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Stabilization of Nonhomogeneous Euler-Bernoulli Beam with a Point Controlled by Combined Feedback Force and Indefinite Damping

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Abstract: The transportation of masses or objects by cranes, necessary in the construction sector to increase productivity, often causes transverse vibrations of the system, which degrades production or process performance, and can even cause breakdowns. Actively controlling or mitigating these vibrations is becoming essential in many fields, such as the construction of satellite panels, etc. Therefore, many efforts have been made in recent years to find effective ways to eliminate these unwanted vibrations. Thus, the vibration behavior of a crane on a construction site, transporting a mass represented by beams, is translated into an equation using the theory of Euler-Bernoulli beam equations. These vibration effects are thus model led in a very reduced manner by a nonhomogeneous Euler-Bernoulli beam fixed at one end and subjected to a point mass at the free end. In this research article, we have generalized Wang's results. We began by defining some fundamental properties of the closed-loop system and then analyzed its spectrum. Using the theory of perturbed problems, we obtained the basic Riesz property. The spectrum-determined growth condition and exponential stability were also derived. Moreover, when the damping was undefined, we provided a condition on the negative value of the damping without destroying the exponential stability of the system.

Keywords: Beam Equation, Semigroup Theory, Asymptotic Analysis, Riesz Basis, Exponential Stability

1. Introduction

We consider a non-uniform Euler-Bernoulli beam with a free-end mass under feedback control, the transverse vibrations are thus defined by the following system:

$$\begin{cases} \kappa(x)\, u_{tt}(x,t) + \left(EI(x)\, u_{xx}(x,t)\right)_{xx} + \gamma(x)\, u_t(x,t) = 0, 0 < x < 1, \;\; t > 0 \\ u(0,t) = u_x(0,t) = u_{xx}(1,t) = 0, \; t > 0 \\ \alpha u_t(1,t) + m u_{tt}(1,t) - \beta EI(1) u_{xxxt}(1,t) - EI(1) u_{xxx}(1,t) = 0, \; t > 0 \end{cases} \tag{1}$$

with $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$, $0 < x < 1$, the initial conditions. Consider $\alpha, \beta,$ and m four given non-zero positive constants where m is a tip mass.

Here and below, $u(x,t)$ stands for a transversal deviation of the beam at position x and time t; a subscript letter denotes the partial derivation with respect variable. The length of the beam is chosen to be unity, $EI(x) = E(x)I(x)$ is the flexural rigidity (or bending stiffness) of the beam while, $E(x) > 0$ is the elasticity modulus and $I(x) > 0$ is the moment of inertia of the cross section. Finally $\kappa(x) > 0$ is the mass density and $\gamma(x)$ is a continuous coefficient function of feedback damping.

Moreover, we shall always assume that:

$$\kappa(.),\ EI(.) \in C^4(0,1)\ and\ \gamma(1) \neq 0 \tag{2}$$

$$\int_0^1 \frac{\gamma(x)}{\kappa(x)} \left(\frac{\kappa(x)}{EI(x)}\right)^{\frac{1}{4}} dx > 0. \tag{3}$$

The Riesz basis property, meaning that the generalized eigenvectors of the system (1) form an unconditional basis for the state Hilbert space, is one of the fundamental properties of a linear vibrating system.

For this kind of system, the stability is usually determined by the spectrum of the associated operator and one can also use the theory of nonharmonic Fourier series to obtain important properties such as the optimal decay rate of the energy.

The same system of Euler-Bernoulli beam viscous damping has been studied with $(m = 0)$ in Touré al [6] with position control. Using the tools in Birkhoff [4, 5] and Naimark [24] the authors proved the Riesz basis property and concluded that the spectrum determines the exponential decay rate if $\gamma > 0$. This system was also recently studied by Hasanov in [12], then by Baysal et al. in [2], with boundary conditions different from those of system (1). In [2, 12], the authors obtained stability results based on an approach that relies on the use of regular weak solutions, an energy identity and a Lyapunov function. Considering a system consisting of an Euler-Bernoulli beam on a star-shaped network with indefinite damping term, Bouallagui et al in [7] prove the exponential stability of the system by the Riesz basis theory that we had applied in our work.

The problems of stabilizing Euler-Bernoulli beams with a tip mass controlled by combined feedback forces have been the subject of several works, we could cite [13, 23]. The asymptotic technique appears to be essential for the study. There are two steps usually fund in the study of linear systems with variable coefficients with damping. The first is to transform the ”dominant term” of the system under study into a uniform ”dominant equation” by space scaling and state transformation where no variable coefficient is involved any longer. The second step is to approximate the eigenfunctions of the system by those of uniform ”dominant equation”. This fundamental idea comes essentially from Birkoff works [4] and [5]. This approach has been used in dealing with the beam equations of variable coefficients see Bao [14], Wang [32]. The system (1) was also studied in the uniform case without the damping by Wang see [32]. The authors obtained exponential stability using a method based on the Riesz basis. In this work, we generalize the uniform system studied by Wang in [32] by adding a damping term $\gamma(x)$ and varying the system coefficients. The main objective here is to obtain the exponential stability of the system (1) thanks to the Riesz basis property. To achieve our goal, we assume $\gamma(1) \neq 0$ and transform the beam eigenvalue problem into an ordinary differential equation $Lu = \lambda u$ with polynomial boundary conditions λ (see Shkalikov [26] Tretter [29], Wang [32]). The rest of the paper is structured as follows. The well-posedness of system (1) in the context of the C_0 semigroup theory are derived in Section 2. In Section 3, we then study the spectrum and the Riesz basis property for the considered system is derived. The exponential stability of system (1) is presented in Section 4. Finally, some concluding remarks are given in the final Section 5.

2. The Well-posedness of the Problem (1)

In this section, we will discuss the well-posedness and some basic properties of system (1). For the purpose, firstly we formulate system (1) into an appropriate Hilbert space.

Set

$$V_E^2 = \left\{ f \in H^2(0,1) : f(0) = f'(0) = 0 \right\}, \tag{4}$$

$$\mathbb{H} = V_E^2(0,1) \times L^2(0,1) \times \mathbb{C} \tag{5}$$

$D(0,1)$ the space of smooth functions with compact support,
$D'(0,1)$ the space of continuous linear functions
$f : D(0,1) \to \mathbb{C}$.
The superscript T stands for the transpose and the spaces $L^2(0,1)$ and $H^k(0,1)$ are defined as

$$L^2(0,1) = \left\{ f : [0,1] \to \mathbb{C} : \int_0^1 |f|^2 < \infty \right\}$$

$$H^k(0,1) = \left\{ f : [0,1] \to \mathbb{C} : f,\ f^2, ..., f^{(k)} \in L^2(0,1) \right\}$$

In the space $\mathbb{H}$, we define the inner-product

$$c\left\langle u,v\right\rangle_{\mathbb{H}}=\int_0^1\left(\kappa(x)g_1\left(x\right)\overline{g_2\left(x\right)}+EI\left(x\right)f_1''\left(x\right)\overline{f_2''\left(x\right)}\right)dx+\frac{\beta^2}{m+\alpha\beta}\eta_1\overline{\eta_2},$$

where $u=\left(f_1,g_1,\eta_1\right)^T\in\mathbb{H}$ and $u=\left(f_2,g_2,\eta_2\right)^T\in\mathbb{H}$, α, β and m are positive constants.

We define an unbounded linear operator $\mathbb{A}_\gamma$:$D\left(\mathbb{A}_\gamma\right)\subset\mathbb{H}\to\mathbb{H}$ as follows:

$$\mathbb{A}_\gamma\begin{pmatrix}f\\g\\\eta\end{pmatrix}=\begin{pmatrix}g\left(x\right)\\-\dfrac{1}{\kappa\left(x\right)}\left(\left(EI\left(x\right)f''\left(x\right)\right)''+\gamma\left(x\right)g\left(x\right)\right)\\-\beta^{-1}\eta-\beta^{-1}\left(\alpha-m\beta^{-1}\right)g(1)\end{pmatrix},\tag{6}$$

for $\begin{pmatrix}f\\g\\\eta\end{pmatrix}\in D\left(\mathbb{A}_\gamma\right)$ and the domain of operator $\mathbb{A}_\gamma$ is given by:

$$D\left(\mathbb{A}_\gamma\right)=\left\{\begin{array}{c}\left(f,g,\eta\right)^T\in H^4\left(0,1\right)\cap V_E^2\left(0,1\right)\times V_E^2\left(0,1\right)\times\mathbb{C}:\\f''(1)=0,\ \eta=-E(1)f^{(3)}(1)+\beta^{-1}mg(1)\end{array}\right\}.\tag{7}$$

With these notations, the set of system (1) can be written as an abstract evolution equation in the Hilbert space $\mathbb{H}$.

$$\left\{\begin{array}{l}\dfrac{dY\left(t\right)}{dt}=\mathbb{A}_\gamma Y\left(t\right)\\Y\left(0\right)=Y_0\in\mathbb{H},\end{array}\right.\tag{8}$$

where

$$Y\left(t\right)=\begin{pmatrix}u\left(.,t\right)\\u_t\left(.,t\right)-EI(1)u_{xxx}(1,t)+\beta^{-1}mu_t(1,t)\end{pmatrix}$$

and

$$Y\left(0\right)=\begin{pmatrix}u_0\\u_1-EI(1)u_{xxx}(1,0)+\beta^{-1}mu_t(1,0)\end{pmatrix}$$

We consider the operator $\mathbb{A}_0$ designating the case without the viscous damping term, i.e.

$$\gamma(x)=0,\ \forall x\in[0,1].$$

Let's put

$$\mathbb{A}_\gamma=\mathbb{A}_0+\Gamma_\gamma\text{ with }\Gamma_\gamma\begin{pmatrix}f\\g\\\eta\end{pmatrix}=\begin{pmatrix}0\\-\dfrac{1}{\kappa\left(x\right)}\gamma\left(x\right)g\left(x\right)\\0\end{pmatrix},\ \forall\begin{pmatrix}f\\g\\\eta\end{pmatrix}\in\mathbb{H}.$$

Γ_γ is a boundary linear operator on $\mathbb{H}$.

