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Research Article

# Error Approximation of the Second Order Hyperbolic Differential Equationby Using DG Finite Element Method

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## Abstract

This article presents a simple efficient and asynchronously correcting a posteriori error approximation for discontinuous finite element solutions of the second-order hyperbolic partial differential problems on triangular meshes. This study considersthe basis functions for error spaces corresponding to some finite element spaces. The discretization error of each triangle is estimated by solving the local error problem. It also shows global super convergence for discontinuous solution on triangular lattice. In this article, the triangular elements are classify into three types: (i) elements with one inflow and two outflow edges are of type I, (ii) elements with two inflows and one outflow edges are of type II and (iii) elements with one inflow edge, one outflow edge, and one edge parallel to the characteristics are of type III. The article investigated higher-dimension discontinuous Galerkin methods for hyperbolic problems on triangular meshes and also studied the effect of finite element spaces on the superconvergence properties of DG solutions on three types of triangular elements and it showed that the DG solution is $O(h^{p+2})$ superconvergent at Legendre points on the outflow edge on triangles having one outflow edge using three polynomial spaces. A posteriori error estimates are tested on a number of linear and nonlinear problems to show their efficiency and accuracy under lattice refinement for smooth and discontinuous solutions.

## Keywords

Finite Element Method, Hyperbolic Problems, Triangular Meshes, Basis Function, Discontinuous Galerkin

## 1. Introduction

The discontinuous Galerkin method was first applied to solve the neutron equation [11] after that it was planned for initial-value problems [6, 7, 10]. Cockburn and Shu [1, 12] prolonged the conservation law method to explain first-order hyperbolic partial differential equations. Super convergence properties for DG methods have been planned for ordinary differential equations [5, 7, 11], for hyperbolic problems [3, 4, 6] and for diffusion and convection-diffusion problems [2, 8, 9]. DG methods permit discontinuous bases, which simplify both h-refinement and p-refinement. The solution space consists of piecewise continuous polynomial functions relative to a structured or unstructured mesh. As such, it can sharply confinement solution discontinuities relative to the computational mesh. It upholds local conservation on an elemental basis. The DG method has a simple assertion pattern between elements with a common face that makes it useful for parallel computation. Recently, Adjerid et al. [13, 14] proved that DG solutions of one-dimensional linear and nonlinear hyperbolic problems using p-degree polynomial approximations exhibit an $O(h^{p+2})$super convergence rate at

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the roots of Radau polynomial of degree $p + 1$ on each element. They furtherproven a strong $O(h^{p+2})$ superconvergence at the downwind end of every element. Krivodonova and Flaherty [6] assembled a posteriori error estimates that converge to the true error under mesh refinement on unstructured triangular meshes. Adjerid and Massey [2, 3] shownsuper convergence results for multi-dimensional problems using rectangular meshes where they showed that the top term in the true local error is spanned by two $(p + 1)$ −degree Radau polynomials in the $x$ and $y$ directions, respectively. They further showed that a p-degree discontinuous finite element solution exhibits $O(h^{p+2})$ ) superconvergence at Radau points obtained as a tensor product of the roots of Radau polynomial of degree $p + 1$. In this paper, we extend the study of Flaherty and Krivodonova [6] to show new super convergence results for DG solutions. The triangular elements are classify into three types: (i) elements with one inflow and two outflow edges are of type I, (ii) elements with two inflows and one outflow edges are of type II and (iii) elements with one inflow edge, one outflow edge, and one edge parallel to the characteristics are of type III. This arrangement will be defined more precisely later. The article presents several new $O(h^{p+2})$point wisesuper convergence results for the three types of elements and three polynomial spaces. In particular, it shows that the solution on elements of type I is $O(h^{p+2})$super convergent at the two vertices of the inflow edge using an appropriate space. Moreover, for some spaces superiorto the space of polynomials of degree p and smaller than the polynomial space of degree p + 1. It exposed additional super convergence points in the interior of each triangle. On elements of type II, the DG solution is $O(h^{p+2})$super convergent at the Legendre points on the outflow edge as well as at interior problem-dependent points. On elements of type III, the DG solution is $O(h^{p+2})$super convergent at the Legendre points on the outflow edge and for some polynomial spaces the DG solution is $O(h^{p+2})$at every point of the outflow edge. This study will extant a super convergence investigation of the local error. These super convergence results still hold on meshes consisting of elements of type III only. In order to hold these super convergence rates for the global solution on general meshes one needs to use estimates of the boundary conditions at the inflow boundary of each element. This is possible on elements whose inflow edges are on the inflow boundary of the domain while on the remaining elements. It accurate the solution by adding an error estimate and use it as an inflow boundary condition.

## 2. DG Formulation and Preliminary Results

Consider a linear first order hyperbolic scalar problem on a bounded convex polygonal domain $\Omega \in R^2$. Let $\beta = [\beta_1, \beta_2]^T$ denote a constant non zero velocity vector. If $n$denotes the outward unit normal vector, the domain boundary $\partial\Omega = \partial\Omega^+ \cup \partial\Omega^- \cup \partial\Omega^0$, where

$\partial\Omega^- = \{(x, y) \in \beta.n < 0\}$, is the inflow boundary.

$\partial\Omega^+ = \{(x, y) \in \beta.n > 0\}$, is the outflow boundary.

$\partial\Omega^0 = \{(x, y) \in \beta.n = 0\}$, is the characteristic boundary.

Let $u(x, y)$ denote a smooth function on $\Omega$ and consider the following hyperbolic boundary value problem

$$\beta.\nabla u + bu = f(x, y), (x, y) \in \Omega = [0,1] \times [0,1] \quad (1)$$

Subject to the boundary conditions

$$u(x, 0) = g_0(x), u(0, y) = g_1(x)$$

Where the function $f(x, y), g_0(x)$, and $g_1(x)$ are selected such that the exact solution $u(x, y) \in C^\infty(\Omega)$. Let $b, \beta_1 \geq 0, \beta_2 \geq 0, {\beta_1}^2 + {\beta_2}^2 \geq 0$, be real constants.

The domain $\Omega$ is partitioned into a regular mesh having $N$ triangular elements $\Delta_j, j = 1 \dots \dots N$ of diameter $h > 0$. In the remainder of this study, it omit the element index and refer to an arbitrary element by $\Delta$ whenever confusion is unlikely.

