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# Positive Definite Solution of a Class of Matrix Equations

**Ran Yang**

School of Mathematics and Statistics, Shandong Normal University, Jinan, P. R. China

**Email address:**
2056653669@qq.com

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**Abstract:** The sources and application fields of nonlinear matrix equations are quite extensive, including control theory, statistics, dynamic programming, etc. A large number of problems can be transformed into solving matrix equations. In this paper we establish some necessary and sufficient conditions for the existence of positive definite solutions to the nonlinear matrix equation $A^T XA = \eta X$. The existence of positive definite solutions to corresponding inequalities were discussed too. In addition, some examples are presented to illustrate the main results of this paper.

**Keywords:** Positive Definite Solution, Matrix Equations, Nonlinear Equations

## 1. Introduction

In practical applications such as bioscience, physics [1], dynamic programming, engineering technology, and economic theory, nonlinear phenomena [2–6] are very common. In recent years, the study of nonlinear matrix equations has become one of the most active topics in mathematics. Nonlinear matrix equations play an important role in control theory [7–9], network optimization, statistics and difference methods for partial differential equations [10–12] and other fields.

The existence of the positive definite solution of the matrix equation is one of the hot point problems studied in the field of matrix theory. There have been some research results on positive definite solutions of nonlinear matrix equations [13–17]. For instance, Hermitian positive definite solutions of matrix equation

$$A - A^* X^{-2} A = I$$

was studied by Zhang Yuhai [13], the necessary conditions and sufficient conditions for the existence of positive definite solutions for the matrix equation

$$X \pm A^* X^{-q} A = Q$$

was derived by

Hasanov [14]. More, the nonlinear matrix equation

$$X + A^* X^{-n} A = Q$$

and the properties of its positive definite solutions are studied by Ivanov [15].

In this paper, we study the positive definite solution of the nonlinear matrix equation

$$A^T XA = \eta X,$$

where $A$ is an $n \times n$ invertible real matrix, $\eta > 0$. We also discussed the existence of positive definite solutions to corresponding inequalities. Finally, several examples are provided to illustrate the main conclusions obtained.

Throughout this paper, we denote the real number field by $R$. The notations $R^{m\times n}$, $OR^{n\times n}$stand for the sets of all $m \times n$ real matrices and all $n \times n$ orthogonal matrices, respectively. The identity matrix of order $n$ is denoted by $I_n$. For a matrix $A$, the symbols $A^T$, $A^{-1}$ and $r(A)$ stand for the transpose, the inverse, the rank of $A$, respectively. If $A$ is a square matrix , $det(A)$ means to take the determinant of $A$. We denote $A > 0$ stands for $A$ is a positive definite matrix. If $A - B$ is a positive definite (semidefinite) matrix, then we write $A > B(A \geq B)$. Similarly, if $A - B$ is a negative definite (semidefinite) matrix, then we denote $A < B(A \leq B)$. Furthermore, the symbol $p_\lambda(A)$, $|\lambda|$ represent the geometric multiplicity and the module of eigenvalue $\lambda$ of $A$, respectively. And the symbols $\lambda_{max}(A)$ and $\lambda_{min}(A)$ denote the maximal and minimal eigenvalues of real symmetric matrix $A$, respectively.

# 2. Existence of Positive Definite Solutions to Inequalities $A^TXA \leq \eta X$

In this section, we consider the existence question of positive definite solutions for the following inequality equation

$$A^TXA \leq \eta X$$

where $A$ is an $n \times n$ real matrix, $\eta > 0$.

Firstly, based on the orthogonal similarity standard form of the real symmetric matrix and orthogonal similarity transformation preserves the semi positive determinacy of real symmetric matrices, we have the following lemma directly.

*Lemma* 2.1. Suppose $A \in R^{n\times n}$ is a real symmetric matrix, there

$$\lambda_{min}(A)I_n \leq A \leq \lambda_{max}(A)I_n,$$

where $\lambda_{max}(A)$ and $\lambda_{min}(A)$ respectively represent the maximum and minimum eigenvalues of matrix $A$.

*Theorem* 2.1. For any $A \in R^{n\times n}$, there exist $0 < X \in R^{n\times n}$ and $0 < \eta \in R$ such that

$$A^TXA \leq \eta X.$$

*Proof* For any $A \in R^{n\times n}$, then $A^TA \in R^{n\times n}$ is obviously a positive semidefinite matrix. So the eigenvalues of $A^TA$ are all non negative real numbers. Set

$$X = I_n, \eta = \lambda_{max}(A^TA) + 1,$$

there $X > 0, \eta \geq 1 > 0$.