Thus, we obtain the following results:

Lemma 2.1. Let $\mathbb{A}_0$ be defined as (6) and (7) with $\gamma(x)=0\ \forall x\in[0,1]$. Then $\mathbb{A}_0$ is a closed and densely defined linear operator in $\mathbb{H}$. For any $\alpha,\beta>0$ and $m>0$, $0\in\rho\left(\mathbb{A}_0\right)$ and $\mathbb{A}_0^{-1}$ is compact on $\mathbb{H}$. Hence $\left(\mathbb{A}_0\right)$ consists of all isolated eigenvalues of finite multiplicity.

Proof. The proof is the same as that of theorem 2.2. □

Theorem 2.1. The operator $\mathbb{A}_0$, defined by (6) and (7) for $\gamma(x)=0\ \forall x\in[0,1]$, is a closed, densely defined m-dissipative operator. Hence the operator $\mathbb{A}_0$ generates a C_0 semigroup of contractions on $\mathbb{H}$.

Proof. First, we prove that the operator $\mathbb{A}_0$ is dissipative. For any $u=\left(f,g,\eta\right)^T\in D\left(\mathbb{A}_0\right)=D\left(\mathbb{A}_\gamma\right)$, we get:

$$\begin{aligned}\left\langle\mathbb{A}_0u,u\right\rangle_{\mathbb{H}}&=\left\langle\left(g\left(x\right)-\frac{1}{\kappa\left(x\right)}\left(\left(EI\left(x\right)f''\left(x\right)\right)''\right)-\beta^{-1}\eta-\beta^{-1}\left(\alpha-m\beta^{-1}\right)g(1)\right),\begin{pmatrix}f\\g\\\eta\end{pmatrix}\right\rangle\\&=\int_0^1-\left(EI(x)f''\left(x\right)\right)''\overline{g\left(x\right)}+EI\left(x\right)\overline{f''\left(x\right)}g''\left(x\right)dx+\frac{\beta^2}{m+\alpha\beta}\left(-\beta^{-1}\eta-\beta^{-1}\left(\alpha-m\beta^{-1}\right)g(1)\right)\overline{\eta}.\end{aligned}$$

By integration by parts and the boundary conditions and (7), we get:

$$\begin{aligned}\langle \mathbb{A}_0 u, u\rangle_{\mathbb{H}} &= \int_0^1 EI(x)[\overline{f''(x)}g''(x) - f''(x)\overline{g''(x)}]dx - E(1)f^{(3)}(1)\overline{g(1)} + \frac{-\beta|\eta|^2 - \alpha\beta g(1)\overline{\eta} + mg(1)\overline{\eta}}{m+\alpha\beta} \\ &= \int_0^1 EI(x)[\overline{f''(x)}g''(x) - f''(x)\overline{g''(x)}]dx + \frac{m\overline{g(1)}\eta + mg(1)\overline{\eta}}{m+\alpha\beta} \\ &- \frac{\left(\beta^{-1}m^2+\alpha m\right)|g(1)|^2}{m+\alpha\beta} + \frac{\alpha\beta\overline{g(1)}\eta - \alpha\beta g(1)\overline{\eta} - \beta|\eta|^2}{m+\alpha\beta}.\end{aligned} \tag{9}$$

The reorganization of the computations gives us:

$$\begin{aligned}\langle \mathbb{A}_0 u, u\rangle_{\mathbb{H}} &= \int_0^1 EI(x)[\overline{f''(x)}g''(x) - f''(x)\overline{g''(x)}]dx + \frac{2m\left(\overline{g(1)}\eta + g(1)\overline{\eta}\right)}{2(m+\alpha\beta)} \\ &- \frac{\left(\beta^{-1}m^2+\alpha m\right)|g(1)|^2}{m+\alpha\beta} + \frac{\alpha\beta\overline{g(1)}\eta - \alpha\beta g(1)\overline{\eta} - \beta|\eta|^2}{m+\alpha\beta}.\end{aligned} \tag{10}$$

As a result:

$$Re\,\langle \mathbb{A}_0 u, u\rangle_{\mathbb{H}} = \frac{\left(2mRe(g(1)\eta) - \left(\beta^{-1}m^2+\alpha m\right)|g(1)|^2 - \beta|\eta|^2\right)}{m+\alpha\beta}, \tag{11}$$

with Re the real part of a complex number.

Since

$$4\beta\left(\beta^{-1}m^2+\alpha m\right) > 4m^2, \text{ then } Re\,\langle \mathbb{A}_0 u, u\rangle_{\mathbb{H}} < 0.$$

It follows from previous result that the operator $\mathbb{A}_0$ is dissipative.

Next, we show that the operator $\mathbb{A}_0$ is m-dissipative.

We prove that the operator $(I - \mathbb{A}_0) : D(\mathbb{A}_0) \subset \mathbb{H} \to \mathbb{H}$ is onto.

Let us show that for all $z = \begin{pmatrix} f \\ g \\ \eta \end{pmatrix} \in \mathbb{H}$, there exists $y = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in D(\mathbb{A}_0)$, such that $z = (I - \mathbb{A}_0)\,y$. So

$$v = u - f \tag{12}$$

$$\kappa v + (EIu_{xx})_{xx} = \kappa g \tag{13}$$

$$\eta = \left(1+\beta^{-1}\right)w + \beta^{-1}\left(\alpha - m\beta^{-1}\right)v(1) \tag{14}$$

$$u_{xx}(1) = u_x(0) = u(0) = 0 \tag{15}$$

$$w = -E(1)u_{xxx}(1) + \beta^{-1}mv(1) \tag{16}$$

where $u \in H^2(0,1)$, $f = u - v \in L^2(0,1)$.

Multiplying both sides of equation (13) by $\varphi \in V_E^2(0,1)$ and integrating from 0 to 1 with respect to x, we have

$$\int_0^1 \kappa u\varphi dx + \int_0^1 (EIu_{xx})_{xx}\,\varphi dx = \int_0^1 \kappa(g+f)\,\varphi dx. \tag{17}$$

By integrating formally by parts twice the expression $\int_0^1 (EIu_{xx})_{xx}\,\varphi dx$ using boundary conditions, we get

$$\int_0^1 EIu_{xx}\varphi_{xx}dx + \int_0^1 \kappa u\varphi dx + EI(1)u_{xxx}(1)\varphi(1) = \int_0^1 \kappa(g+f)\,\varphi dx.$$

This is the weak formulation of $(13) - (16)$. The left-hand side of (17) is a continuous coercive bilinear form a of φ and u. Moreover, the right-hand side of (17) is a continuous linear form L of φ on $V_E^2(0,1)$.

Using the Well-known Lax-Milgram theorem, there exists a unique $u \in V_E^2(0,1)$ such that

$$a(u,\varphi) = L(\varphi) \;\; for\ all\ \varphi \in V_E^2(0,1) \tag{18}$$

Regularity of the solution:
Let $\varphi \in D(0,1)$. Then equation (18) becomes

$$\int_0^1 \kappa u\varphi dx + \int_0^1 (EIu_{xx})_{xx}\,\varphi dx = \int_0^1 l\varphi dx, \tag{19}$$

with $l = \rho(g+f)$.

This can be written as

$$\int_0^1 (\kappa u - l + (EIu_{xx})_{xx} = 0,\ with\ \ l = \kappa(g+f)$$

and lead to

$$\kappa u + (EIu_{xx})_{xx} = l \in\ D'(0,1), \tag{20}$$

the same equality hold in $L^2(0,1)$

because $(EIu_{xx})_{xx} = l - \kappa u \in L^2(0,1)$.

By using particular φ in $V_E^2(0,1)$, we recover the boundary conditions in u.

Hence, we have found a unique solution

$u \in H^4(0,1) \cap V_E^2(0,1)$ of $(13)-(16)$. This shows that the operator $I - \mathbb{A}_0$ is onto. We can thus conclude that the operator A is m-dissipative according to the Lumer-Phillips theorem see [25]. □

As a result we obtain the following important theorem for the continuation of the work:

Theorem 2.2. Let the operator

$\mathbb{A}_\gamma = \mathbb{A}_0 + \Gamma_\gamma : D(\mathbb{A}_\gamma) \subset \mathbb{H} \to \mathbb{H}$ defined by (6)-(7), then $\mathbb{A}_\gamma^{-1}$ exists and is compact on $\mathbb{H}$ and $\mathbb{A}_\gamma$ is generates a C_0-semigroup $\left\{e^{\mathbb{A}_\gamma t}\right\}_{t\geq 0}$ on $\mathbb{H}$. Hence, system (8) is well posed.

Proof. A direct calculation shows that: for all $\begin{pmatrix} f \\ g \\ \eta \end{pmatrix} \in \mathbb{H}$, there is a unique $\begin{pmatrix} f_1 \\ g_1 \\ \eta_1 \end{pmatrix} \in D(\mathbb{A}_\gamma)$ such that

$$\begin{pmatrix} f \\ g \\ \eta \end{pmatrix} = \mathbb{A}_\gamma \begin{pmatrix} f_1 \\ g_1 \\ \eta_1 \end{pmatrix},$$

where

$$\begin{cases} g_1 &=& f \\ \eta_1 &=& -\beta\eta - \left(\alpha - m\beta^{-1}\right) f(1) \\ f_1 &=& \int_0^x \int_0^y \frac{1}{EI(s)} \int_s^1 \int_r^1 \gamma(t)f(t) + \kappa(t)g(t)dtdrdsdy + (\beta\eta + \alpha f(1)) \int_0^x \int_0^y \frac{1-s}{EI(s)} dsdy. \end{cases}$$

So $\mathbb{A}_\gamma$, is invertible so $\mathbb{A}_\gamma^{-1}$ exists and, $0 \in \rho(\mathbb{A}_\gamma)$ where $\rho(\mathbb{A}_\gamma)$ is resolvent of operator $\mathbb{A}_\gamma$.