Multiply (1) by a test function$v$, integrate over an arbitrary element $\Delta$,

$$\iint_\Delta (\beta.\nabla u + bu)\, v\, dx\, dy = \iint_\Delta f v\, dx\, dy$$

Apply Green's theorem to write

$$\int_{\Gamma_-} (\beta.n)uvds + \int_{\Gamma_+} (\beta.n)uvds + \iint_\Delta (-\beta.\nabla u + bu)\, v\, dx\, dy = \iint_\Delta f v\, dx\, dy \quad (2)$$

Where $\Gamma_+$ and $\Gamma_-$ denote the outflow boundary and inflow boundary, respectively, of $\Delta$. Next we approximate $u(x, y)$ by a piecewise polynomial function $U(x, y)$ whose restriction to $\Delta$ is in $P_p$ consisting of complete polynomial of degree $p$

$$P_k = \{q | q = \sum_{m=0}^{k} \sum_{i=0}^{m} c_i^m x^i y^i\}, k = 0,1, \dots \dots .., p \quad (3)$$

Here $U(x, y)$ is a piecewise polynomial not necessarily continuous across inter-element boundaries.

In our error analysis we will also use the following spaces

$$V_k = P_k \cup \{x^i y^{k+1-i}, i = 1,2, \dots \dots ., k\}, k = 0,1, \dots \dots, p \quad (4)$$

$$U_k = P_k \cup \{x^{k+1} y^{k+1},\}, k = 0,1, \dots \dots, p \quad (5)$$

And

$$P_k = \{q | q = \sum_{i=0}^{k} c_i x^i\}, k = 0,1, \dots \dots, p \quad (6)$$

Note that$V_0 = P_0, U_0 = P_1$and

$$P_{k+1} \subset V_p \cup span\{x^{p+1}, y^{p+1}\}, p \geq 1 \quad (7)$$

These spaces have the following dimensions $\dim(P_p) = (p+1)(p+2)/2$, $\dim\ (V_p) = \frac{(p+2)(p+3)}{2} - 2$, and $\dim(U_p) = \frac{(p+1)(p+2)}{2} + 2$.

Let $S^{N,p}$ denote the space of piecewise polynomial functions $U$ such that the restriction of $U$ to an element $\Delta$ is in $W_p$ which denotes $P_p$, $V_p$, or $U_p$. The discrete DG formulation consists of determining $U \in S^{N,p}$such that

$$\int_{\Gamma_-} (\beta.n)U^- V ds + \int_{\Gamma_+} (\beta.n)UV ds + \iint_{\Delta} (-\beta.\nabla V + bV)\, U\, dx\, dy = \iint_{\Delta} fV\, dx\, dy, \forall V \in W_p \quad (8)$$

Where$U^-$ is the limit from the inflow element sharing $\Gamma_-$, i.e., if $(x,y) \in \Gamma_-$, then

$$U^-(x,y) = \lim_{s\to 0^+} U^-((x,y)+sn)$$

Next, consider the problem (8) on an element $\Delta$ such that $\Gamma_- \subset \partial\Omega^-$. Let $U^-$ be an approximation of the true solution $u$on $\Gamma_-$ and subtract (8) from (2) with $v = V$ to obtain the DG orthogonality condition for the local error $\epsilon = u - U$

$$\int_{\Gamma_-} (\beta.n)\epsilon^- V ds + \int_{\Gamma_+} (\beta.n)\epsilon V ds + \iint_{\Delta} (-\beta.\nabla V + bV)\, \epsilon\, dx\, dy = 0, \forall V \in W_p \quad (9)$$

The map of a physical triangle $\Delta$ having vertices $(x_i, y_i), i = 1,2,3$ into the canonical triangle $(0,0), (1,0)$ and $(0,1)$ by the standard affine mapping

$$\begin{pmatrix} x(\zeta,\rho) \\ y(\zeta,\rho) \end{pmatrix} = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} \zeta \\ \rho \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad (10)$$

For simplicity, consider the DG orthogonality on the right angle with vertices $(0,0), (h,0)$ and $(0,h)$ which applying the affine mapping (3.10) with $\epsilon(\zeta,\rho)$ and $V(\zeta,\rho)$ leads to

$$\int_{\Gamma_-} (\beta.n)\epsilon^- V ds + \int_{\Gamma_+} (\beta.n)\epsilon V ds + \iint_{\Delta} (-\beta.\nabla V + hbV)\, \epsilon\, d\zeta\, d\rho = 0, \forall V \in W_p \quad (11)$$

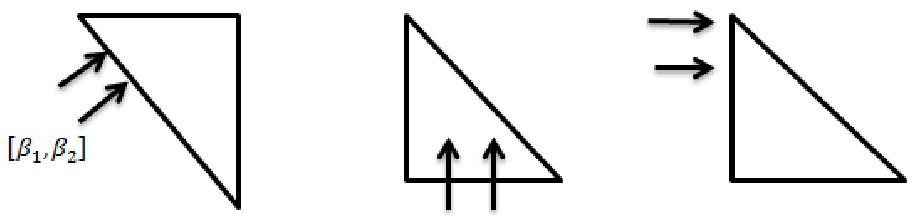


***Figure 1**. Three types of element.*

The triangular elements are classify into three types: (i) elements with one inflow and two outflow edges are of type I, (ii) elements with two inflows and one outflow edges are of type II and (iii) elements with one inflow edge, one outflow edge, and one edge parallel to the characteristics are of type III

In the analysis, it will use orthogonal polynomials given on the canonical triangle defined by the vertices $(0,0), (1,0)$ and $(0,1)$ as

$$\varphi_k^l(\zeta,\rho) = 2^k \hat{L}_k\left(\frac{2\zeta}{1-\rho} - 1\right)(1-\rho)^k \hat{P}_l^{2k+1,0}(2\rho - 1), k,l \geq 0 \quad (12)$$

Where $\hat{P}_n^{\beta_1,\beta_2}(x), -1 \leq x \leq 1$, is the Jacobi Polynomial.

$$\hat{P}_n^{\beta_1,\beta_2}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\beta_1}(1+x)^{-\beta_2}\frac{d^n}{dx^n}\left[(1-x)^{\beta_1+n}(1+x)^{\beta_2+n}\right], \beta_1, \beta_2 > -1. \quad (13)$$

And $\hat{L}_n(x) = \hat{P}_n^{0,0}(x) \in P_n, -1 \leq x \leq 1$, is the legendre polynomial.

$$(\varphi_k^l, \varphi_p^q) = \int_0^1 \int_0^{1-\rho} \varphi_k^l \varphi_p^q\, d\zeta d\rho = b_{kp}^{lq}\delta_{kp}\delta_{lq} \quad (14)$$

And is complete in the space $P_p$. In our analysis we also need the Radua polynomials

$$\hat{R}_{p+1}(x) = (1-x)\hat{P}_p^{1,0}(x) = C\left(\hat{L}_{p+1}(x) - \hat{L}_p(x)\right) \quad (15)$$

The $(k+1)$-degree polynomials

$$\psi_k^l(\zeta,\rho) = \hat{P}_k^{1,0}\left(\frac{2\zeta}{1-\rho} - 1\right)(1-\rho)^k \hat{P}_l^{2k+2,0}(2\rho - 1), k,l \geq 0, k+l \leq p \quad (16)$$

Satisfy the orthogonality condition

$$(\psi_k^l, \psi_p^q) = \int_0^1 \int_0^{1-\rho} (\zeta - (1-\rho))\psi_k^l \psi_p^q\, d\zeta d\rho = b_{kp}^{lq}\delta_{kp}\delta_{lq} \quad (17)$$

And thus provide a basis for $P_p$.