Applying Lemma 2.1 to matrix $A^TA$, we can immediately obtain the following inequality

$$A^TXA \leq \eta X.$$

# 3. Existence of Positive Definite Solutions to Equation $A^TXA = \eta X$

In this section, we derive some necessary and sufficient conditions for the existence of positive definite solutions to the nonlinear matrix equation $A^TXA = \eta X$.

For the sake of simplicity, we provide the following definition.

*Definition* 3.1. Let $A \in R^{n\times n}$ be given. $A$ is said to be positive definite solvable $(PDS)$ if there exist $0 < X \in R^{n\times n}$ and $0 < \eta \in R$ satisfy

$$A^TXA = \eta X. \quad (1)$$

Because the positive definiteness of the matrix is invariant under orthogonal similarity transformation, we have the following Lemma.

*Lemma* 3.1. Let $A \in R^{n\times n}$ be given. $A$ is $PDS$ if and only if there exists $Q \in OR^{n\times n}$ such that $Q^TAQ$ is $PDS$.

In order to derive our main results, we provide the following two lemmas that will be used.

*Lemma* 3.2. (See [18]) Let $A \in R^{n\times n}$ be given, $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the real eigenvalues of $A$, $a_j \pm ib_j$ are the imaginary eigenvalues of $A$, $a_j, b_j \in R$, $j = 1, 2, \ldots, m$, $r + 2m = n$. Then there exists $Q \in OR^{n\times n}$, such that

$$Q^TAQ = \begin{pmatrix} \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_r \end{pmatrix} & * \\ 0 & \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_m \end{pmatrix} \end{pmatrix} \triangleq D, \quad (2)$$

where $A_j$ are real matrix of order 2, $a_j \pm ib_j$ are the eigenvalues of $A_j$, $j = 1, 2, \ldots, m$. And through the suitable selection of the matrix $Q$, we can enable the diagonal block of $D$ to be arranged in any specified order.

*Lemma* 3.3. (See [19])
Let
$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}$
is a real symmetric matrix,
where $A_{11} \in R^{s\times s}, A_{22} \in R^{t\times t}$. Then $A > 0$ is equivalent to

$$A_{11} > 0, A_{22} - A_{12}^TA_{11}^{-1}A_{12} > 0.$$

### *3.1. Sufficient Conditions*

*Proposition* 3.1. Let $A \in R^{n\times n}$ be an orthogonal matrix, then $A$ is $PDS$.

*Proof* For any $0 < \eta \in R$, set $X = \eta I_n$, we always get

$$A^TXA = \eta X,$$

so $A$ is $PDS$.

*Proposition* 3.2. Let $A_{11} \in R^{s\times s}, A_{12} \in R^{s\times t}$ and $A_{22} \in R^{t\times t}$ be given. Suppose that $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ is $PDS$, then $A_{11}$ is $PDS$.

*Proof* Assume $A$ is $PDS$, then there exist $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix} > 0$ and $\eta > 0$
such that

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}^T \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} = \eta \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix}. \quad (3)$$

On the one hand, it follows from Lemma 3.3 that $X_{11} > 0$. On the other hand, based on computation and equation (3), we derive

$$A_{11}^TX_{11}A_{11} = \eta X_{11}.$$

Therefore $A_{11}$ is $PDS$.

### *3.2. Necessary Conditions*

*Theorem* 3.1. Let $A \in R^{n\times n}$ be a $PDS$ matrix that satisfies equation (1), then $\eta = (det(A))^{\frac{2}{n}}$, $det(A) \neq 0$.

*Proof* By taking the determinant at both ends of equation (1), it can be obtained that

$$det(A^T XA) = \eta^n det(X).$$

Since

$$det(A^T XA) = det(A^T)det(X)det(A)$$

and

$$det(X) > 0, \eta > 0 \ ,$$

then we derive Theorem 3.1 immediately.

*Theorem* 3.2. Let $A \in R^{n\times n}$ be a $PDS$ matrix, then the module of each eigenvalue of matrix $A$ is $(|det(A)|)^{\frac{1}{n}}$.

*Proof* According to Lemma 3.2, there exists $Q \in OR^{n\times n}$ satisfies equation (2), where $\lambda_1$, $\lambda_2$, ..., $\lambda_r$ are the real eigenvalues of $A$, and each $2 \times 2$ matrix $A_j$ on the diagonal has a pair of conjugate imaginary roots $a_j \pm ib_j$, $a_j$, $b_j \in R$, $j = 1, 2, \ldots, m$.