Since

$$|\eta_1| \leq \beta\,|\eta| + \left|\alpha - m\beta^{-1}\right| \|f\|_{H^2(0,1)},$$

it follows that

$$\left\| \mathbb{A}_\gamma^{-1} \begin{pmatrix} f \\ g \\ \eta \end{pmatrix} \right\|_{H^4(0,1)\times H^2(0,1)\times\mathbb{C}} \leq N \left\| \begin{pmatrix} f \\ g \\ \eta \end{pmatrix} \right\|_{\mathbb{H}}$$

for some constant $N > 0$. By the Sobolev embedding theorem [1], $\mathbb{A}_\gamma^{-1}$ is compact on $\mathbb{H}$. Therefore, according to the spectral theory of compact operator, the spectrum $\sigma(\mathbb{A}_\gamma)$ consists entirely of isolated eigenvalues finite multiplicity.

Finally, since the operator $\mathbb{A}_0$ is infinitesimal generator of a C_0-semigroup $S(t) = e^{\mathbb{A}_0 t}\ \forall t \geq 0$ on $\mathbb{H}$, satisfying

$$\|S(t)\| \leq Me^{\omega t}\ \ \forall M \in \mathbb{R}_*^+.$$

As we have $\mathbb{A}_\gamma = \mathbb{A}_0 + \Gamma_\gamma$ with Γ_γ is a boundary linear operator on $\mathbb{H}$.

Then using the perturbation by bounded linear operator, we deduce that the operator $\mathbb{A}_\gamma = \mathbb{A}_0 + \Gamma_\gamma$ is infinitesimal generator of a C_0-semigroup $T(t) = e^{\mathbb{A}_\gamma t}\ \forall t \geq 0$ on $\mathbb{H}$, satisfying

$$\|T(t)\| \leq Me^{(\omega + M\|\Gamma_\gamma\|)t}$$

(see A. Pazy [25], Theorem 1.1]). Hence system (1) is well-posed. □

3. Spectral Analysis and the Riesz Basis Property of Eigenfunctions of Operator $\mathbb{A}_\gamma$

3.1. Spectral Analysis of $\mathbb{A}_\gamma$

In this section, we use of the following result which deals with the eigenvalue problem of beams in the form of an ordinary differential equation $L(u) = \lambda u$ with λ polynomial boundary conditions see Shkalikov [26]; Tretter [29]. We used this same technique in [27] and in this work we apply this method to prove the basic properties of the system (1).

We begin by recalling some notations and definitions. Let $L(u)$ be an ordinary differential operator of order $n = 2q \in \mathbb{N}$,

$$L(u) = u^{(n)}(x) + \sum_{\nu=1}^{n} f_\nu(x)\, u^{(n-\nu)}(x), \ \ 0 < x < 1, \tag{21}$$

and let the boundary conditions defined at the two points $x = 0$, and $x = 1$ be

$$B_j(u) = \sum_{\nu=0}^{k_j} \left(\alpha_{j_\nu} u^{(k_j-\nu)}(0) + \beta_{j_\nu} u^{(k_j-\nu)}(1) \right), \ \ 1 \le j \le n, \tag{22}$$

where $k_j \in \mathbb{N}$, $1 \le k_j \le n-1$ and $\alpha_{j_\nu}, \beta_{j_\nu} \in \mathbb{C}$, $\alpha_{j_0} + \beta_{j_0} > 0$.
We suppose that the coefficient functions $f_\nu(x)$ $(1 \le \nu \le n)$ in (1) are sufficiently smooth in $[0,1]$, and that the boundary conditions are normalized in the sense that $\sum\limits_{\nu=0}^{n} k_j$ is a minimal with respect to all equivalent boundary conditions (see Naimark, 1967).

Let $u_k(x,\rho)$ $(k = 1,2,...,n)$ be the fundamental solution for the equation:

$$L(u) + \rho^n u + \rho^q \mu(x)\, u(x) = 0, \quad \rho \in \mathbb{C} \tag{23}$$

where $\mu(x)$ being continuous in $[0,1]$, and let ω_k $(k = 1,2,...,n)$ be the noth roots of $\omega^n + 1 = 0$. If we denote by $\Delta(\rho)$ the characteristic determinant of (23) with respect to (22)

$$\Delta(\rho) = \det\left[B_j(u_k(.,\rho))\right]_{j,k=1,2,...,n}$$

then $\Delta(\rho)$ can be expressed asymptotically in the form $(for\ r \ge 1)$

$$\Delta(\rho) = \rho^k \sum_{\mathbb{k}_r} e^{\rho\mu_{\mathbb{k}_r}} \left[F^{\mathbb{k}_r}\right]_r, \tag{24}$$

whenever ρ is large enough see Shkalikov [26]; Naimark [24]. Here, $\mathbb{k}_k$ is a k element subset of $\{1,2,...,n\}$, $\mu_{\mathbb{k}_k} = \sum\limits_{j \in \mathbb{k}_k} \omega_j$,

$$\left[F^{\mathbb{k}_k}\right]_r = F_0^{\mathbb{k}_k} + \rho^{-1} F_1^{\mathbb{k}_k} + ... + \rho^{-r+1} F_{r-1}^{\mathbb{k}_k} + \mathcal{O}\left(\rho^{-r}\right), \tag{25}$$

and the sum runs over all possible selection of $\mathbb{k}_k$.

Since, according to the previous section, the spectrum $\sigma(\mathbb{A}_\gamma)$ consists entirely of isolated eigenvalues. Thus, the conditions are met for studying the problem of the eigenvalues of operator $\mathbb{A}_\gamma$.

Let $\lambda \in \sigma(\mathbb{A}_\gamma)$ and Φ be an eigenfunction of $\mathbb{A}_\gamma$ corresponding to λ. Then each eigenfunction corresponding to $\lambda \in \sigma(\mathbb{A}_\gamma)$ with $\lambda \neq -\beta^{-1}$ is of the form

$$\Phi = \begin{pmatrix} \lambda^{-1}\phi \\ \phi \\ -\frac{\beta^{-1}(\alpha - m\beta^{-1})}{\lambda + \beta^{-1}} \phi(1) \end{pmatrix}$$

satisfies the following equation for $\phi \neq 0$, see [23]:

$$\begin{cases} \lambda^2 \kappa(x)\phi(x) + \left(EI(x)\,\phi''(x)\right)'' + \lambda\gamma(x)\,\phi(x) = 0, \quad 0 < x < 1, \\ \phi(0) = \phi'(0) = \phi''(1) = 0 \\ \phi'''(1) = \dfrac{\alpha\lambda + \lambda^2 m}{(1+\beta\lambda)EI(1)} \phi(1). \end{cases} \tag{26}$$

Expanding (26) yields

$$\begin{cases} \phi^{(4)}(x) + \dfrac{2EI'(x)}{EI(x)}\phi'''(x) + \dfrac{EI''(x)}{EI(x)}\phi''(x) + \lambda\dfrac{\gamma(x)}{EI(x)}\phi(x) + \dfrac{\lambda^2\kappa(x)}{EI(x)}\phi(x) = 0, \ 0 < x < 1, \\ \phi(0) = \phi'(0) = \phi''(1) = 0 \\ \phi'''(1) = \dfrac{\alpha\lambda + \lambda^2 m}{(1+\beta\lambda)EI(1)} \phi(1). \end{cases} \tag{27}$$

In order to simplify our computations, we introduce a spatial scale transformation in:

$$f(z) = \phi(x), \quad z = \frac{1}{h}\int_0^x \left(\frac{\kappa(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta$$

where

$$h = \int_0^1 \left(\frac{\kappa(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta$$

Then, (27) together with its boundary conditions can be transformed into

$$\begin{cases} f^{(4)}(z) + a(z) f'''(z) + b(z) f''(z) + c(z) f'(z) + \lambda h^4 \dfrac{\gamma(x)}{\kappa(x)} f(x) + \lambda^2 h^4 f(z) = 0, \ 0 < z < 1, \\ f(0) = f'(0) = 0 \\ z_x^2(1) f''(1) + z_{xx}(1) f'(1) = 0 \\ f'''(1) + \dfrac{3z_{xx}(1)}{z_x^2(1)} f''(1) + \dfrac{z_{xxx}(1)}{z_x^3(1)} f'(1) - \dfrac{\alpha\lambda + \lambda^2 m}{(1+\beta\lambda) z_x^3(1) EI(1)} f(1) = 0, \end{cases} \tag{28}$$

with

$$a(z) = \frac{6z_{xx}}{z_x^2} + \frac{EI'(x)}{z_x EI(x)} \tag{29}$$

$$b(z) = \frac{3z_{xx}^2}{z_x^4} + \frac{6z_{xx}EI'(x)}{z_x^3 EI(x)} + \frac{EI'(x)}{z_x^2 EI(x)} + \frac{4z_{xx}}{z_x^3} \tag{30}$$

$$c(x) = \frac{z_{xxxx}}{z_x^4} + \frac{2z_{xxx}EI'(x)}{z_x^4 EI(x)} + \frac{z_{xx}EI''(x)}{z_x^4 EI(x)} \tag{31}$$

$$z_x = \frac{1}{h}\left(\frac{\kappa(x)}{EI(x)}\right)^{\frac{1}{4}}, \quad z_x^4 = \frac{1}{h^4}\frac{\kappa(x)}{EI(x)} \tag{32}$$

and

$$z_{xx} = \frac{1}{4h}\left(\frac{\kappa(x)}{EI(x)}\right)^{\frac{-3}{4}} \frac{d}{dx}\left(\frac{\kappa(x)}{EI(x)}\right)^{\frac{1}{4}}. \tag{33}$$

If we define

$$d(x) = \frac{\gamma(x)}{\kappa(x)}$$

The equation in (28) is for any $0 < z < 1$

$$f^{(4)}(z) + a(z) f'''(z) + b(z) f''(z) + c(z) f'(z) + \lambda h^4 d(x) f(x) + \lambda^2 h^4 f(z) = 0.$$