Drop the hat and let $\hat{L}_p, \hat{P}_n^{\beta_1,\beta_2}$ and $\hat{R}_p$ denote the shifted Jacobi, Legendre and Radua polynomials, respectively, on [0,1]

The finite element spaces $V_p$ and $U_p$ are suboptimal, i.e., they contain $p+1$-degree terms that do not contribute to global convergence rate, however, they yield $O(h^{p+2})$ superconvergence rates at some additional interior points which simplifies the a posteriori error estimation procedures described in [13]

If the exact solution is an analytic function, then the local error can be written as a Maclaurin series

$$\epsilon(\zeta,\rho)=\sum_{k=0}^{\infty}Q_k(\zeta,\rho)h^k,) \quad (18)$$

Where $Q_k\in P_k$

Lemma 1: If $Q_k\in P_k, k=0,\ldots\ldots,p$ satisfies

$$\int_{\Gamma_+}(\beta.n)Q_0Vds+\iint_{\Delta}(-\beta.\nabla V)\,Q_0\,d\zeta\,d\rho=0\ \forall V\in W_p \quad (19)$$

And for $1\le k\le p$,

$$\int_{\Gamma_+}(\beta.n)Q_kVds+\iint_{\Delta}(-\beta.\nabla VQ_k+bVQ_{k-1})\,d\zeta\,d\rho=0,\forall V\in W_p \quad (20)$$

Then,

$$Q_k=0, 0\le k\le p \quad (21)$$

Furthermore, Let $u\in C^{\infty}(\Delta)$ and $U\in U_p(\Delta)$ be the solution of (1) and (8), respectively, with $U^-|_{\Gamma_-}=u$. If $\beta_1,\beta_2\ge 0$ such that $\Delta$ is either a triangle, then the local finite error can be written as

$$\epsilon(\zeta,\rho)=\sum_{k=p+1}^{\infty}h^k\,Q_k(\zeta,\rho) \quad (22)$$

Where

$$\iint_{\Delta}Q_{p+1}V\,d\zeta\,d\rho=0\ \forall V\in P_{p-1}$$

$$\int_{\Gamma_+}(\beta.n)Q_{p+1}Vds=0,\forall V\in P_p$$

$$\int_{\Gamma_+}(\beta.n)Q_kds=0, k\ge p+1$$

$$Q_{p+1}(\zeta,\rho)=\sum_{i=0}^{p}b_i^p\,\varphi_{p-i}^i(\zeta,\rho)+\sum_{i=0}^{p+1}b_i^{p+1}\varphi_{p+1-i}^i(\zeta,\rho) \quad (23)$$

Furthermore, at the outflow boundary of the physical element $\Delta$ the local error satisfies

$$\int_{\Gamma_+}(\beta.n)\epsilon ds=O(h^{p+2}) \quad (24)$$

Local error

Now, the first new results for the local error using space $P_p$ in element $\Delta$ can be stated as:

Theorem 1:

Under the same assumption as in $\epsilon(\zeta,\rho)=\sum_{k=p+1}^{\infty}h^k\,Q_k(\zeta,\rho)$ there exist two constants $C_1$ and $C_2$ such that on the outflow edge

$$Q_{p+1}(1-\rho,\rho)=C_1L_{p+1}(\rho) \quad (25)$$

$$Q_{p+1}(\zeta,1-\zeta)=C_2L_{p+1}(\zeta) \quad (26)$$

Furthermore,

$$\iint_{\Delta}\frac{\partial Q_{p+1}}{\partial\zeta}\zeta^i\rho^jd\zeta d\rho=0, i=1,\ldots..p, j=0,\ldots\ldots,p-1, i+j\le p \quad (27)$$

$$\iint_{\Delta}\frac{\partial Q_{p+1}}{\partial\rho}\zeta^i\rho^jd\zeta d\rho=0, i=0,\ldots..p-1, j=1,\ldots\ldots,p, i+j\le p) \quad (28)$$

$$\iint_{\Delta}\beta.\nabla(1-\zeta-\rho)^i\,Q_{p+1}d\zeta d\rho=0, i=1,\ldots..,p \quad (29)$$

Proof: First note that (25) is a direct consequence of

$$\int_{\Gamma_+}(\beta.n)Q_{p+1}Vds=0,\forall V\in P_p$$

In order to prove (27),

$$I=\iint_{\Delta}\frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\zeta}\zeta^i\rho^jd\zeta d\rho=\int_0^1\zeta^i\left(\int_0^{1-\zeta}\frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\zeta}\rho^jd\,\rho\right)d\zeta \quad (30)$$

Differentiating the auxiliary polynomial

$$q(\zeta)=\int_0^{1-\zeta}Q_{p+1}(\zeta,\rho)\,\rho^jd\rho,$$

Leads to

$$q'(\zeta)=-(1-\zeta)^jQ_{p+1}(\zeta,1-\zeta)d\zeta+\int_0^{1-\zeta}\frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\zeta}\rho^jd\,\rho. \quad (31)$$

Combining this with (30) yields

$$I=\int_0^1\zeta^i(1-\zeta)^jQ_{p+1}(\zeta,1-\zeta)d\zeta+\int_0^1\zeta^i\,q'(\zeta)1d\zeta) \quad (32)$$

The orthogonally condition for $i+j\le p$ infers that the first term in the right side of (32) is zero. Now, integrate the second term in (32) by parts and use $q(\zeta)$ to write

$$I=\zeta^i\int_0^1Q_{p+1}(\zeta,\rho)\rho^jd\rho\Big|_{\zeta=0}^{\zeta=1}-i\int_0^1\int_0^{1-\zeta}\zeta^{i-1}\rho^jQ_{p+1}(\zeta,\rho)\,d\zeta d\rho. \quad (33)$$

For $i+j\le p$ and $i>0$, apply the orthogonally condition

$$\iint_{\Delta}Q_{p+1}V\,d\zeta\,d\rho=0\ \forall V\in P_{p-1}$$

To establish (27). The proof of (28) follows the same line reasoning.

Now, substitute the maclaurin series of the local error in the DGM orthogonally condition on the canonical element (11) with $W_p = P_p$ and follow the reasoning of the $O(h^{p+1})$ term to write

$$\int_{\Gamma_+}(\beta.n)Q_{p+1}Vds + \iint_\Delta (-\beta.\nabla V)\, Q_{p+1}\, d\zeta\, d\rho = 0\ \forall V \in P_p. \quad (34)$$

Since on the canonical element the outflow edge is the segment $\rho = 1-\zeta, 0 \le \zeta \le 1$ becomes

$$\int_0^1(\beta.n)Q_{p+1}(1-\rho,\rho)V(1-\rho,\rho)ds - \iint_\Delta (\beta.\nabla V)\, Q_{p+1}\, d\zeta\, d\rho = 0, \forall V \in P_p. \quad (35)$$

Testing against $V = (1-\zeta-\rho)^i, 1 \le i \le p$ the first in (35) is zero which establishes (29).

