligned} f(x) &= E_{\alpha(x),1}(x^{\alpha(x)}) + 2E_{\alpha(x),2}(x^{\alpha(x)}), \\ g(x) &= E_{\alpha(x),1}(x^{\alpha(x)}) + 2E_{\alpha(x),2}(x^{\alpha(x)}) + x^{(\alpha(x)+1)} E_{\alpha(x),\alpha(x)+2}(x^{\alpha(x)}), \end{aligned}$$

where $E_{\alpha,\beta}(z)$ is the Mittag Leffler function defined by (3). We know that $f(x) < g(x)$ for $x > 0$. This is consistent with theorem 4.2.

This is a simple example, but it demonstrates the basic idea of the comparison principle when applied to a FDIs.

5. Conclusion

This research provides an investigation into FDIs using the comparison principle with Caputo fractional derivative of variable order $\alpha(x)$ $(0 \leq n-1 < \alpha(x) \leq n)$ under strict and nonstrict inequalities. The study's significance lies in analyzing solutions to FDIs, showcasing its versatility as a fundamental tool in nonlinear analysis. As fractional calculus advances, these findings provide valuable contributions to both theory and practical applications, guiding future research in the field. This work can be extended to explore additional applications of FDIs and conduct numerical simulations to validate theoretical findings.

Abbreviations

FDEs	Fractional Differential Equations
FDIs	Fractianal Differential Inequalities

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Conflicts of Interest

The authors declare no conflicts of interest.

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