This can be further simplified by applying another invertible transformation:

$$g(z) = \exp\left(\frac{1}{4}\int_0^z a(\zeta)\, d\zeta\right) f(z), \quad 0 < z < 1,$$

and we arrive at the following eigenvalue problem that is equivalent to the origins one:

$$\begin{cases} g^{(4)}(z) + b_1(z) g''(z) + c_1(z) g'(z) + d_1 g(z) + \lambda h^4 d(x) g(x) + \lambda^2 h^4 g(z) = 0, \quad 0 < z < 1, \\ g(0) = g'(0) = 0 \\ g''(1) + b_{11} g'(1) + b_{12} g(1) = 0 \\ g'''(1) + b_{21} g''(1) + b_{22} g'(1) + b_{23} g(1) = 0, \end{cases} \tag{34}$$

where

$$b_1(1) = -\frac{3}{2}a'(z) - \frac{3}{8}a^2(z) + b(z)$$

$$c_1(z) = \frac{1}{8}a^3(z) - \frac{1}{2}a(z)b(z) - a'(z) + c(z)$$

$$d_1(z) = \frac{3}{16}a'^2(z) - \frac{1}{4}a'''(z) + \frac{3}{32}a'(z)a^2(2) - \frac{3}{256}a^4(z) + b(z)\left(\frac{1}{16}a^2(z) - \frac{1}{4}a'(z)\right) - \frac{a(z)c(z)}{4}$$

$$b_{11} = -\frac{1}{2}a(1) + \frac{z_{xx}(1)}{z_x^2(1)}$$

$$b_{12} = \frac{\frac{1}{16}z_x^2(1)a^2(1) - \frac{1}{4}z_x^2(1)a'(1) - \frac{1}{4}z_{xx}(1)a(1)}{z_x^2(1)}$$

$$b_{21} = -\frac{3}{4}a(1) + \frac{3z_{xx}(1)}{z_x^2(1)}$$

$$b_{22} = -\frac{3}{4}a'(1) + \frac{3}{16}a^2(1) - \frac{3z_{xx}(1)a(1)}{2z_x^2(1)} + \frac{z_{xxx}(1)}{z_x^3(1)}$$

$$b_{23} = -\frac{1}{4}a''(1) + \frac{3}{16}a'(1)a(1) - \frac{1}{64}a^3(1) - \frac{3z_{xx}(1)a'(1)}{4z_x^2(1)} + \frac{3z_{xx}(1)a^2(1)}{16z_x^2(1)} - \frac{z_{xxx}(1)a(1)}{4z_x^3(1)} - \frac{\alpha\lambda + \lambda^2 m}{(1+\beta\lambda)z_x^3\,(1)\,EI\,(1)}.$$

We will now solve the eigenvalue problem (34) by dividing the complex plane into eight different sectors as Birkhoff $(1908a, b)$ [3], [4] and Naimark (1967) [24]. This technique was used by Wang et al in [30–32], Touré al in [27] and My Driss et al in [23], then recently by other authors see [16].

Let us consider the following eight sectors:

$$S_k = \left\{z \in \mathbb{C} : \frac{k\pi}{4} \leq \arg z \leq \frac{(k+1)\,\pi}{4}\right\}, \quad k = 0, 1, 2, ..., 7$$

and let $\omega_1, \omega_2, \omega_3, \omega_4$ be the roots of equation $\vartheta^4 + 1 = 0$ that are arranged so that

$$\mathrm{Re}\,(\rho\omega_1) \leq \mathrm{Re}\,(\rho\omega_2) \leq \mathrm{Re}\,(\rho\omega_3) \leq \mathrm{Re}\,(\rho\omega_4)\,, \quad \forall\rho \in S_k. \tag{35}$$

Obviously, in sector S_1, we can choose

$\omega_1 = \exp(i\frac{3}{4}\pi) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \omega_2 = \exp(i\frac{1}{4}\pi) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,$

$\omega_3 = \exp(i\frac{5}{4}\pi) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \omega_4 = \exp(i\frac{7}{4}\pi) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,$

such that the inequalities in (35) are verified and we make these same choices in other sectors. In the rest of this section, we shall derive the asymptotic behavior of the eigenvalue of the sectors S_1 and S_2 because the same will hold for the other sectors with similar proofs.

Setting $\lambda = \dfrac{\rho^2}{h^2}$, in each sector S_k.

The following lemmas will be used for the fundamental asymptotic solutions of the system (34) , see [32].

Lemma 3.1. For $\rho \in S_k$ with ρ large enough, the equation: $g^{(4)}(z) + b_1(z)\,g''(z) + c_1(z)\,g'(z) + d_1 g(z) + \rho^2 h^2 d(z)\,g(z) + \rho^4 g(z) = 0, \ \ 0 < z < 1$, has four linearly independent asymptotic fundamental solutions,

$$\Phi_s(z,\rho) = \exp\rho\omega_s z\left(1 + \frac{\Phi_{s,1}(z)}{\rho} + \mathcal{O}\left(\rho^{-2}\right)\right), \quad s = 1, 2, 3, 4 \tag{36}$$

and hence their derivatives for $s = 1, 2, 3, 4$ and $j = 1, 2, 3$ are given by

$$\frac{d^j}{dz^j}\Phi_s(z,\rho) = (\rho\omega_s)^j \exp\rho\omega_s z\left(1 + \frac{\Phi_{s,1}(z)}{\rho} + \mathcal{O}\left(\rho^{-2}\right)\right) \tag{37}$$

where

$$\Phi_{s,1}(z) = -\frac{1}{4\omega_s^3}\int_0^z \omega_s^2 b(\zeta)\,d\zeta, \quad \Phi_{s,1}(0) = 0,$$

for s=1,2,3,4 and

$$\Phi_{s,1}(z) = -\frac{1}{4\omega_s}\int_0^1 b_1(\zeta)\,d\zeta - \frac{h^2}{4\omega_s^3}\int_0^1 d(\zeta)\,d\zeta$$

$$= \frac{1}{\omega_s}\mu_1 + \frac{1}{\omega_s^3}\mu_2,$$

$with\ \ \mu_1 = -\dfrac{1}{4}\int_0^1 b_1(\zeta)\, d\zeta\ \ et\ \ \mu_2 = -\dfrac{h^2}{4}\int_0^1 d(\zeta)\, d\zeta.$

Lemma 3.2. For $\rho \in S_1$, if we set $\delta = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, then we have following inequalities:

$$\operatorname{Re}(\rho\omega_1) \le -|\rho|\,\delta,\ \ \operatorname{Re}(\rho\omega_4) \ge |\rho|\,\delta,\ \omega_4 = -\omega_1 \text{ and}$$

$$e^{\rho\omega_1} = \mathcal{O}\left(\rho^{-2}\right) \text{ when } |\rho| \to +\infty.$$

By applying the two lemmas above, we obtain the following Theorem:

Theorem 3.1. Let $\Delta(\rho)$ be the characteristic determinant of the eigenvalue problem (34). In sector S_1, an asymptotic expression of $\Delta(\rho)$ is given by:

$$\begin{array}{rcl} \Delta(\rho) & = & \rho^6 e^{\rho\omega_4}\{2e^{\rho\omega_2} + 2e^{-\rho\omega_2} + 2\mu_3\rho^{-1}e^{\rho\omega_2} \\ & + & 2i\mu_4\rho^{-1}e^{-\rho\omega_2} + \mathcal{O}\left(\rho^{-2}\right)\}, \end{array} \tag{38}$$

where

$$\left\{ \begin{array}{l} \mu_3 = \sqrt{2}(\mu_1 + b_{11}) - \sqrt{2}\mu_2 + \sqrt{2}\dfrac{m}{\beta z_x^3(1)\,EI(1)\,h^2} \\ \mu_4 = \sqrt{2}(\mu_1 + b_{11}) + \sqrt{2}\mu_2 - \sqrt{2}\dfrac{m}{\beta z_x^3(1)\,EI(1)\,h^2}. \end{array} \right. \tag{39}$$

Problem (34) is strongly regular. Therefore the system of the generalized eigenfunctions forms a Riesz basis in the state Hilbert space $\mathbb{H}$.

Proof. According to lemmas 3.1 and 3.2 and substituting (36) and (37) into the boundary conditions (34), we obtain asymptotic expressions for the boundary conditions for large enough $|\rho|$:

$$U_4(\Phi_s,\rho) = \Phi_s(0,\rho) = 1 + \mathcal{O}\left(\rho^{-2}\right) = [1]_2\,,\quad s = 1,2,3,4, \tag{40}$$

$$\begin{array}{rcl} U_3(\Phi_s,\rho) & = & \Phi_s'(0,\rho) \\ & = & \rho\omega_S\left(1 + \mathcal{O}\left(\rho^{-2}\right)\right) \\ U_3(\Phi_s,\rho) & = & \rho\omega_S[1]_2\,,\quad s = 1,2,3,4, \end{array}$$

$$\begin{array}{rcl} U_2(\Phi_s,\rho) & = & \Phi_s''(1,\rho) + b_{11}\Phi_s'(1,\rho) + b_{12}\Phi_s(1,\rho)\,U_2(\Phi_s,\rho) \\ & = & (\rho\omega_S)^2 e^{\rho\omega_S}[1 + \left(\omega_S^2\mu_1 + \mu_2\right)\rho^{-1}\omega_S^{-3} + b_{11}\rho^{-1}\omega_S^{-1} + \mathcal{O}\left(\rho^{-2}\right)] \\ U_2(\Phi_s,\rho) & = & (\rho\omega_S)^2 e^{\rho\omega_S}[1 + (\left(\omega_S^2\mu_1 + \mu_2\right)\omega_S^{-3} + b_{11}\omega_S^{-1})\rho^{-1}]_2,\ \ s = 1,2,3,4. \end{array} \tag{41}$$

$$\begin{array}{rcl} U_1(\Phi_s,\rho) & = & \Phi_s'''(1,\rho) + b_{21}\Phi_s''(1,\rho) + b_{22}\Phi_s(1,\rho) + b_{23}\Phi_s(1,\rho) \\ & = & (\rho\omega_S)^3 e^{\rho\omega_S}(1 + \left(\omega_S^2\mu_1 + \mu_2\right)\rho^{-1}\omega_S^{-3} + b_{21}\rho^{-1}\omega_S^{-1}) + (\rho\omega_S)^3 e^{\rho\omega_S}\left(b_{23}(\rho\omega_S)^{-3} + \mathcal{O}\left(\rho^{-2}\right)\right) \\ & = & (\rho\omega_S)^3 e^{\rho\omega_S}[1 + (\omega_S^2(\mu_1 + b_{21}) + \mu_2)\rho^{-1}\omega_S^{-3}] - (\rho\omega_S)^3 e^{\rho\omega_S}[\dfrac{\omega_S^{-3}\left(\rho m + \alpha\rho^{-1}h^2\right)}{(h^2 + \beta\rho^2)\,z_x^3(1)\,EI(1)\,h^2} \\ & + & \mathcal{O}\left(\rho^{-2}\right)] \\ U_1(\Phi_s,\rho) & = & (\rho\omega_S)^3 e^{\rho\omega_S}[1 + (\omega_S^2(\mu_1 + b_{21}) + \mu_2)\rho^{-1}\omega_S^{-3} - \chi]_2,\ s = 1,2,3,4, \end{array} \tag{42}$$

where $\chi = \dfrac{m\omega_S^{-3}\rho^{-1}}{\beta z_x^3(1)\,EI(1)\,h^2}$.