Equation (25) infers that the local error is $O(h^{p+2})$super-convergent at the roots of Legendre polynomial on the outflow edge.

The following theorem state and prove the same results as for $U^- = \pi u$, an interpolant of the exact boundary condition at the roots of $p+1$ degree Legendre on the inflow edges.

Theorem 2: Under the same assumptions as in $U^-|_{\Gamma_-} = \pi g$ on each inflow boundary edge the properties (22) and (25-29) still hold.

Proof: Since the inflow term in the orthogonally condition (11) is not zero in general, substituting the series (22) in the DG orthogonally condition (11), using

$$w(x(\zeta)) - \pi w(x(\zeta)) = \sum_{k=p+131}^{\infty} Q_k^-(\zeta)h^k$$

And collecting terms having the same power of h we obtain (11)

$$\sum_{k=p+1}^{\infty} h^k\left(\int_{\Gamma_-}(\beta.n)Q_k^- Vds + \int_{\Gamma_+}(\beta.n)Q_k Vds + \iint_\Delta (-\beta.\nabla V Q_k + bVQ_{k-1})\, d\zeta\, d\rho\right) = 0, \forall V \in P_p \ldots\ldots\ldots\ldots\ldots\ldots(36)$$

A direct application reveals that $Q_{p+1}^-$ satisfies

$$\int_{\Gamma_-}(\beta.n)Q_{p+1}^- Vds = 0, \forall V \in P_p$$

Thus, the $O(h^{p+1})$ term yields

$$\int_{\Gamma_+}(\beta.n)Q_{p+1}Vds + \iint_\Delta \left(-\beta.\nabla V Q_{p+1}\right) d\zeta\, d\rho = 0, \forall V \in P_p. \ldots\ldots\ldots\ldots..(37)$$

From this point on the proof is the same as for theorem 1. Next, consider elements type II and III using the spaces $U_p$.

Nothing that $P_p \subset U_p$ and $P_p \subset V_p$, for $p \ge 1$, apply the same proof to establish the result of $U_p$ and $V_p$. However, the DG error in the larger spaces $U_p$ and $V_p$ satisfies additional orthogonality conditions on elements of type II and III as stated in the next theorem.

Theorem 3: Let $p \ge 1$ and $\Delta = \{(\zeta,\rho), \zeta,\rho \ge 0, \zeta+\rho \le 1\}$. Let $u \in C^\infty(\Delta)$ and $U \in U_p(\Delta)$ be the solution of (1) and (8) respectively with $U^-|_{\Gamma_-} = u$. If $\beta_1, \beta_2 \ge 0$ such that $\Delta$ is either of type II and III, then the local finite element error can be written as in (25) where the leading term $Q_{p+1}$, satisfies the following conditions

$$\int_0^1 \beta_2\, Q_{p+1}(\zeta, 1-\zeta)\zeta^{p+1}d\zeta + \int_0^1\int_0^{1-\zeta} \beta_1\, \zeta^{p+1}\frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\zeta}d\zeta d\rho = 0, \quad (38)$$

And

$$\int_0^1 \beta_1\, Q_{p+1}(1-\rho,\rho)\rho^{p+1}d\rho + \int_0^1\int_0^{1-\rho} \beta_2\, \rho^{p+1}\frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\rho}d\zeta d\rho = 0, \quad (39)$$

If either $\beta_1 = 0, \beta_2 > 0\ or\ \beta_1 > 0, \beta_2 = 0$, i.e.,$\Delta$ is of type III, then the leading term of the local error is zero on the outflow edge

$$Q_{p+1}(1-\rho,\rho) = 0, 0 \le \rho \le 1$$

Furthermore, if $\beta_1 > 0, \beta_2 = 0$, then

$$\int_0^1\int_0^{1-\zeta} \zeta^k Q_{p+1}(\zeta,\rho)d\zeta d\rho = 0, 0 \le k \le p$$

$$\int_0^{1-\zeta}\frac{\partial^k Q_{p+1}(\zeta,\rho)}{\partial\zeta^k}d\rho = 0, k = 0,1, 0 \le \zeta \le 1$$

$$\iint_\Delta \zeta^{p+1}\frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\zeta}d\zeta d\rho = 0$$

Similarly, if $\beta_2 > 0, \beta_1 = 0$, then

$$\int_0^1\int_0^{1-\zeta} \rho^k Q_{p+1}(\zeta,\rho)d\zeta d\rho = 0, 0 \le k \le p$$

$$\int_0^{1-\rho}\frac{\partial^k Q_{p+1}(\zeta,\rho)}{\partial\rho^k}d\rho = 0, k = 0,1, 0 \le \zeta \le 1$$

And

$$\iint_\Delta \rho^{p+1}\frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\rho}d\zeta d\rho = 0$$

Proof: Inserting the Maclaurin series for the local error (22) in this DGM orthogonally condition (35) with $U^-|_{\Gamma_-} = u$ and $W_p = U_p$ the $O(h^{p+2})$term leads to (34) for all $V \in U_p$.

Testing against $V = \zeta^{p+1}$ we obtain

$$\int_0^1(\beta_1+\beta_2)\, Q_{p+1}(\zeta, 1-\zeta)\zeta^{p+1}d\zeta + \iint_\Delta (-\beta.\nabla V Q_{p+1})\zeta^{p+1}\, d\zeta\, d\rho = 0, \quad (40)$$

Which in turn, can be written as

$$\int_0^1 (\beta_1+\beta_2)\, Q_{p+1}(\zeta,1-\zeta)\zeta^{p+1} d\zeta - \int_0^1 \left[(\beta_1(p+1)\zeta^p) \int_0^{1-\zeta} Q_{p+1}(\zeta,\rho)d\rho\right] d\zeta = 0 \quad (41)$$

Consider the polynomials

$$q(\zeta) = \int_0^{1-\zeta} Q_{p+1}(\zeta,\rho)\, d\rho,$$

Leads to

$$q'(\zeta) = -Q_{p+1}(\zeta,1-\zeta) + \int_0^{1-\zeta} \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\zeta} d\,\rho, 0 \le \zeta \le 1 \quad (42)$$

Now, integrate the second term in (41) by parts and using (42) leads to

$$\int_0^1 (\beta_1+\beta_2)\, Q_{p+1}(\zeta,1-\zeta)\zeta^{p+1} d\zeta - \beta_1\zeta^{p+1} \int_0^{1-\zeta} Q_{p+1}(\zeta,\rho)d\rho\Big|_{\zeta=0}^{\zeta=1} + \int_0^1 \beta_1\zeta^{p+1}\left(-Q_{p+1}(\zeta,1-\zeta) + \int_0^{1-\zeta} \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\zeta} d\,\rho\right) d\zeta = 0. \quad (43)$$