By Lemma 3.1 we obtain that $A$ is $PDS$ iff $D$ is $PDS$, notice that the eigenvalues of matrices $A$ and $D$ are identical, without loss of generality, let's assume $A = D$.

Firstly, we prove the case $m = 0$. The conclusion is immediate for the case $r = 1$. Suppose that the conclusion is true for the matrix of order $r$, now let's prove the case $r + 1$. Let $A_1 \in R^{r\times r}$ be given, set

$$A = \begin{pmatrix} A_1 & * \\ 0 & \lambda_{r+1} \end{pmatrix}.$$

Assume $A$ is $PDS$, from Proposition 3.2, we know that $A_1$ is $PDS$. Then by induction hypothesis and Theorem 3.1, we obtain

$$|\lambda_1| = |\lambda_2| = \ldots = |\lambda_r| = \eta^{\frac{1}{2}}. \tag{4}$$

In view of Theorem 3.1,

$$\eta = (\lambda_1\lambda_2 \ldots \lambda_r\lambda_{r+1})^{\frac{2}{r+1}}. \tag{5}$$

As can be seen from (4) and (5),

$$|\lambda_{r+1}| = \eta^{\frac{1}{2}} = (|det(A)|)^{\frac{1}{r+1}}.$$

Secondly, suppose that the conclusion is true for the matrix of order $r + 2m$, now let's prove the case $r + 2(m + 1)$. Let $A_2 \in R^{(r+2m)\times(r+2m)}$ be given, set

$$A = \begin{pmatrix} A_2 & * & * \\ 0 & a_{m+1} & b_{m+1} \\ 0 & -b_{m+1} & a_{m+1} \end{pmatrix}.$$

Suppose $A$ is $PDS$, by Proposition 3.2 we obtain that $A_2$ is $PDS$. Then by induction hypothesis and Theorem 3.1, we derive

$$|\lambda_1| = |\lambda_2| = \ldots = |\lambda_r| = \sqrt{a_1^2 + b_1^2} = \cdots = \sqrt{a_m^2 + b_m^2} = \eta^{\frac{1}{2}}. \tag{6}$$

Using Theorem 3.1, we have

$$\eta = [\lambda_1 \cdots \lambda_r(a_1^2 + b_1^2) \cdots (a_m^2 + b_m^2)(a_{m+1}^2 + b_{m+1}^2)]^{\frac{2}{r+2(m+1)}}. \tag{7}$$

It follows from (6) and (7) that

$$\sqrt{a_{m+1}^2 + b_{m+1}^2} = \eta^{\frac{1}{2}} = (|det(A)|)^{\frac{1}{r+2(m+1)}}.$$

The proof is completed.

# 4. Examples

In this section, we give two numerical examples to verify the efficiency of the results.

*Example* 4.1. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. There exists $X = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} > 0$, such that $A^T XA = X$, $A$ is $PDS$. Notice that $det(A) = 1 \neq 0$, all characteristic values of $A$ are $1, -1, -1$, i.e. the module of each eigenvalue of matrix $A$ is $(|det(A)|)^{\frac{1}{3}} = 1$.

*Example* 4.2. Let $A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \in R^{n\times n}$ be a diagonal matrix. If $A$ is $PDS$, i.e. there exist $0 < X = (x_{ij} - n \times n) \in R^{n\times n}$ and $0 < \eta \in R$ satisfy

$$A^T XA = \eta X.$$

By comparing the elements in the i-th row and jth column on both sides of the matrix equation mentioned above, it can be concluded that

$$\lambda_i\lambda_j x_{ij} = \eta x_{ij}$$

for any $1 \leq i, j \leq n$. Because $X$ is a positive definite matrix, then $x_{ii} > 0$ for any $1 \leq i \leq n$. So $\lambda_1^2 = \lambda_2^2 = \cdots = \lambda_n^2 = \eta > 0$, i.e. $det(A) \neq 0$, $\eta = (det(A))^{\frac{2}{n}}$, the module of each eigenvalue of matrix $A$ is $(|det(A)|)^{\frac{1}{n}}$.

# 5. Conclusions

In this paper, we obtain that for any real matrix $A$, the inequality $A^TXA \leq \eta X$ always has a positive definite solution. Secondly, we derive some necessary and sufficient conditions for the existence of positive definite solutions to the nonlinear matrix equation $A^TXA = \eta X$. After that, we give some examples to illustrate the main results of this paper. Next, we will continue to explore, hoping to find necessary and sufficient conditions that are easy to distinguish the existence of positive definite solutions to the above matrix equation.

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# Conflicts of Interest

The authors declare no conflicts of interest.

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