Note that $\lambda = \dfrac{\rho^2}{h^2} \neq 0$ is the eigenvalue (34) if and only if ρ satisfies the characteristic determinant $\Delta(\rho) = 0$, with

$$\Delta(\rho) = \left| \begin{array}{cccc} U_4(\Phi_1,\rho) & U_4(\Phi_2,\rho) & U_4(\Phi_3,\rho) & U_4(\Phi_4,\rho) \\ U_3(\Phi_1,\rho) & U_3(\Phi_2,\rho) & U_3(\Phi_3,\rho) & U_3(\Phi_4,\rho) \\ U_2(\Phi_1,\rho) & U_2(\Phi_2,\rho) & U_2(\Phi_3,\rho) & U_2(\Phi_4,\rho) \\ U_1(\Phi_1,\rho) & U_1(\Phi_2,\rho) & U_1(\Phi_3,\rho) & U_1(\Phi_4,\rho) \end{array} \right|, \tag{43}$$

so substituting $(40)-(42)$ into (43) , using lemma 3.2, we get:

$$\Delta(\rho)=\left|\begin{matrix} [1]_2 & [1]_2 \\ \rho\omega_1[1]_2 & \rho\omega_2[1]_2 \\ 0 & (\rho\omega_2)^2 e^{\rho\omega_2}\left[1+\left(\omega_2^2(\mu_1+b_{11})+\mu_2\right)\rho^{-1}\omega_2^{-3}\right]_2 \\ 0 & (\rho\omega_2)^3 e^{\rho\omega_2}\left[1+\left(\omega_2^2(\mu_1+b_{21})+\mu_2\right)\rho^{-1}\omega_2^{-3}-\chi\right]_2 \end{matrix}\right.$$

$$\left.\begin{matrix} [1]_2 & 0 \\ \rho\omega_3[1]_2 & 0 \\ (\rho\omega_3)^2 e^{\rho\omega_3}\left[1+\left(\omega_3^2(\mu_1+b_{11})+\mu_2\right)\rho^{-1}\omega_3^{-3}\right]_2 & (\rho\omega_4)^2 e^{\rho\omega_4}\left[1+\left(\omega_4^2(\mu_1+b_{11})+\mu_2\right)\rho^{-1}\omega_4^{-3}\right]_2 \\ (\rho\omega_3)^3 e^{\rho\omega_3}\left[1+\left(\omega_3^2(\mu_1+b_{21})+\mu_2\right)\rho^{-1}\omega_2^{-3}-\chi\right]_2 & (\rho\omega_4)^3 e^{\rho\omega_3}\left[1+\left(\omega_4^2(\mu_1+b_{21})+\mu_2\right)\rho^{-1}\omega_4^{-3}-\chi\right]_2 \end{matrix}\right|.$$

Expanding the above determinant, we obtain the following expression

$$\begin{aligned}\Delta(\rho)=\rho^6 e^{\rho\omega_4}\{&(\omega_2-\omega_1)\omega_3^2\omega_4^2 e^{\rho\omega_3}[\omega_4-\omega_3+\left(\omega_3^{-2}-\omega_4^{-2}\right)\omega_4\omega_3(\mu_1+b_{11})\rho^{-1}\\ &+\left(\omega_4^{-2}-\omega_3^{-2}+\left(\omega_3^{-4}-\omega_4^{-4}\right)\omega_4\omega_3\right)\mu_2\rho^{-1}+\frac{m\rho^{-1}}{\beta z_x^3(1)EI(1)h^2}\left(\omega_3^{-2}-\omega_4^{-2}\right)]\\ &+(\omega_1-\omega_3)\omega_2^2\omega_4^2 e^{\rho\omega_2}[\omega_4-\omega_2+\left(\omega_2^{-2}-\omega_4^{-2}\right)\omega_4\omega_2(\mu_1+b_{11})\rho^{-1}\\ &+\left(\omega_4^{-2}-\omega_2^{-2}+\left(\omega_2^{-4}-\omega_4^{-4}\right)\omega_4\omega_2\right)\mu_2\rho^{-1}+\frac{m\rho^{-1}}{\beta z_x^3(1)EI(1)h^2}\left(\omega_2^{-2}-\omega_4^{-2}\right)]+\mathcal{O}\left(\rho^{-2}\right)\}.\end{aligned}$$

In sector S_1, the choices are:
$\omega_1^2=-i,\ \omega_2^2=i,\ \omega_3^2=i,\ \omega_4^2=-i,\ \omega_3^{-4}-\omega_4^{-4}=0,$
$\omega_2^{-4}-\omega_4^{-4}=0,\ \omega_2^{-1}\omega_4=-i,\ \omega_3=-\omega_2,$
$\omega_4-\omega_3=\sqrt{2},\ \omega_1-\omega_3=\sqrt{2}i,\ \omega_2-\omega_1=\sqrt{2},\ \omega_4-\omega_2=-i\sqrt{2},$
$\omega_2^{-2}-\omega_4^{-2}=-2i,\ \omega_3^{-2}-\omega_4^{-2}=-2i,\ \omega_3^2\omega_4^2=1,\ \omega_2^2\omega_4^2=1.$
So a straightforward simplification will arrive at

$$\Delta(\rho)=\rho^6 e^{\rho\omega_4}\{2e^{\rho\omega_2}+2e^{-\rho\omega_2}+2\mu_3\rho^{-1}e^{\rho\omega_2}+2i\mu_4\rho^{-1}e^{-\rho\omega_2}+\mathcal{O}\left(\rho^{-2}\right)\}.$$

This expansion is also true for other sectors S_k by applying the exact same arguments above.

Then, from the determinant $\Delta(\rho)$, we have

$$\Theta_{10}=2\neq 0,\ \ \Theta_{00}=0,\ \ \Theta_{-10}=2\neq 0.$$

From ([32], Definition 3.2.5), we conclude that the eigenvalue problem (34) is strongly regular. □

We can further deduce a necessary condition for the stability of (1) because the spectrum determined growth condition is true now. For this purpose, we look for an asymptotic expansion for the eigenvalues of the system (34).

Theorem 3.2. Let $\mathbb{A}_\gamma$ be defined by (6) and (7), then an asymptotic expression of the eigenvalues of the problem (34) is given by

$$\lambda_n \quad = \quad \frac{\sqrt{2}}{2h^2}(\mu_4-\mu_3)\pm\frac{1}{h^2}\left(\frac{\sqrt{2}}{2}(\mu_4+\mu_3)+\left(n+\frac{1}{2}\right)^2\pi^2\right)i+\mathcal{O}\left(n^{-1}\right), \tag{44}$$

where $n=N,N+1,...$ with N large enough, and

$$\begin{aligned}\mu_4-\mu_3&=\sqrt{2}\mu_2-2\sqrt{2}\frac{m}{\beta z_x^3(1)EI(1)h^2}=-\frac{2\sqrt{2}mh}{\beta EI(1)}\left(\frac{\kappa(1)}{EI(1)}\right)^{\frac{-3}{4}}\\ \mu_4+\mu_3&=2\sqrt{2}(\mu_1+b_{11}).\end{aligned} \tag{45}$$

We have

$$\mu_2=-\frac{h^2}{4}\int_0^1 d(\zeta)\,d\zeta,\ \ d(x)=\frac{\gamma(x)}{\kappa(x)},\frac{dz}{dx}=\frac{1}{h}\left(\frac{\kappa(x)}{EI(x)}\right)^{\frac{1}{4}},$$

so,

$$\mu_2=-\frac{h^2}{4}\int_0^1\frac{\gamma(x)}{\kappa(x)}\frac{1}{h}\left(\frac{\kappa(x)}{EI(x)}\right)^{\frac{1}{4}}dx=-\frac{h}{4}\int_0^1\frac{\gamma(x)}{\kappa(x)}\left(\frac{\kappa(x)}{EI(x)}\right)^{\frac{1}{4}}dx.$$

Moreover, λ_n $(n = N, N+1, ...)$ with sufficiently large modulus are simple and distinct except for finitely many of them, and satisfy

$$\lim_{n\to+\infty} \mathrm{Re}\lambda_n = -\frac{2m}{\beta h EI(1)}\left(\frac{\kappa(1)}{EI(1)}\right)^{\frac{-3}{4}} - \frac{1}{2h}\int_0^1 \frac{\gamma(x)}{\kappa(x)}\left(\frac{\kappa(x)}{EI(x)}\right)^{\frac{1}{4}} dx. \tag{46}$$

Proof. Using (38), we can deduce an asymptotic expression for the eigenvalues of problem (34). The equation $\Delta(\rho) = 0$ and (38) imply that

$$2e^{\rho\omega_2} + 2e^{-\rho\omega_2} + 2\mu_3\rho^{-1}e^{\rho\omega_2} + 2i\mu_4\rho^{-1}e^{-\rho\omega_2} + \mathcal{O}\left(\rho^{-2}\right) = 0$$

which equivalent to

$$e^{\rho\omega_2} + e^{-\rho\omega_2} + \mu_3\rho^{-1}e^{\rho\omega_2} + i\mu_4\rho^{-1}e^{-\rho\omega_2} + \mathcal{O}\left(\rho^{-2}\right) = 0 \tag{47}$$

and can be rewritten as

$$e^{\rho\omega_2} + e^{-\rho\omega_2} + \mathcal{O}\left(\rho^{-1}\right) = 0. \tag{48}$$