Using $q(1) = 0$, it can be obtained

$$\int_0^1 (\beta_1+\beta_2)\, Q_{p+1}(\zeta,1-\zeta)\zeta^{p+1} d\zeta - \int_0^1 \beta_1\zeta^{p+1}\left(Q_{p+1}(\zeta,1-\zeta)\right) d\zeta + \int_0^1 \beta_1\zeta^{p+1} \int_0^{1-\zeta} \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\zeta} d\,\rho d\zeta = 0. \quad (44)$$

Now (44) simplifies to

$$\int_0^1 \beta_2\, Q_{p+1}(\zeta,1-\zeta)\zeta^{p+1} d\zeta + \int_0^1 \beta_1\zeta^{p+1} \int_0^{1-\zeta} \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\zeta} d\,\rho d\zeta = 0.$$

This establishes (38).
In order to prove (39) we set $V = \rho^{p+1}$ in (35) to obtain

$$\int_0^1 (\beta_1+\beta_2)\, Q_{p+1}(\rho,1-\rho)\rho^{p+1} d\zeta + \iint_\Delta \left(-\beta.\nabla V Q_{p+1}\right)\rho^{p+1}\, d\zeta\, d\rho = 0, \quad (45)$$

This in turn, can be written as

$$\int_0^1 (\beta_1+\beta_2)\, Q_{p+1}(1-\rho,\rho)\rho^{p+1} d\rho - \int_0^1 \left[(\beta_2(p+1)\rho^p) \int_0^{1-\rho} Q_{p+1}(\zeta,\rho)d\zeta\right] d\rho = 0 \quad (46)$$

Consider the polynomials

$$q(\rho) = \int_0^{1-\rho} Q_{p+1}(\zeta,\rho)\, d\zeta,$$

And

$$q'(\rho) = -Q_{p+1}(1-\rho,\rho) + \int_0^{1-\rho} \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\rho} d\,\zeta, 0 \le \rho \le 1 \quad (47)$$

Integrate the second term in (46) by parts and using (47) leads to

$$\int_0^1 (\beta_1+\beta_2)\, Q_{p+1}(1-\rho,\rho)\rho^{p+1} d\rho - \beta_2\rho^{p+1} \int_0^{1-\rho} Q_{p+1}(\zeta,\rho)d\zeta\Big|_{\rho=0}^{\rho=1} + \int_0^1 \beta_2\rho^{p+1}\left(-Q_{p+1}(1-\rho,\rho) + \int_0^{1-\rho} \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\rho} d\,\zeta\right) d\rho = 0. \quad (48)$$

Using $q(1) = 0$

$$\int_0^1 (\beta_1+\beta_2)\, Q_{p+1}(1-\rho,\rho)\rho^{p+1} d\rho - \int_0^1 \beta_2\rho^{p+1}\left(Q_{p+1}(1-\rho,\rho)\right) d\rho + \int_0^1 \beta_2\rho^{p+1} \int_0^{1-\rho} \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\rho} d\,\rho d\zeta = 0 \quad (49)$$

Now (44) simplifies to

$$\int_0^1 \beta_1\, Q_{p+1}(1-\rho,\rho)\rho^{p+1} d\rho + \int_0^1 \beta_2\rho^{p+1} \int_0^{1-\rho} \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\rho} d\,\zeta d\rho = 0.$$

Which establishes (39).
Continue the proof by considering (37) with $\beta = [\beta_1, 0]^t$ leading to

$$\int_0^1 Q_{p+1}(1-\rho,\rho)\, V d\rho + \int_0^1 \int_0^{1-\zeta} \frac{\partial V}{\partial\zeta} Q_{p+1} d\zeta d\rho = 0, \forall V \in U_p. \quad (50)$$

Testing against $V = \rho^i, 0 \le i \le p+1$, obtain the orthogonality condition on the outflow edge

$$\int_0^1 Q_{p+1}(1-\rho,\rho)\, \rho^i d\rho = 0, 0 \le i \le p+1$$

Since $Q_{p+1}(1-\rho,\rho) \in P_{p+1}$, it can be established

$$Q_{p+1}(1-\rho,\rho) = 0, 0 \le \rho \le 1$$

As a result, (50) becomes

$$\int_0^1 \int_0^{1-\zeta} \frac{\partial V}{\partial\zeta} Q_{p+1} d\zeta d\rho = 0, \forall V \in U_p \quad (51)$$

Testing against $V = \zeta^{k+1}, 0 \le k \le p$, (3.51) yields

$$\int_0^1 \int_0^{1-\zeta} \zeta^k Q_{p+1}(\zeta,\rho) d\zeta d\rho = 0, 0 \le k \le p$$

Consider the $(p+2)$ degree polynomial

$$q(\zeta) = \int_0^{1-\zeta} Q_{p+1}(\zeta,\rho)\, d\rho,$$

And its derivative

$$q'(\zeta) = -Q_{p+1}(\zeta, 1-\zeta) + \int_0^{1-\zeta} \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial \zeta} d\rho = \int_0^{1-\zeta} \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial \zeta} d\rho \quad (52)$$

Where, It is used $Q_{p+1}(1-\rho,\rho) = 0$.

Noting that $q(1) = q'(1) = 0$, write $q(\zeta) = (1-\zeta)^2 r(\zeta)$, where $r(\zeta)$ is a polynomial of degree $p$. Then

$$0 = \int_0^1 \int_0^{1-\zeta} (1-\zeta)^k Q_{p+1}(\zeta,\rho) d\zeta d\rho = \int_0^1 (1-\zeta)^k \left( \int_0^{1-\zeta} Q_{p+1}(\zeta,\rho) d\rho \right) d\zeta$$

$$= \int_0^1 (1-\zeta)^k q(\zeta) d\zeta, 0 \le k \le p$$

Hence,

$$\int_0^1 (1-\zeta)^k (1-\zeta)^2 r(\zeta) d\zeta = 0, 0 \le k \le p. \quad (53)$$

Which infers that $r(\zeta)$ is orthogonal to all polynomial in $P_p$ with respect to the weight function$(1-\zeta)^2$. Thus,$q(\zeta) = 0$ which completes the proof.

The proof for the case $\beta = [0, \beta_2]^t$ is similar.

The next theorem is state and prove the super convergence results for elements of type II and III using the space $V_p$.