Note that the following equation:

$$e^{\rho\omega_2} + e^{-\rho\omega_2} = 0$$

has solutions

$$\rho_n = \left(n + \frac{1}{2}\right)\frac{\pi i}{\omega_2},\ n = 1, 2, ... \tag{49}$$

Let $\widetilde{\rho_n}$ be the solutions of (48). Applying Rouch?s theorem see Naimark, 1967, p.70 in [24] to (48), we get the following expression:

$$\widetilde{\rho_n} = \rho_n + \alpha_n = \left(n + \frac{1}{2}\right)\frac{\pi i}{\omega_2} + \alpha_n, \quad \alpha_n = \mathcal{O}\left(n^{-1}\right), \tag{50}$$

$$n = N, N+1, ...$$

where N is a large positive integer. Substituting $\widetilde{\rho_n}$ into (47), and using the fact that $e^{\rho\omega_2} = -e^{-\rho\omega_2}$.
We obtain:

$$e^{\alpha_n\omega_2} - e^{-\alpha_n\omega_2} + \mu_3\widetilde{\rho_n}^{-1}e^{\alpha_n\omega_2} - i\mu_4\widetilde{\rho_n}^{-1}e^{-\alpha_n\omega_2} + \mathcal{O}(\widetilde{\rho_n}^{-2}) = 0.$$

Expanding the exponential function according to its Taylor series, we get:

$$\alpha_n = -\frac{\mu_3}{2\omega_2\rho_n} + \frac{\mu_4}{2\omega_2\rho_n}i + \mathcal{O}\left(n^{-2}\right), \quad n = N, N+1, ...$$

Therefore, we have

$$\widetilde{\rho_n} = \left(n + \frac{1}{2}\right)\frac{\pi i}{\omega_2} + \frac{\mu_3}{2\left(n + \frac{1}{2}\right)\pi}i + \frac{\mu_4}{2\left(n + \frac{1}{2}\right)\pi} + \mathcal{O}\left(n^{-2}\right), \quad n = N, N+1, ...$$

Note that $\lambda_n = \frac{\rho_n^2}{h^2} \neq 0$, $\omega_2 = e^{i\frac{\pi}{4}}$ and $\omega_2^2 = i$. So we have

$$\lambda_n = \frac{\sqrt{2}}{2h^2}(\mu_4 - \mu_3) + \frac{1}{h^2}(\frac{\sqrt{2}}{2}(\mu_4 + \mu_3) + \left(n + \frac{1}{2}\right)^2\pi^2)i + \mathcal{O}\left(n^{-1}\right) \tag{51}$$

where $n = N, N+1, ...$ with N large enough.
The same proof can be applied to sector S_2 because the eigenvalues of the problem (34) can obtained by a similar calculation with the choices
$\omega_1 = \exp(i\frac{1}{4}\pi) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \omega_2 = \exp(i\frac{3}{4}\pi) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,$
$\omega_3 = \exp(i\frac{7}{4}\pi) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \omega_4 = \exp(i\frac{5}{4}\pi) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,$
so that inequality (35) holds:

$$\mathrm{Re}\left(\rho\omega_1\right) \leq \mathrm{Re}\left(\rho\omega_2\right) \leq \mathrm{Re}\left(\rho\omega_3\right) \leq \mathrm{Re}\left(\rho\omega_4\right), \quad \forall\rho \in S_2.$$

Hence, in sector S_2, we have the following asymptotic expression of $\Delta(\rho)$:

$$\Delta(\rho) = \rho^6 e^{\rho\omega_4}\{2e^{\rho\omega_2} + 2e^{-\rho\omega_2} - 2\mu_3\rho^{-1}e^{\rho\omega_2} + 2i\mu_4\rho^{-1}e^{-\rho\omega_2} + \mathcal{O}\left(\rho^{-2}\right)\}.$$

By a direct calculation, we have for $n = N, N+1, ...$

$$\widetilde{\rho_n} = \left(n + \frac{1}{2}\right)\frac{\pi i}{\omega_2} - \frac{\mu_3}{2\left(n+\frac{1}{2}\right)\pi} i + \frac{\mu_4}{2\left(n+\frac{1}{2}\right)\pi} + \mathcal{O}\left(n^{-2}\right), \tag{52}$$

with N large enough. Again

$$\lambda_n = \frac{\sqrt{2}}{2h^2}\left(\mu_4 - \mu_3\right) - \frac{1}{h^2}\left(\frac{\sqrt{2}}{2}\left(\mu_4 + \mu_3\right) + \left(n + \frac{1}{2}\right)^2\pi^2\right)i + \mathcal{O}\left(n^{-1}\right), \tag{53}$$

where $n = N, N+1, ...$ with N large enough.

Here we should point out that the eigenvalues generated from the other sectors S_k coincide with those from S_1 and S_2 the detailed argument can be found in [24]. Combining with (51) and (53), we obtain the result on the eigenvalues. □

Remark 3.1. From the above theorem, taking $EI(x) = \kappa(x) = 1,\ \gamma(x) = 0\ \forall x \in [0,1]$, we obtain the result of Wang see ([32] pages 82-86). We thus generate this result of Wang.

Remark 3.2. In Theorem 3.2, we obtain the asymptotic expression of $\mathbb{A}_\gamma$. If $\lambda \in \sigma(\mathbb{A}_\gamma)$ with λ large enough, then λ has an asymptotic form given by (44). So, $\sigma(\mathbb{A}_\gamma) \diagdown \left\{\lambda_n,\ \overline{\lambda_n}\ \ n \geq N\right\}$ has finitely many eigenvalues.

3.2. Riesz Basis Property of Eigenfunctions of $\mathbb{A}_\gamma$

Definition 3.1. The boundary problem (23) with (22) is said to be regular if the coefficients $F_0^{\mathbb{k}_k}$ in (24) are nonzero. Furthermore, the regular boundary problem (23) with (22) is said to be strongly regular if the zeros of $\Delta(\rho)$ are asymptotically simple and isolate one from another.

We consider the following theorem see [32]:

Theorem 3.3. If the ordinary differential system with parameter $\lambda = \rho^q$

$$\begin{cases} L(u,\lambda) = L(u) + \lambda^2 u + \lambda\mu(x)u \\ B_j(u) = 0,\ \ 1 \leq j \leq 2q \end{cases} \tag{54}$$

has strongly regular boundary conditions, then the generalized eigenfunction system of $\mathbb{A}$ ($\mathbb{A}$ the operator associated to (54)) form a Riesz basis in the Hilbert space $\mathbb{H}$.

In this subsection, we discuss the Riesz basis property of the eigenfunctions of operator $\mathbb{A}_\gamma$ of the system (8). We begin with showing that the generalized eigenfunctions of $\mathbb{A}$ defined by (54) form an unconditional basis in Hilbert state space $\mathbb{H}$.

For this aim, we introduce a transformation $\mathcal{L}$ via

$$\mathcal{L}(f,g,\xi) = (\phi,\psi,\xi^*)\ \ whith\ \ \xi,\xi^* \in \mathbb{C},$$

where

$$\phi(x) = f(z),\ \ \psi(x) = g(z),\ \ z = \frac{1}{h}\int_0^x \left(\frac{\kappa(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta,$$

with

$$h = \int_0^1 \left(\frac{\kappa(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta.$$

It is easily seen that $\mathcal{L}$ is a bounded invertible operator on $\mathbb{H}$.

Now we define the following ordinary differential operator:

$$\begin{cases} L(f) = f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z), \\ \mu(z) = h^2 d(z) = h^2\dfrac{\gamma(x)}{\kappa(x)} \\ B_1(f) = f(0),\ \ B_2(f) = f'(0), \\ B_3(f) = z_x^2(1)f''(1) + z_{xx}(1)f'(1) = 0 \\ B_4(f) = f'''(1) + \dfrac{3z_{xx}(1)}{z_x^2(1)}f''(1) + \dfrac{z_{xxx}(1)}{z_x^3(1)}f'(1) - \dfrac{\alpha\lambda + \lambda^2 m}{(1+\beta\lambda)z_x^3(1)EI(1)}f(1) = 0, \end{cases} \tag{55}$$

where the coefficients are given by (29)-(33). Let $\mathbb{A}$, $\eta \in \sigma(\mathbb{A})$ be an eigenvalue of $\mathbb{A}$ and $\left(\eta^{-1}g, g, -\frac{\beta^{-1}(\alpha - m\beta^{-1})}{\eta+\beta^{-1}}g(1)\right)^T$ be an eigenfunction, then we have $g = \eta f$ and f will satisfy the following equation:

$$f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z) + \eta u(z)f(z) + \eta^2 f(z) = 0,$$

with boundary conditions $B_j(f) = 0,\ j = 1,2,3,4$. Now by taking $\lambda = \dfrac{\eta}{h^2}$ and

$$\mathcal{L}(f, g, \xi) = (\phi(x), \psi(x), \xi^*)$$

we see that $\psi = \lambda\phi$ and ϕ satisfies the equation

$$\begin{cases} \phi^{(4)}(x) + \dfrac{2EI'(x)}{EI(x)}\phi'''(x) + \dfrac{EI''(x)}{EI(x)}\phi''(x) + \dfrac{\lambda^2\kappa(x)}{EI(x)}\phi(x) + \lambda\dfrac{\gamma(x)}{\kappa(x)} = 0, 0 < x < 1, \\ \phi(0) = \phi'(0) = \phi''(1) = 0 \\ \phi'''(1) = -\dfrac{\alpha\lambda + \lambda^2 m}{(1+\beta\lambda)z_x^3(1)EI(1)}\phi(1). \end{cases}$$

Hence we have the following result.

Theorem 3.4. Let operator $\mathbb{A}_\gamma$ of the system (8). Then the eigenvalues of operator $\mathbb{A}_\gamma$ are all simple except for finitely many of them, and the generalized eigenfunctions of operator $\mathbb{A}_\gamma$ form a Riesz basis for the Hilbert state space $\mathbb{H}$.

Proof. From the previous subsection, we have shown that the boundary problem (34) is strongly regular because the coefficients of $F_0^{\mathbb{k}_k}$ are nonzero in $\Delta(\rho)$ see Definition 3.1, which implies that the eigenvalues are separated and simple except for finitely many of them. Thus the first assertion is true. Next, according to Theorem 3.3, the strongly regular boundary conditions ensure that the generalized eigenfunction sequence $F_n = (f_n, \eta_n f_n, \xi_n)$ of operator $\mathbb{A}$ forms a Riesz basis for $\mathbb{H}$. Since $\mathcal{L}$ is bounded and invertible on $\mathbb{H}$, it follows that $\Psi_n = (\phi_n, \lambda_n\phi_n, \xi_n^*) = \mathcal{L}F_n$ also forms a Riesz basis on $\mathbb{H}$. □

4. Exponential Stability

We are now in a position to investigate the stability of system (8). Since the Riesz basis property implies the spectrum determined growth condition (see Curtain & Zwart, 1995), and (46) describes the asymptote of $\sigma(\mathbb{A}_\gamma)$, for any small $\varepsilon > 0$ there are only finitely many eigenvalues of $\mathbb{A}_\gamma$ in the following half-plane:

$$\Sigma : \mathrm{Re}\lambda > -\frac{2m}{\beta h EI(1)}\left(\frac{\kappa(1)}{EI(1)}\right)^{\frac{-3}{4}} - \frac{1}{2h}\int_0^1 \frac{\gamma(x)}{\kappa(x)}\left(\frac{\kappa(x)}{EI(x)}\right)^{\frac{1}{4}} dx. + \varepsilon \tag{56}$$

The following stability result that describe how stability depends upon damping function γ.

We recall the following definition to begin see [32].

Definition 4.1. Let $\mathcal{A}$ be the infinitesimal generator of a strongly continuous semigroup $T(t)$ on a Hilbert space $\mathcal{H}$. Consider

$$\omega(\mathcal{A}) = \lim_{t\to\infty}\frac{1}{t}\|T(t)\|,$$

the growth exponent bound of $T(t)$, and

$$s(\mathcal{A}) = sup\{Re(\lambda) : \lambda \in \sigma(\mathcal{A})\},$$

the spectral bound of the operator $\mathcal{A}$. If

$$\omega(\mathcal{A}) = s(\mathcal{A})$$

We say that the spectrum determined growth condition holds. If $\omega(\mathcal{A}) < 0$, then $T(t)$ is exponentially stable.

Theorem 4.1. If γ is continuous and nonnegative on $[0,1]$ and $\beta > 0$, then the system (8) is exponentially stable.

Proof. For any $u = (f, g, \eta)^T \in D(\mathbb{A}_\gamma)$,

$$\langle \mathbb{A}_\gamma u, u\rangle_{\mathbb{H}} = \left\langle \begin{pmatrix} g(x) \\ -\dfrac{1}{\kappa(x)}\left((EI(x)f''(x))'' + \gamma(x)g(x)\right) \\ -\beta^{-1}\eta - \beta^{-1}\left(\alpha - m\beta^{-1}\right)g(1) \end{pmatrix}, \begin{pmatrix} f \\ g \\ \eta \end{pmatrix} \right\rangle$$

$$= \int_0^1 -(EI(x)f''(x))''\overline{g(x)} + EI(x)\overline{f''(x)}g''(x) - \gamma(x)|g(x)|^2 dx + \frac{\beta^2}{m+\alpha\beta}\left(-\beta^{-1}\eta - \beta^{-1}\left(\alpha - m\beta^{-1}\right)g(1)\right)\overline{\eta}.$$

As a result

$$Re\langle \mathbb{A}_\gamma u, u\rangle_{\mathbb{H}} = -\int_0^1 \gamma(x)|g(x)|^2 dx + \frac{1}{m+\alpha\beta}(2mRe(g(1)\eta) - \left(\beta^{-1}m^2 + \alpha m\right)|g(1)|^2 - \beta|\eta|^2), \tag{57}$$

with

$$\left(2mRe\left(g(1)\eta\right)-\left(\beta^{-1}m^{2}+\alpha m\right)\left|g(1)\right|^{2}-\beta\left|\eta\right|^{2}\right)<0$$

since

$$4\beta\left(\beta^{-1}m^{2}+\alpha m\right)>4m^{2}.$$

where we derive the last equation by integrating twice by parts and using the boundary conditions of the system (1).

It follows from previous result that the operator $\mathbb{A}_{\gamma}$ is dissipative.

Moreover, the spectrum of $\mathbb{A}_{\gamma}$ has an asymptotic

$$Re\lambda\simeq-\frac{2m}{\beta hEI\left(1\right)}\left(\frac{\kappa\left(1\right)}{EI\left(1\right)}\right)^{\frac{-3}{4}}-\frac{1}{2h}\int_{0}^{1}\frac{\gamma\left(x\right)}{\kappa\left(x\right)}\left(\frac{\kappa\left(x\right)}{EI\left(x\right)}\right)^{\frac{1}{4}}dx.$$

If we can show that there is no eigenvalue on the imaginary axis then the exponential stability holds.

Let $\lambda=ir$ with $r\in\mathbb{R}$ be an eigenvalue of operator $\mathbb{A}_{\gamma}$ on the imaginary axis and $\Phi=(\psi,\phi,\xi)$ be the corresponding eigenfunction.

We have

$$\Phi=\left(\lambda^{-1}\phi,\phi,-\frac{\beta^{-1}(\alpha-m\beta^{-1})}{\lambda+\beta^{-1}}\phi(1)\right).$$

We get

$$Re(\langle\mathbb{A}_{\gamma}\Phi,\Phi\rangle)=\left\|\Phi\right\|_{\mathbb{H}}^{2}Re(\lambda)=0,$$

and

$$\int_{0}^{1}\gamma\left(x\right)\left|\phi\left(x\right)\right|^{2}dx-\frac{\left(2mRe\left(\phi(1)\xi\right)-\left(\beta^{-1}m^{2}+\alpha m\right)\left|\phi(1)\right|^{2}-\beta\left|\xi\right|^{2}\right)}{m+\alpha\beta}=0.$$

Since

$$4\beta\left(\beta^{-1}m^{2}+\alpha m\right)>4m^{2},$$

then

$$-\frac{\left(2mRe\left(\phi(1)\xi\right)-\left(\beta^{-1}m^{2}+\alpha m\right)\left|\phi(1)\right|^{2}-\beta\left|\xi\right|^{2}\right)}{m+\alpha\beta}=0$$

and

$$\gamma\left(x\right)\left|\phi\left(x\right)\right|^{2}=0,\ \forall x\in\left[0,1\right],$$

moreover $\gamma(x)\geq 0$ in $[0,1]$, and $\gamma(1)\neq 0$, we have

$$\phi(1)=0.$$

Which implies that the eigenvalue Φ of $\mathbb{A}_{\gamma}$ is zero because $\psi=\lambda^{-1}\phi$ and $\phi\left(x\right)$ is the uniqueness solution of the differential equation:

$$\left\{\begin{array}{l}\lambda^{2}m(x)\phi\left(x\right)+\left(EI\left(x\right)\phi''\left(x\right)\right)''+\lambda\gamma\left(x\right)\phi\left(x\right)=0,\\ \quad\text{for }\ 0<x<1,\\ \phi\left(0\right)=\phi'\left(0\right)=\phi''\left(1\right)=\phi'''\left(1\right)=\phi(1)=0.\end{array}\right.\tag{58}$$

The above equation has a zero solution only see [27].

However, $\Phi=0$ contradicts Φ being an eigenfunction and so there is no eigenvalue on the imaginary axis. Then $Re(\lambda)<0$.

From the Theorem 3.4 and the spectrum determined growth condition Definition 4.1, the system is exponentially stable. □

Now we can consider the case where $\gamma\left(.\right)$ is continuous and indefinite in $[0,1]$. We follow a Wang's idea [32]. Consider $\gamma\left(x\right)=\gamma^{+}\left(x\right)-\gamma^{-}\left(x\right)$ for all $x\in[0,1]$ with

$$\begin{aligned}\gamma^{+}\left(x\right)&=\max\left\{\gamma\left(x\right),0\right\}=\left\{\begin{array}{l}\gamma\left(x\right),\text{ if }\gamma\left(x\right)>0\\ 0\quad\text{elsewhere}\end{array}\right.,\\ \gamma^{-}\left(x\right)&=\max\left\{-\gamma\left(x\right),0\right\}=\left\{\begin{array}{l}-\gamma\left(x\right),\text{ if }\gamma\left(x\right)<0\\ 0\quad\text{elsewhere}\end{array}\right.,\end{aligned}$$

let

$$\mathbb{A}_{\gamma^+}(f,g,\eta)^T = \begin{pmatrix} g(x) \\ -\dfrac{1}{\kappa(x)}\left((EI(x)f''(x))'' + \gamma^+(x)g(x)\right) - \beta^{-1}\eta - \beta^{-1}\left(\alpha - m\beta^{-1}\right)g(1) \end{pmatrix},$$

$$\forall (f,g,\eta)^T \in D\left(\mathbb{A}_{\gamma^+}\right) = D\left(\mathbb{A}_{\gamma}\right)$$

and

$$\Gamma_-(f,g,\eta)^T = \left(0, \frac{\gamma^-(x)}{\kappa(x)}g(x), 0\right)^T, \ \forall (f,g,\eta)^T \in \mathbb{H}.$$

So $\mathbb{A}_\gamma$ can be rewritten as follows: $\mathbb{A}_\gamma = \mathbb{A}_{\gamma^+} + \Gamma_-$.