Theorem 5: Under the assumption of theorem 1 and using the polynomial space $V_p$ the leading term in the local DG error on an element of type II and III satisfies

$$\iint_\Delta (\beta.\nabla Q_{p+1}) \zeta^i \rho^{p+1-i} d\zeta\, d\rho = 0, i = 1, \dots .p. \quad (54)$$

Moreover, on an element of type III with $\beta_1 > 0, \beta_2 = 0$

$$\iint_\Delta \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial \zeta} \zeta^i \rho^{p+1-i} d\zeta d\rho = 0, i = 1, \dots .p. \quad (55)$$

And

$$Q_{p+1}(\zeta,\rho) = C_1 L_{p+1}(\rho) + C_2 (1-\rho)^{p+1} R_{p+1}\left(\frac{\zeta}{1-\rho}\right) \quad (56)$$

Similarly, if $\beta_2 > 0, \beta_1 = 0$

$$\iint_\Delta \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial \rho} \zeta^i \rho^{p+1-i} d\zeta d\rho = 0, i = 1, \dots .p. \quad (57)$$

And

$$Q_{p+1}(\zeta,\rho) = C_1 L_{p+1}(\zeta) + C_2 (1-\zeta)^{p+1} R_{p+1}\left(\frac{\rho}{1-\zeta}\right) \quad (58)$$

Proof: Inserting the Maclaurin series for the local error (22) in this DGM orthogonally condition (35) with $U^-|_{\Gamma_-} = u$ and $W_p = V_p$ the $O(h^{p+1})$term leads to

$$\int_{\Gamma_+} (\beta.n) Q_{p+1} V ds + \iint_\Delta \left(-\beta.\nabla V Q_{p+1}\right) d\zeta\, d\rho = 0, \forall V \in V_p$$

On the canonical element the outflow edge is segment $\rho = 1-\zeta, 0 \le \zeta \le 1$, becomes

$$\int_0^1 (\beta.n) Q_{p+1}(\zeta, 1-\zeta) V(\zeta, 1-\zeta) d\zeta + \iint_\Delta \left(-\beta.\nabla V Q_{p+1}\right) d\zeta\, d\rho = 0, \forall V \in V_p \quad (59)$$

Testing against $V = \zeta^i \rho^{p+1-i}, 1 \le i \le p$

$$\int_0^1 (\beta_1 + \beta_2) Q_{p+1}(\zeta, 1-\zeta) \zeta^i \rho^{p+1-i} d\zeta - \iint_\Delta \left(\beta.\nabla(\zeta^i \rho^{p+1-i}) Q_{p+1}\right) d\zeta\, d\rho = 0, 1 \le i \le p \quad (60)$$

This can be written as

$$\int_0^1 (\beta_1 + \beta_2) Q_{p+1}(\zeta, 1-\zeta) \zeta^i \rho^{p+1-i} d\zeta - \int_0^1 (\beta_1 i \zeta^{i-1}) q(\zeta) d\zeta$$

$$- \int_0^1 (\beta_2 (p+1-i) \rho^{p-i}) g(\rho) d\rho = 0, 1 \le i \le p \quad (61)$$

Where

$$q(\zeta) = \int_0^{1-\zeta} \rho^{p+1-i} Q_{p+1}(\zeta,\rho)\, d\rho, and g(\rho) = \int_0^{1-\rho} \zeta^i Q_{p+1}(\zeta,\rho)\, d\zeta$$

Differentiating $q(\zeta)$ and $g(\rho)$ yields

$$q'(\zeta) = -(1-\zeta)^{p+1-i} Q_{p+1}(\zeta, 1-\zeta) + \int_0^{1-\zeta} \rho^{p+1-i} \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial \zeta} d\rho, 0 \le \zeta \le 1$$

$$g'(\rho) = -(1-\rho)^i Q_{p+1}(1-\rho,\rho) + \int_0^{1-\rho} \zeta^i \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial \rho} d\zeta, 0 \le \zeta \le 1$$

Integrating the second and third term in (61) by parts and using$q'(\zeta)$ and $g'(\rho)$ leads to

$$\int_0^1 (\beta_1 + \beta_2) Q_{p+1}(\zeta, 1-\zeta) \zeta^i (1-\zeta)^{p+1-i} d\zeta - \beta_1 \zeta^i q(\zeta)\Big|_{\zeta=0}^{\zeta=1} + \int_0^1 \beta_1 \zeta^i q'(\zeta) d\zeta$$

$$-\beta_2 \rho^{p+1-i} g(\rho)\Big|_{\rho=0}^{\rho=1} + \int_0^1 \beta_2 \rho^{p+1-i} g'^{(\rho)} d\rho = 0, 1 \le i \le p. \quad (61)$$

Using $q(1) = g(1) = 0$

$$\int_0^1 (\beta_1 + \beta_2) Q_{p+1}(\zeta, 1-\zeta) \zeta^i (1-\zeta)^{p+1-i} d\zeta - \int_0^1 \beta_1 \zeta^i (1-\zeta)^{p+1-i} Q_{p+1}(\zeta, 1-\zeta) d\zeta$$

$$+\int_0^1 \beta_1 \zeta^i \int_0^{1-\zeta} \rho^{p+1-i} \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial \zeta} d\rho\, \partial\zeta - \int_0^1 \beta_2 \rho^{p+1-i}(1-\rho)^i Q_{p+1}(1-\rho,\rho) d\rho$$

$$+\int_0^1 \beta_2 \rho^{p+1-i} \int_0^{1-\rho} \zeta^i \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial \rho} d\zeta \partial\rho = 0, 1 \le i \le p. \quad (62)$$

Note that outflow boundary terms cancel out and (62) becomes

$$\beta_1 \int_0^1 \int_0^{1-\zeta} \zeta^i \rho^{p+1-i} \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial \zeta} d\zeta \partial\rho + \beta_2 \int_0^1 \int_0^{1-\rho} \zeta^i \rho^{p+1-i} \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial \rho} d\zeta \partial\rho = 0) \quad (63)$$

Thus, it is established (54).

Corollary 1: Under the statement of theorem 3 on a triangle of type III with $\beta_1 > 0$ *and* $\beta_2 = 0$ and $U$ is the DG solution, the following results hold.

If $U \in P_p$, the superconvergence rates are $O(h^{p+2})$ of $U$ at the points

$$(1-\rho_i, \rho_i), i = 1, \dots\dots\dots, p+1. \quad (64)$$

Where $\rho_1 < \cdots < \rho_{p+1}$ are the roots of $L_{p+1}$ in [0,1].

If $U \in V_p$ we have $O(h^{p+2})$ superconvergence rates of $U$ at the points

$$((1-\rho_i)\zeta_k, \rho_i), i = 1, \dots\dots\dots, p+1, k = 1, \dots\dots\dots., p+1 \quad (65)$$

Where $\zeta_1 < \cdots < \zeta_{p+1} = 1$ are the roots of Radua polynomial $L_{p+1}(\rho) - L_p(\rho)$ shifted to [0,1] and $\rho_1 < \cdots < \rho_{p+1}$ are the roots of $L_{p+1}$ shifted to [0,1].

Finally, if $U \in U_p$, the superconvergence rates are having $O(h^{p+2})$at every point on the outflow edge

$$(\zeta, 1-\zeta), 0 \le \zeta \le 1. \quad (66)$$

Proof: Note that (64) and (66) follows directly from theorem 1 and 3, respectively. In order to prove (64) it is noted that

$$Q_{p+1} = C_1 L_{p+1}(\rho) + C_2 \prod_{k=1}^{p+1} (\zeta - (1-\rho)\zeta_k)$$

Thus, each line $\rho = 1 - \frac{\zeta}{\zeta_k}, Q_{p+1} = C_1 L_{p+1}(\rho)$

For elements of type I consider the DG formulation (8) on the right angle $\Delta$ with vertices $(h,0), (h,h)$ and $(0,h)$ mapped to the reference triangle defined by the vertices $(1,0), (1,1)$ and $(0,1)$. The next theorem state and prove new super convergences results for elements of type I using the spaces $U_p$.

Theorem 7: Let $p \ge 1$ and $\Delta$ denote the reference triangle defined by the vertices $(1,0), (1,1)$ and $(0,1)$. Let $u \in C^\infty(\Delta)$ and $U \in U_p(\Delta)$ be a solution of (1) and (8), respectively with $U^-|_{\Gamma_-} = u$. If $\beta_1 > 0$ and $\beta_2 > 0$, then the local error can be written as (22) where the leading term $Q_{p+1}$, satisfies the following orthogonality conditions

$$\int_{\Gamma_+} (\beta.n) Q_{p+1} (\beta_2\zeta - \beta_1\rho)^i ds = 0, i = 0, \dots\dots\dots, p, \quad (67)$$

$$\iint_\Delta \beta.\nabla(\zeta-1)^i(\rho-1)^{j-1} Q_{p+1}\, d\zeta\, d\rho = 0, 1 \le i \le j \le p, i+j \le p \quad (68)$$

$$\int_0^1 \beta_2 Q_{p+1}(\zeta,1)(\zeta-1)^i d\zeta - i \iint_\Delta \beta_1 (\zeta-1)^{i-1} Q_{p+1}\, d\zeta\, d\rho = 0, i = 1, \dots\dots p+1 \quad (69)$$

$$\int_0^1 \beta_1 Q_{p+1}(1,\rho)(\rho-1)^i d\rho - i \iint_\Delta \beta_2 (\rho-1)^{i-1} Q_{p+1}\, d\zeta\, d\rho = 0, i = 1, \dots\dots p+1 \quad (70)$$

Furthermore

$$\beta_2 Q_{p+1}(\zeta,1) + \beta_1 Q_{p+1}(\zeta, 1-\zeta) + \beta_1 \int_{1-\zeta}^1 \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial \zeta} d\rho = 0, 0 \le \zeta \le 1, \quad (71)$$

$$\beta_1 Q_{p+1}(1,\rho) + \beta_1 Q_{p+1}(1-\rho,\rho) + \beta_2 \int_{1-\rho}^1 \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial \rho} d\zeta = 0, 0 \le \rho \le 1, \quad (72)$$

Finally, the point wisesuper convergence

$$Q_{p+1}(0,1) = Q_{p+1}(1,0) = 0$$

Proof: The DG orthogonally (11) can be written as

$$\int_{\Gamma_-} (\beta.n)\epsilon^- V ds + \int_{\Gamma_+} (\beta.n)\epsilon V ds + \iint_\Delta (-\beta.\nabla V + hbV)\, \epsilon\, d\zeta\, d\rho = 0, \forall V \in U_p \quad (73)$$

Let, use $U^-|_{\Gamma_-} = u$ and substituting (22) in (73) and collecting the terms with same powers in $h$, the $O(h^{p+1})$ term yields

$$\int_{\Gamma_+} (\beta.n) Q_{p+1} V ds + \iint_\Delta (-\beta.\nabla V)\, Q_{p+1}\, d\zeta\, d\rho = 0, \forall V \in U_p, \quad (74)$$

Testing against $(\beta_2\zeta - \beta_1\rho)^i = 0, i = 0, \dots\dots, p$, establish (67).

Since the two outflow edges of $\Delta$ are $\zeta = 1$ and $\rho = 1$, (74) may be written as

$$\int_0^1 \beta_2 Q_{p+1}(\zeta,1) V(\zeta,1) d\zeta + \int_0^1 \beta_1 Q_{p+1}(1,\rho) V(1,\rho) d\rho - \iint_\Delta (\beta.\nabla V)\, Q_{p+1}\, d\zeta\, d\rho = 0,$$

$$\forall V \in U_p \quad (75)$$

Testing against $V = (\zeta-1)^i(\rho-1)^{j-1}, 1 \le i+j \le p$, the boundary terms in (75) are zero which yields (68)

Next, obtain (69) and (70) from (75) by testing against $V=(\zeta-1)^i, 1\le i\le p+1$ and $V=(\rho-1)^i, 1\le i\le p+1$ respectively.

Finally, prove (71) by considering the polynomial

$$q(\zeta)=\int_{1-\zeta}^{1} Q_{p+1}(\zeta,\rho)d\rho,$$

And

$$q'(\zeta)=Q_{p+1}(\zeta,1-\zeta)+\int_{1-\zeta}^{1}\frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\zeta}d\,\rho, 0\le\zeta\le 1. \quad (76)$$

Testing against $V=(\zeta-1)^i, 0\le i\le p+1$, in (75)

$$\int_0^1 \beta_2 Q_{p+1}(\zeta,1)d\zeta=0$$

And

$$\int_0^1 \beta_2 Q_{p+1}(\zeta,1)(\zeta-1)^i d\zeta - \int_0^1\int_{1-\zeta}^{1} \beta_1 i(\zeta-1)^{i-1}Q_{p+1}\,d\zeta\,d\rho=0, 0\le i\le p \quad (77)$$

This can be written as

$$\int_0^1 \beta_2 Q_{p+1}(\zeta,1)(\zeta-1)^i d\zeta - \int_0^1\left[\beta_1 i(\zeta-1)^{i-1}\int_{1-\zeta}^{1} Q_{p+1}(\zeta,\rho)\,d\rho\right]d\zeta$$

$$=\int_0^1 \beta_2 Q_{p+1}(\zeta,1)(\zeta-1)^i d\zeta - \int_0^1 \beta_1 i(\zeta-1)^{i-1}q(\zeta)d\zeta = 0, \quad (78)$$

Integration by parts

$$\int_0^1 \beta_2 Q_{p+1}(\zeta,1)(\zeta-1)^i d\zeta - \beta_1(\zeta-1)^i\Big|_{\zeta=0}^{\zeta=1} + \int_0^1 \beta_1 i(\zeta-1)^{i-1}q'(\zeta)d\zeta=0.) \quad (79)$$

Noting that the following holds

$$-\beta_1(\zeta-1)^0 q(\zeta)\Big|_{\zeta=0}^{\zeta=1}+\int_0^1\beta_1(\zeta-1)^0 q'(\zeta)d\zeta = -\beta_1 q(\zeta)\Big|_{\zeta=0}^{\zeta=1}+\beta_1 q(\zeta)\Big|_{\zeta=0}^{\zeta=1}=0$$

Prove (79) for $i=0$.