Theorem 4.2. Let $s\left(\mathbb{A}_{\gamma^+}\right) = \sup\left\{\mathrm{Re}\lambda | \lambda \in \sigma\left(\mathbb{A}_{\gamma^+}\right)\right\}$. if

$$\max_{x\in[0,1]}\left\{\frac{\gamma^-(x)}{\kappa(x)}\right\} < \left|s\left(A_{\gamma^+}\right)\right|,$$

then the system (8) is exponentially stable.

Proof. Referring to Section 2, we see that the operator Γ_- is bounded and symmetric, that is, that:

$$\langle \Gamma_-(u), v\rangle_{\mathbb{H}} = \langle u, \Gamma_-(v)\rangle_{\mathbb{H}} \quad \forall u,\ v \in \mathbb{H},$$

so the operator Γ_- is self-adjoint (see H. Brezis [8], page 113) and we have:

$$\|\Gamma_-\| = \max_{x\in[0,1]}\left\{\frac{\gamma^-(x)}{\kappa(x)}\right\}. \tag{59}$$

By Theorem 2.2 and the definition of the operator $\mathbb{A}_{\gamma^+}$, $\{e^{\mathbb{A}_{\gamma^+}t}\}_{t\geq 0}$ is a semigroup of contractions and $s\left(\mathbb{A}_{\gamma^+}\right) < 0$.

By applying the theory of disturbance of linear operators, we have $\lambda \in \rho\left(\mathbb{A}_\gamma\right)$ every time $Re\lambda > s\left(\mathbb{A}_{\gamma^+}\right) + \|\Gamma_-\|$. Again, the theorem 3.4 gives

$$\omega\left(\mathbb{A}_\gamma\right) = s\left(\mathbb{A}_\gamma\right) \leq s\left(\mathbb{A}_{\gamma^+}\right) + \|\Gamma_-\| < 0,$$

where $\omega\left(\mathbb{A}_\gamma\right)$ designates the rate Optimal of the system's energy decline. So the system, (8) is exponentially stable. □

5. Conclusion

In this paper, using the basic Riesz theory, we discussed the exponential stabilization of a variable coefficients Euler-Bernoulli beam equation with viscous damping and a tip mass. Different from the earlier papers, we imposed a very important condition on the damping term γ to prove that no eigenvalue of the operator $\mathbb{A}_\gamma$ of system (1) is on the imaginary axis. This led us to the exponential stability of the system through the basic Riesz property and the spectrum determined growth condition. We note that the condition used in this paper can also be applied to the study of the exponential stability of other systems.

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Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] R. Adams, "Sobolev spaces," Academic Press, New York, 1975.

[2] O. Baysala, A. Hasanov and A. Kawano, "Exponential stability of damped Euler-Bernoulli beam controlled by boundary springs and dampers," arXiv: 2307.06170v1 [math.AP] 12 Jul 2023.

[3] A. Benaddi, B. P. Rao, "Energy decay rate of wave equations with indefinite damping," J. Diff. Eqns, Vol. 161, issue 2, 1 March 2000, pp. 337-357. https://doi.org/10.1006/jdeq.2000.3714

[4] G. D. Birkhoff, "On the asymptotic character of the solutions of certain linear differential equations containing a parameter," Trans. Amer. Math. Soc., Vol. 9, No. 2 Apr., 1908, pp. 219-231. https://doi.org/10.2307/1988652

[5] G. D. Birkhoff, "Boundary value and expansion problems of ordinary linear differential equations," Trans. Amer. Math. Soc. Vol. 9, No. 4 (Oct., 1908), pp. 373-395. https://doi.org/10.2307/1988661

[6] S. K. Bose, "On initial conditions for a boundary stabilized hybrid Euler-Bernoulli beam," Proc.Indian Acad. Sci. (Math. Sci.), Vol. 111, No. 3, August 2001, pp. 365-370. https://doi.org/10.1007/BF02829602

[7] Z. Bouallagui and M. Jellouli, "Stability of Euler-Bernoulli beam equation on a star-shaped network with indefinite damping," Applicable Analysis, 20 Feb 2025, pp. 1-23. https://doi.org/10.1080/00036811.2025.2468518

[8] H. Brezis, "Analyse fonctionnelle: Théorie et applications," Masson. 1983.

[9] R. F. Curtain and H. J. Zwart, "An Introduction to Infinite Dimensional Linera System Theory," New York: Springer. Vol 38, iss. 3 1996. https://doi.org/10.1137/1038096

[10] B. N. Data, S. Elhay, Y. M. Ram and D. R. Sarkissian, "Partial eigenstructure assignment for the quadratic pencil," Journal of Sound and Vib, Vol. 230, Iss. 1, 10 February 2000, pp. 101-110. https://doi.org/10.1006/jsvi.1999.2620

[11] B. N. Data, Y. M. Ram and D. R. Sarkissian, "Spectrum modification for gyroscopic system," Z. Angew. Math. Mech. Vol. 82 Iss. 3, March 2002, pp. 191-200. https://doi.org/10.1002/1521-4001(200203)82:3<191::AID-ZAMM191>3.0.CO;2-L

[12] A. Hasanov, "Exponential stability of Euler-Bernoulli beam under boundary controls in rotation and angular velocity," arXiv: 2307.01518v1 [math.OC] 4 Jul 2023.

[13] B. Z. Guo, "On the boundary control for a hybrid system with variable coefficients," J. Optim: Theor. Appl., Vol. 114, No. 2, August 2002, pp. 373-395. https://doi.org/10.1023/A:1016039819069

[14] B. Z. Guo, "Riesz basis property and exponential stability of controlled Euler-Bernoulli beam equation with variable coefficients," SIAM J. Control Optim., Vol. 40, Iss. 6, 2002, pp. 1905-1923. https://doi.org/10.1137/S0363012900372519

[15] A. Haraux "Semi-Linear Hyperbolic Problems in Bounded Domain," Harwood Academic and Breach New York, Vol. 3 January 1987.

[16] A. A. H. Kouassi, F. N. Diop, S. A. J. Yapi, and K. A. Touré "Stability of variable coefficients Rayleigh beams with indefinite damping under force control" Far East Journal of Dynamical Systems. Vol. 38, No 1, 2025, pp. 127-157. https://doi.org/10.17654/0972111825006

[17] V. Komornik, "Exact Contrllability and Stabilization (the Multiplier Method)," Masson, Paris: Wiley. 1995.

[18] S. Li, Y. Wang, Z. Liang, J. Yu and G. Zhu, "Stabilization of Vibrating Beam with a Tip Mass Controlled by Combined Feedback Forces," Vol. 256, Iss. 1, 1 April 2001, pp. 13-38. https://doi.org/10.1006/jmaa.2000.7217

[19] B. R. Li, "The perturbation theory of a class of linear operators with applications," Acta Math. Sin. (Chinese), Vol. 21, Iss. 3, 1078, pp. 206-222. https://doi.org/10.12386/A1978sxxb0023

[20] K. M. Liew, Y. Xiang and S. Kitipornchai, "Research on thick plate vibration: a literature survey," J. Sound Vib., Vol. 180, Iss. 1, 9 February 1995, pp. 163-176. https://doi.org/10.1006/jsvi.1995.0072

[21] K. Liu, Z. Liu and B. Rao, "Exponential stability of an abstract non-dissipative linear system," SIAM. J. Control Optim., Vol. 40, Iss. 1, July 2001, pp. 149-165. https://doi.org/10.1137/S0363012999364930

[22] T. Mizusawa, "Vibration of rectangular mindlin plates with tapered thickness by the spline strip method," Comput. Struct., Vol. 46, Iss. 3, 3 February 1993, pp. 451-463. https://doi.org/10.1016/0045-7949(93)90215-Y

[23] A. My Driss and E. B. Abderrahman, "Stabilization of variable coefficients Euler-Bernoulli beam equation with a tip mass controlled by combined feedback forces," Annals of the University of Craiova, Mathematics and Computer Science Series. Vol. 42 No: 1, 2015, pp. 238-248. https://doi.org/10.52846/ami.v42i1.754

[24] M. A. Naimark, "Linear Differential Operators," 1968, New York: Ugar.

[25] A. Pazy, "Semigroup of Linear Operators and Applications to Partial Differential Equations," 1983, New York: Springer.

[26] A. A. Shkalikov, "Boundary value problems for ordinary differential equations with a parameter in the boundary conditions," J. Sov. Math., Vol. 33, june 1986, pp. 1311-1342. https://doi.org/10.1007/BF01084754

[27] A. K. Touré A. Coulibaly and H. A. A. Kouassi, "Riesz basis property and exponential stability of Euler-Bernoulli beams with variable coefficients and indefinite damping under a force control in position and velocity," Electronic Journal of Differential Equations, Vol. 2015, February 2015 No: 54, pp 1-20.

[28] K. A. Touré E. Mensah and M. Taha, "Riesz Basis Property and Exponential Stability of a Flexible Euler-Bernoulli Beam with a Force Control in Position and Velocity," Far East Journal of Applied Mathematics, Vol. 69, No 1, 2012, pp 29-56.

[29] C. Tretter, "On $\lambda-$Nonlinear Boundary Eigenvalue Problem," Mathematical Research, Vol. 71, 1993, Berlin: Akademie.

[30] J. Wang and S. P. Yung, “Stability of a nonuniform Rayleigh beam with indefinite damping,” Vol. 55, Iss. 10, October 2006, pp. 863-870. https://doi.org/10.1016/j.sysconle.2006.04.002

[31] J. Wang, G. Q. Xu, and S. P. Yung, “Exponential stability of variable coefficients Rayleigh beams under boundary feedback control: a Riesz basis approach,” Syst. Control Lett., VOl. 51, Iss. 1 January 2004, pp. 33-50. https://doi.org/10.1016/S0167-6911(03)00205-6

[32] J. Wang, “Riesz basis property of some infinite-dimensional control problems and its applications,” The HKU Scholars Hub The University of Hong Kong (thesis) 2004. http://dx.doi.org/10.5353/th_b3016350

[33] G. Q. Xu and D. X. Feng, “The Riesz basis property of a Timoshenko beam with boundary feedback and application,” IMA J. Appl. Math., Vol. 67, Iss: 4, August 2002, pp. 357-370. https://doi.org/10.1093/imamat/67.4.357