Thus

$$\int_0^1 \beta_2 Q_{p+1}(\zeta,1)(\zeta-1)^i d\zeta - \beta_1(\zeta-1)^i\Big|_{\zeta=0}^{\zeta=1} + \int_0^1 \beta_1 i(\zeta-1)^{i-1}q'(\zeta)d\zeta=0, 0\le i\le p+1$$

Combining $q(0)=0$, (76) and (79), it is obtained

$$\int_0^1 \beta_2 Q_{p+1}(\zeta,1)(\zeta-1)^i d\zeta + \int_0^1 \beta_1(\zeta-1)^i\left(Q_{p+1}(\zeta,1-\zeta)+\int_{1-\zeta}^{1}\frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\zeta}d\,\rho\right)d\zeta=0, 0\le i\le p+1 \quad (80)$$

Which can be written as the originality condition

$$\int_0^1 H_1(\zeta)(\zeta-1)^i d\zeta=0, i=0,\ldots\ldots.., p+1 \quad (81)$$

Where

$$H_1(\zeta)= \beta_2 Q_{p+1}(\zeta,1)+\beta_1 Q_{p+1}(\zeta,1-\zeta)+\beta_1\int_{1-\zeta}^{1}\frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\zeta}d\,\rho$$

Is a polynomial of degree $p+1$.

Thus, $H_1(\zeta)=0$ which establishes (71)

The proof of (72) follow the same line of reasoning by considering the polynomial

$$s(\rho)=\int_{1-\rho}^{1} Q_{p+1}(\zeta,\rho)d\zeta,$$

And

$$s'(\rho)=Q_{p+1}(1-\rho,\rho)+\int_{1-\rho}^{1}\frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\rho}d\,\zeta, 0\le\rho\le 1 \quad (82)$$

Testing against $V=(\rho-1)^i, 0\le i\le p+1$, in (82)

$$\int_0^1 \beta_2\, Q_{p+1}(\zeta,1)V(\zeta,1)d\zeta+\int_0^1\beta_2\, Q_{p+1}(1,\rho)V(1,\rho)d\rho - \int_0^1\int_{1-\rho}^{1}\beta\nabla V Q_{p+1}\,d\zeta\,d\rho=0\ \forall v\in U_p \quad (83)$$

Yields

$$\int_0^1 \beta_1 Q_{p+1}(1,\rho)d\rho=0$$

And

$$\int_0^1 \beta_1 Q_{p+1}(1,\rho)(\rho-1)^i d\rho - \int_0^1\int_{1-\rho}^{1}\beta_2 i(\rho-1)^{i-1}Q_{p+1}\,d\zeta\,d\rho=0, 0\le i\le p \quad (84)$$

This can be written as

$$\int_0^1 \beta_1 Q_{p+1}(1,\rho)(\rho-1)^i d\rho - \int_0^1\left[\beta_2 i(\rho-1)^{i-1}\int_{1-\rho}^{1}Q_{p+1}(\zeta,\rho)\,d\zeta\right]d\rho$$

$$=\int_0^1 \beta_1 Q_{p+1}(1,\rho)(\rho-1)^i d\rho - \int_0^1\beta_2 i(\rho-1)^{i-1}s(\rho)d\rho = 0, \quad (85)$$

By applying integration by parts

$$\int_0^1 \beta_2 Q_{p+1}(1,\rho)(\rho-1)^i d\rho - \beta_2(\rho-1)^i s(\rho)\Big|_{\rho=0}^{\rho=1} + \int_0^1\beta_2 i(\rho-1)^{i-1}s'(\rho)d\rho=0. \quad (86)$$

Noting that the following holds

$$-\beta_2(\rho-1)^0 s(\rho)\Big|_{\rho=0}^{\rho=1}+\int_0^1\beta_2(\rho-1)^0 s'(\rho)d\rho=$$

$$-\beta_2 s(\rho)|_{\rho=0}^{\rho=1} + \beta_2 s(\rho)|_{\rho=0}^{\rho=1} = 0$$

Prove (86) for $i = 0$.
Thus

$$\int_0^1 \beta_1 Q_{p+1}(1,\rho)(\rho-1)^i d\rho - \beta_2(\rho-1)^i s(\rho)\Big|_{\rho=0}^{\rho=1} + \int_0^1 \beta_2(\rho-1)^{i-1} s'(\rho) d\rho = 0, 0 \le i \le p+1 \quad (87)$$

Combining $s(0) = 0$, (82) and (87)

$$\int_0^1 \beta_1 Q_{p+1}(1,\rho)(\rho-1)^i d\rho + \int_0^1 \beta_2(\rho-1)^i \left(Q_{p+1}(1-\rho,\rho) + \int_{1-\rho}^1 \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\rho} d\zeta\right) d\rho = 0,$$

$$0 \le i \le p+1$$

It can written

$$\int_0^1 H_2(\rho)(\rho-1)^i d\rho = 0, i = 0, \ldots \ldots, p+1$$

Where

$$H_2(\zeta) = \beta_1 Q_{p+1}(1,\rho) + \beta_2 Q_{p+1}(1-\rho,\rho) + \beta_2 \int_{1-\rho}^1 \frac{\partial Q_{p+1}(\zeta,\rho)}{\partial\rho} d\zeta$$

Thus, $H_2(\zeta) = 0$ which establishes (72)

Conclude the proof of this theorem by establishing the point wise super convergence (73) as

$$0 = H_1(0) = \beta_2 Q_{p+1}(0,1) + \beta_1 Q_{p+1}(0,1) = (\beta_1 + \beta_2) Q_{p+1}(0,1)$$

Similarly, $H_2(0) = 0$ leads to

$$H_2(0) = \beta_1 Q_{p+1}(1,0) + \beta_2 Q_{p+1}(1,0) = (\beta_1 + \beta_2) Q_{p+1}(1,0)$$

This completes the proof of the theorem.

## 3. Conclusion

This paper has investigated the error of the numerical solution by applying the Discontinuous Galerkin finite element method for the second-order hyperbolic differential equation. It is a different and straightforward approach to seek error analysis from all other finite element scheme whichis given in the literature. It is investigated higher-dimension discontinuous Galerkin methods for hyperbolic problems on triangular meshes and also studied the effect of finite element spaces on the superconvergence properties of DG solutions on three types of triangular elements and it showed that the DG solution is $O(h^{p+2})$superconvergent at Legendre points on the outflow edge on triangles having one outflow edge using three polynomial spaces.

## Abbreviations

DG Discontinuous Galerkin

## Conflicts of Interest

The authors declare no conflicts of interest.

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